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On the absolute maximum dynamic response of a beam subjected to a moving mass

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Abstract. Taking the mid-span/center-point of the structure as the reference point of capturing the maximum dynamic response is very customary in the available literature of the moving load problems. In this article, the absolute maximum dynamic response of an Euler-Bernoulli beam subjected to a moving mass is widely investigated for various boundary conditions of the base beam. The response of the beam is obtained by utilizing a robust numerical method so-called OPSEM (Orthonormal Polynomial Series Expansion Method). It is underlined that the absolute maximum dynamic response of the beam does not necessarily take place at the mid-span of the beam and thus the conventional analysis needs modifications. Therefore, a comprehensive parametric survey of the base beam absolute maximum dynamic response is represented in which the contribution of the velocity and weight of the moving inertial objects are scrutinized and compared to the conventional version (maximum at mid-span).

Keywords: absolute maximum dynamic response; Euler-Bernoulli beam; moving mass; OPSEM

1. Introduction

The concern of designing safe bridge structures dates back to the ancient era. Nowadays, with the accelerating advance of transportation industries, the speed and the weight of the vehicles has been significantly increased. Consequently, the need of the bridge engineers towards having a more precise insight into the dynamic effects of the vehicular loads on the bridge structures could be sensed. There are diverse effective factors governing the design of the bridge structures; one of the eminent factors that should be considered is the maximum values of the bridge dynamic response under the traversing inertial loads caused by the moving vehicles. Such dynamic loads characterize a broad family of engineering problems as the moving load dynamic problems (Ouyang 2011).

It is noteworthy to highlight that there exists many other instances that the influence of a

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moving inertial body on the structure needs to be inspected, such as walk-ways, rail ways, roads, runways, and the deck of ships on which the aircrafts land (Ding *et al.* 2012, Mofid *et al.* 2012, Vaseghi Amiri *et al.* 2013). In addition to the aforementioned substructures, the moving load problem should be dealt with in connection with the computer disk memories, overhead cranes (Oguamanam *et al*. 2001), railway bridge, high-speed train (Johansson *et al.* 2013), guide ways (Wang and Zhang 2007), tunnels (Andersen and Jones 2006, Forrest and Hunt 2006), layered soils (Siddharthan *et al.* 1993, Xu *et al.* 2008), pipelines (Sofiyev *et al.* 2011), and high-speed machining processes (Cifuentes and Lalapet 1992, Ebrahimzadeh Hassanabadi *et al*. 2014a) which are all exposed to the traversing loads (Ouyang 2011, Ebrahimzadeh Hassanabadi *et al.* 2014).

In view of the facts mentioned above, the computation of the dynamic stresses and displacements in a structure under a moving load has been of interest of different researchers in the recent decades. In this regard, Ouyang (2011) has addressed a variety of applications and many published works on moving load dynamic problems covering an interesting discussion on some fundamental solutions of the problem in classic analytical style.

As previously mentioned, it is very customary to monitor the dynamic behavior of the structure under the action of a moving load at the mid-span/center-point of the sub-structure. In this manner, Esmailzadeh and Ghorashi (1995) studied dynamic behavior of a simply supported beam under the action of uniform partly distributed moving masse/force. Kiani *et al.* (2010) researched on the dynamic response of the multi-span viscoelastic thin beams. They applied generalized moving least square method (GMLSM) for discretization of parameters in spatial domain and presented design parameter in the form of mass or velocity of the moving mass for various values of relaxation rate and number of beam spans. Nikkhoo *et al*. (2014) presented a wide parametric study on the maximum dynamic response of a thin rectangular plate to a series of moving masses. Mofidand and Akin (1996), Mofid and Shadnam (2000) utilized DET (Discrete Element Technique) to determine the response time history of beams for different boundary conditions. They assumed that a series of virtual rigid bars hinged together with springs could be assumed as a model of a flexible continuous beam. Yavari *et al.* (2002) obtained the dynamic response of the Timoshenko beams subjected to moving mass via DET. Fotouhi (2007) applied FEM to investigate the dynamic response of the beams with large deformations; they verified the results with other researches. Ding *et al.* (2012) utilized Galerkin method to study the dynamic response of an elastic beam supported on nonlinear foundation with viscous damping. Ebrahimzadeh Hassanabadi *et al.* (2013) proposed the Orthonormal Polynomial Series Expansion Method (OPSEM) as a convenient numerical approach to analyze non-uniform thin beams vibration due to a moving load.

Considering the mid-span/center point of the structure in order to illustrate the overall structural dynamic behavior could be seen in many other published articles (Vaseghi Amiri *et al.* 2013, Ebrahimzadeh Hassanabadi *et al.* 2014, Lin and Chang 2005, EftekharAzam *et al.* 2012). Regarding the voluminous published studies available on the vibration of continuums under the moving loads, the critical values of the bending moment and the vertical displacement are widely computed at the mid-span of the beams. In this paper, the problem of determining the absolute maximum dynamic response of the structure whole through its spatial domain (along the beam length) is focused. By using OPSEM (Ebrahimzadeh Hassanabadi *et al.* 2013) a robust and convenient computational technique is employed to analyze the dynamic behavior of a uniform beam under a moving mass. By verifying the solutions with the outputs of other researchers, the correctness of the numerical investigations is ascertained. The critical points at which the absolute maximum deformation/flexural moment occur are comprehensively tracked during the analyses.

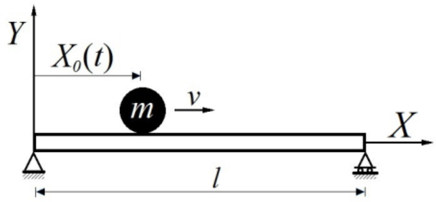


Fig. 1 A beam-type structure under the influence of a traversing lumped inertial body

The conventional approach of illustrating the time history of maximum dynamic response is compared to the spectral curves provided regarding the absolute maximums of values as mentioned in the latter. As a result, it is demonstrated that locations of the critical point of the beams is not necessarily located at the mid-span and it is dependent on the load weight and velocity. In this regard, a comprehensive parametric study is carried out for S-S, S-C and C-C (C: Clamped, S: Simply Supported) beam fixities.

2. Problem formulation and solution

A uniform beam of length l is considered. Its mass density and bending stiffness are ρ and EI correspondingly where E is the modulus of elasticity and I is the beam moment of inertia. The cross-section is signified by A and the vertical displacement at any point along the beam and any time is expressed via $W(X, t)$. A concentrated moving load $P(X, t)$ with the mass and velocity of m and v moves along the beam where $X_0(t)$ stands for the coordinates of the moving object as is shown in Fig. 1.

Considering inertial effects of the moving mass, the governing motion equation of the beam could be expressed as

$$\frac{\partial^2}{\partial t^2}W(X,t)=\frac{1}{\rho A}\left\{P(X,t)-EI\frac{\partial^4}{\partial X^4}W(X,t)\right\} \tag{1}$$

wherein

$$P(X,t)=-m\left(g+\frac{\mathrm{d}^2}{\mathrm{d}t^2}W\left(X_0(t),t\right)\right)\delta\left(X-X_0(t)\right) \tag{2}$$

In the above equation, $\delta(\ )$ is the Dirac Delta and mathematically defines a concentrated pattern of moving load distribution. In this article, the moving mass is assumed to remain fully attached to the base beam during the course of movement. In OPSEM the spatial domain should be transformed to a coordinate system in a norm space composed of a set of BOPs (Basic Orthonormal Polynomials). Thus, by considering $X=\frac{l}{2}(x+1)$ Eq. (1) could be transformed from $X\in[0,l]$ to $X\in[-1,1]$

$$\frac{\partial^2}{\partial t^2}W(x,t)=\frac{1}{\rho A}\left\{\frac{2}{l}P(x,t)-\frac{16EI}{l^4}\frac{\partial^4}{\partial x^4}W(x,t)\right\} \tag{3}$$

$$P(x,t) = -m\left(g + \frac{\mathrm{d}^2}{\mathrm{d}t^2}W\left(x_0(t),t\right)\right)\delta\left(x - x_0(t)\right) \tag{4}$$

where $X_0(t) = \frac{l}{2}\left(x_0(t)+1\right)$. The BOPs are in fact a rearrangement of the Maclaurian series expansion of $W(x, t)$ with respect to x. Hence, the discretization of the spatial domain could be achieved by assuming the below orthonormal polynomial series expansion

$$W\left(x,t\right) = \sum_{j=1}^{n} a_j\left(t\right) w_j\left(x\right) \tag{5}$$

$$\left\langle w_i\left(x\right), w_j\left(x\right)\right\rangle = \int_{-1}^{1} w_i\left(x\right) w_j\left(x\right)\mathrm{d}x = \delta_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \tag{6}$$

in which $w_j(x)$ is the jth BOP and $a_j(t)$ denotes the jth coordinate. OPSEM is a rapidly converging series with the stability of $O(x^{m+n})$ where n is the number of involved BOPs in the truncated Orthonormal Polynomial Series and m denotes the number of fixity constraints where for a single-span beam it is m=4. By introducing Eq. (5) into Eqs. (3) and (4) the problem could be rewritten in terms of the BOPs

$$\sum_{j=1}^{n} w_j\left(x\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}a_j\left(t\right) = \frac{1}{\rho A}\left\{\frac{2}{l}P\left(x,t\right) - \frac{16EI}{l^4}\sum_{j=1}^{n} a_j\left(t\right)\frac{\partial^4}{\partial x^4}w_j\left(x\right)\right\} \tag{7}$$

$$P\left(x,t\right) = -mg - m\sum_{j=1}^{n}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{w_j\left(x_0\left(t\right)\right)a_j\left(t\right)\right\} \tag{8}$$

Assuming $a_j(t)=e^{\mathrm{J}\omega t}$ in Eq. (5) where $\mathrm{J} = \sqrt{-1}$ one can arrive at

$$W\left(x,t\right) = \mathrm{e}^{\mathrm{J}\omega t}\sum_{j=1}^{n} a_j w_j\left(x\right). \tag{9}$$

and replacing Eq. (8) in Eq.(3) regarding $P(x, t)$=0, the relation of the free vibration equation of the beam could be reached

$$\omega^2\sum_{j=1}^{n} a_j w_j\left(x\right) = \frac{16EI}{\rho A l^4}\sum_{j=1}^{n} a_j\frac{\partial^4}{\partial x^4}w_j\left(x\right) \tag{10}$$

The spatial domain could be removed by employing inner product of $w_i(x)$ (aforementioned in Eq. (6)) on both sides of Eqs. (7) and (10). In this manner the Equation of free vibration turns into

$$\mathbf{K}_{\mathrm{b}}\mathbf{a} = \omega^2\mathbf{a} \tag{11}$$

$$\mathbf{a} = \left[a_j\right] \tag{12}$$

$$\mathbf{K}_{\mathrm{b}} = \left[\int_{-1}^{1}\frac{16EI}{\rho A l^4}w_i\left(x\right)\frac{\partial^4}{\partial x^4}w_j\left(x\right)\mathrm{d}x\right] \tag{13}$$

Assuming non-trivial solution to Eq. (11) an eigenvalue problem should be tackled

$$\det\left(\mathbf{K}_{\mathrm{b}}-\omega^2\mathbf{I}\right)=0 \tag{14}$$

which yields the natural mode-shapes and frequencies of the beam. In Eq. (11) **a** denotes the vector form of unknown amplitude factors of BOPs and ω is the natural frequency of free vibration; the dimensions of **I**, $\mathbf{K}_{\mathrm{b}}$, and **a** are $n\times n$, $n\times n$, and $n\times 1$ respectively and **I** signifies the identity matrix.

After the elimination of the space in Eq. (7) one can arrive at (by multiplying both sides of Eq. (7) by $w_i(x)$ and then integrating over the spatial domain [−1, 1])

$$\left(\mathbf{I}+\mathbf{M}_{\mathrm{L}}(t)\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{a}(t)+\mathbf{C}_{\mathrm{L}}(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)+\left(\mathbf{K}_{\mathrm{b}}+\mathbf{K}_{\mathrm{L}}(t)\right)\mathbf{a}(t)=\mathbf{F}(t) \tag{15}$$

in which

The coefficient matrices of the set of coupled ODEs in state-space equations, i.e., $\mathbf{M}_{\mathrm{L}}(t)$, $\mathbf{C}_{\mathrm{L}}(t)$, and $\mathbf{K}_{\mathrm{L}}(t)$, contain the terms related to the convective terms of the moving mass transverse acceleration. These terms could be extracted from the second order total derivative with respect to time in Eq. (8)

$$\begin{aligned}\frac{d^2}{dt^2}\{w_j(x_0(t))a_j(t)\}=\Bigg\{&w_j(x)\frac{d^2}{dt^2}a_j(t)+\left(2\frac{\partial}{\partial t}w_j(x)\frac{d}{dt}x_0(t)\right)\frac{d}{dt}a_j(t)+\\&\left(\frac{\partial^2}{\partial x^2}w_j(x)\left(\frac{d}{dt}x_0(t)\right)^2+w_j(x)\frac{d^2}{dt^2}x_0(t)\right)a_j(t)\Bigg\}_{x=x_0(t)}\end{aligned} \tag{16}$$

therefore

$$\mathbf{M}_{\mathrm{L}}=\left[mw_i(x)w_j(x)\right]_{x=x_0(t)} \tag{17}$$

$$\mathbf{C}_{\mathrm{L}}(t)=\left[2mw_i(x)\frac{\partial}{\partial x}w_j(x)\frac{\mathrm{d}}{\mathrm{d}t}x_0(t)\right]_{x=x_0(t)} \tag{18}$$

$$\mathbf{K}_{\mathrm{L}}(t)=\left[mw_i(x)\left(\frac{\partial^2}{\partial x^2}w_j(x)\left(\frac{\mathrm{d}}{\mathrm{d}t}x_0(t)\right)^2+w_j(x)\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_0(t)\right)\right]_{x=x_0(t)} \tag{19}$$

F(t) is the matrix version for of the gravitational acceleration contribution with $n\times 1$ dimension

$$\mathbf{F}(t)=\left[-mgw_i(x)\right]_{x=x_0(t)} \tag{20}$$

The moving mass/structure inertial interaction could be ignored by neglecting $\mathbf{M}_{\mathrm{L}}(t)$, $\mathbf{C}_{\mathrm{L}}(t)$, and $\mathbf{K}_{\mathrm{L}}(t)$ in the computations

$$\mathbf{I}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{a}(t)+\mathbf{0}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)+\mathbf{K}_{\mathrm{b}}\mathbf{a}(t)=\mathbf{F}(t) \tag{21}$$

The format given in Eq. (21) yields the well-known simulation frame work of moving force. It is significant to notice that the moving force would lead to invalid analysis results for the heavy masses with high velocities. The static deformation w_s (x_0, x) (deformation of the beam at any arbitrary point x due to the application of a load of magnitude $-mg$ on point x_0) of the beam is also straightforward to be concluded from Eq. (20) by ignoring the time derivatives arriving at

$$w_{\mathrm{s}} = \mathbf{\Omega\Xi} \tag{22}$$

$$\mathbf{\Xi} = \mathbf{K}^{-1}\mathbf{F} \tag{23}$$

$$\mathbf{\Omega} = \left[w_j\left(x\right) \right]_{1\times n} \tag{24}$$

Eqs. (15) and (21) could be handled by using a step-by-step matrix exponential based approximation (EftekharAzam *et al.* 2012).

3. Numerical examples

A simply supported beam with length l=5 m, cross sectional area A=0.1l×0.05l, modulus of elasticity E=3.6×10^{10} Pa and mass density ρ=2700 Kg.m^{-3} is considered. The BOPs of a simply supported beam could be conveniently constructed according to Ebrahimzadeh Hassanabadi *et al.* (2013) where in this article the first 18 are involved in the numerical examples. It is assumed that the moving mass starts to traverse the beam (with zero initial conditions of the moving mass/beam dynamics system under study) from left side of the beam.

$$X_0\left(t\right) = vt = \alpha v' t \tag{25}$$

To facilitate the reproduction of the results and in order to obtain a standard normalized format of the data representation, the following parameters are defined

$$T_1 = \frac{1}{\omega_1} \tag{26}$$

$$v' = l/T_1 \tag{27}$$

$$m_{\mathrm{b}} = \rho A l \tag{28}$$

$$w_0 = w_{\mathrm{s}}\left(x_0, 0.5\right) \tag{29}$$

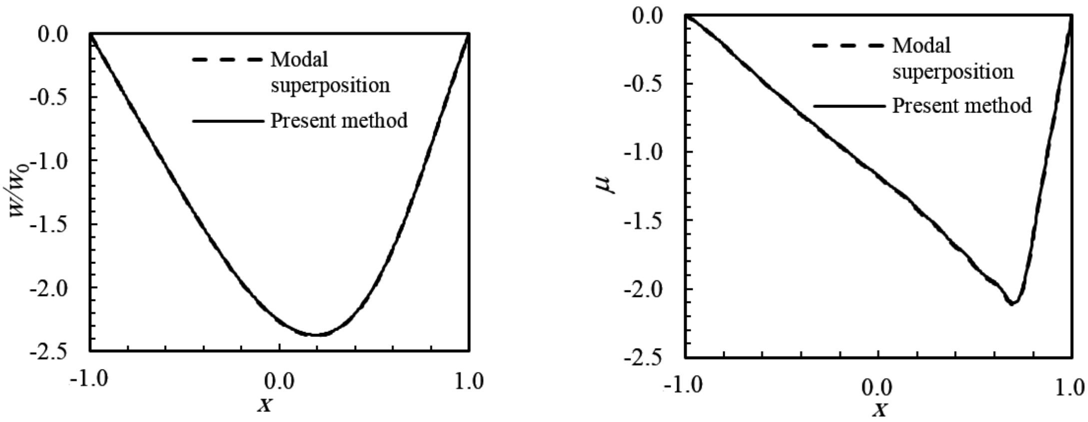
$$M_{\mathrm{b}} = \left[EI \frac{\partial^2}{\partial x^2} w_{\mathrm{s}}\left(x_0, x\right) \right]_{x=0.5} \tag{30}$$

as the normalizing parameters of time, velocity, mass, deformation and bending moment respectively. In this regard,

$$\alpha = v / v' \tag{31}$$

$$\gamma = m / m_{\mathrm{b}} \tag{32}$$

$$\mu = M / M_{\mathrm{b}} \tag{33}$$



(a) beam deformation (b) flexural moment

Fig. 2 Comparisons of the introduced solutions and the modal superposition method; γ=0.45

$$\beta = \frac{\text{maxmum deformation along beam length}}{\text{maxmum deformation at mid-span}} \tag{34}$$

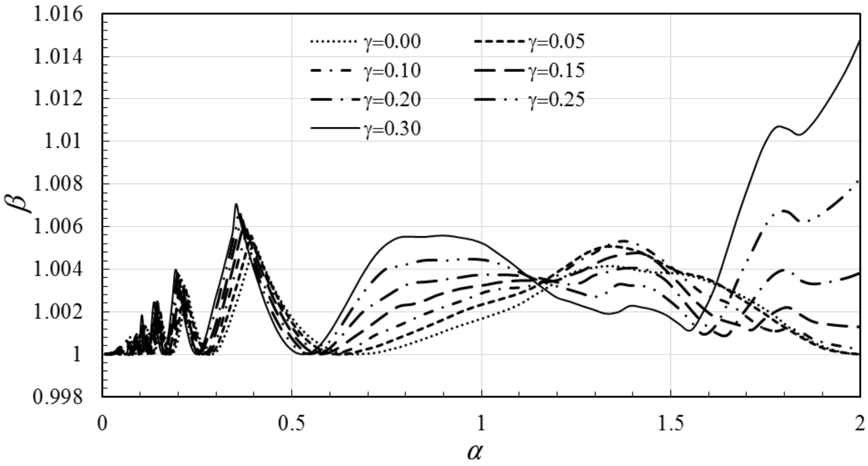
$$\lambda = \frac{\text{maxmum flexural moment along beam length}}{\text{maxmum flexural moment at mid-span}} \tag{35}$$

Are normalized velocity, normalized mass and normalized bending moment. For all of the analysis, the forced vibration of the beam is considered.

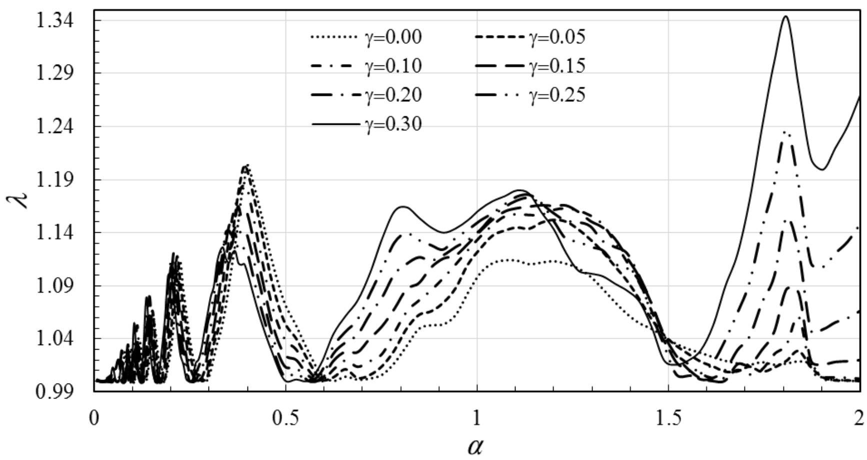
To provide an evaluation of the results precision, a comparison of the present method with the modal superposition method (including the first 20 natural modes based on Mofid *et al*. (2000)) is given in Fig. 2. Given γ=0.45, the graphs represent the deformation and bending moment of the beam along its length at the instant of occurring the absolute maximum response within the range of velocity $0\leq\alpha\leq2$. An excellent agreement of the OPSEM and eigenfunction expansion method could be observed.

Moreover, Fig. 2 clearly proves that the absolute maximum dynamic responses of the beam does not necessarily correspond the maximum response captured at the mid-span. Regarding the weight and the motion velocity of the moving mass, locations of the critical points for bending moment and vertical displacement will be changed.

In this regard, to gain a more clear understanding, β and λ are plotted versus the motion velocity of the moving mass for different weight values in Figs. 3, 5, and 7 for simply supported (S-S), simply supported-clamped (S-C), and clamped (C-C) beams respectively. The locations of the load at which the absolute maximum deformation take place for S-S, S-C, and C-C beams are depicted in Figs. 4, 6, and 8 respectively. The results show that the beam maximum deformation achieved by monitoring the mid-span is no longer valid as the velocity of the traveling object grows. Additionally, it could be clearly concluded that the heavier the mass, the larger the deficiency of the conventional approach for recording the beam maximum response. Although it seems increasing in mass leads to higher β value, with respect to S-C beams Fig. 5(a) indicates it is not always true, actually in this case it is completely inverse because increasing in mass contributes to increase the absolute maximum response but less than of reveled to the mid-span. Figs. 4, 6, and 8



(a) beam deformation

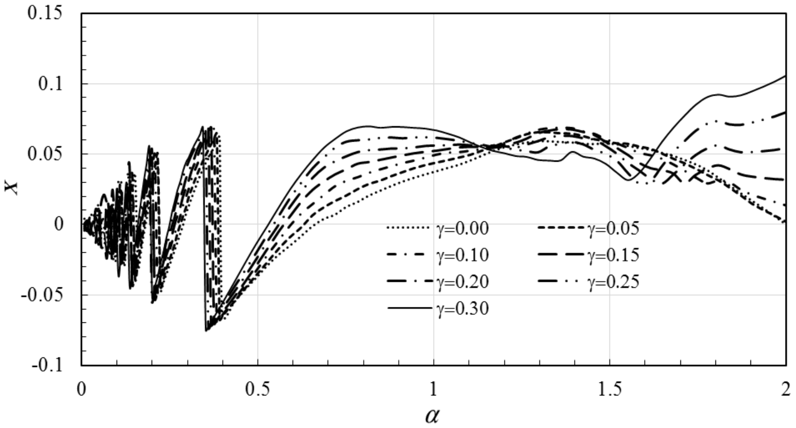


(b) flexural moment

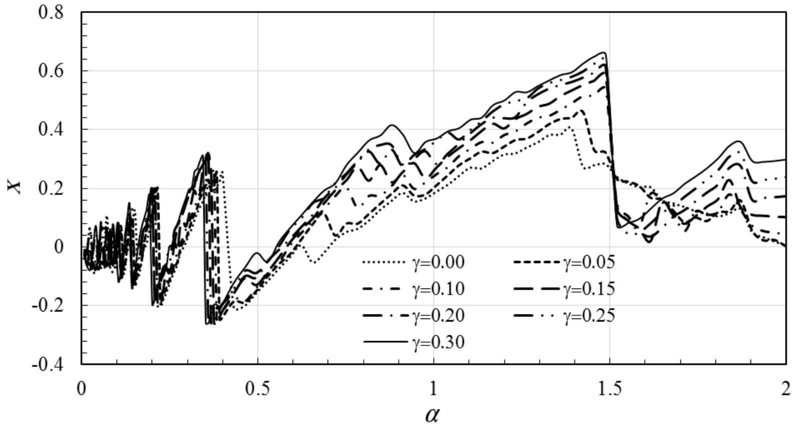
Fig. 3 Absolute maximum dynamic response along beam length versus the maximum deformation at the mid-span for a simply supported beam (S-S)

show the point at which the absolute maximum dynamic response takes place has not a strict location and it is variable along the beam, even though there is an exception. Indeed Figs. 6(b) and 8(b) indicate the absolute maximum dynamic response always takes place on x=−1 or 1 for S-C and C-C beams. Also the results reveled the absolute maximum dynamic responses are oscillatory at low velocities, whereas there are rapid declines and increases in their values.

Identifying the most critical load case is of high priority concerns for the bridge engineers and it should be shed light on with an appropriate precision. Assessing the critical dynamic loading that results in the absolute maximum bending moment of a beam-type bridges is one of the most dominant design factors. Fig. 5 reasonably highlights that the deficiency of the conventional approach in detecting the critical flexural moment becomes notably more error-prone than the beam deformation. Thus, revisiting the previous works on detecting the maximum dynamic response of the beam, specifically the beam bending moment, seems to be necessary.

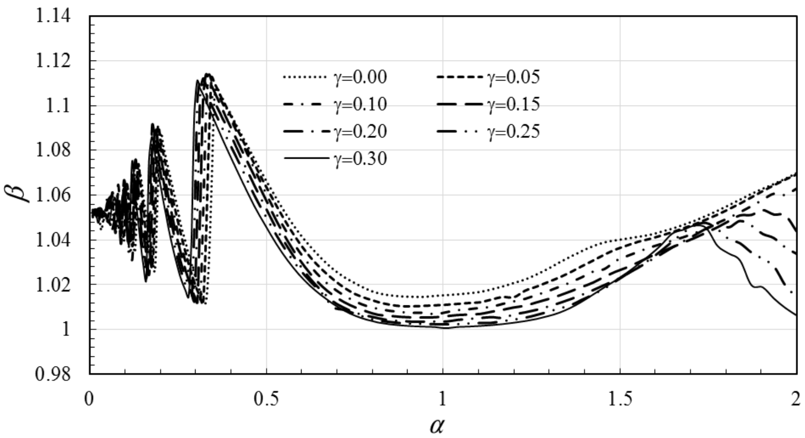


(a) beam deformation



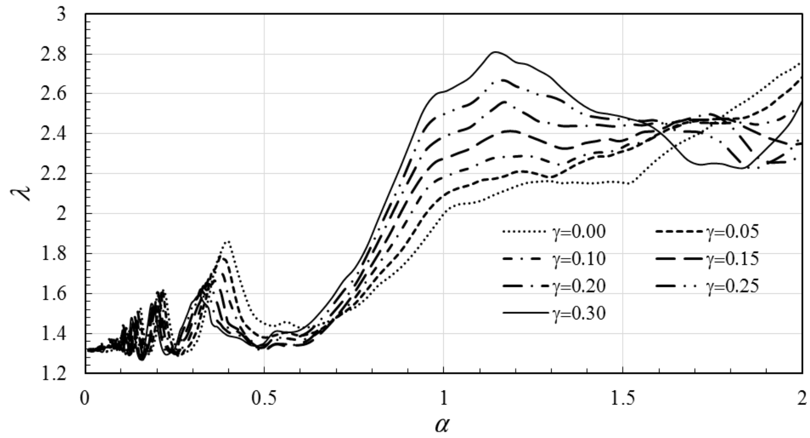
(b) flexural moment

Fig. 4 The point at which the absolute maximum dynamic response takes place for a simply supported beam (S-S)



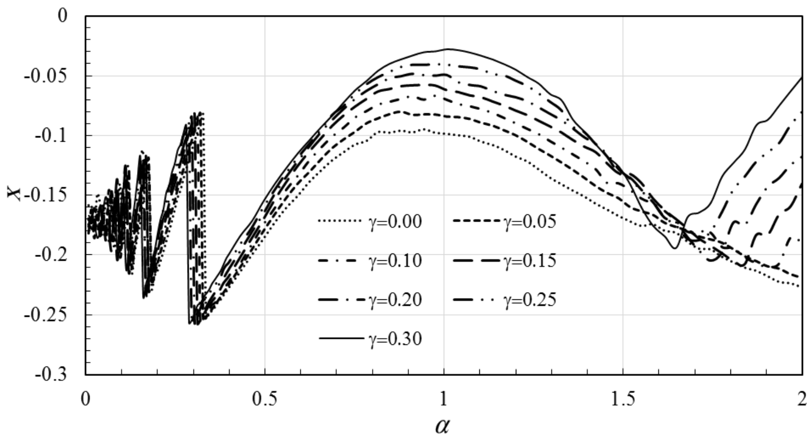
(a) beam deformation

Fig. 5 Absolute maximum dynamic response along beam length versus the maximum deformation at the mid-span for a simply supported-Clamped beam (S-C)

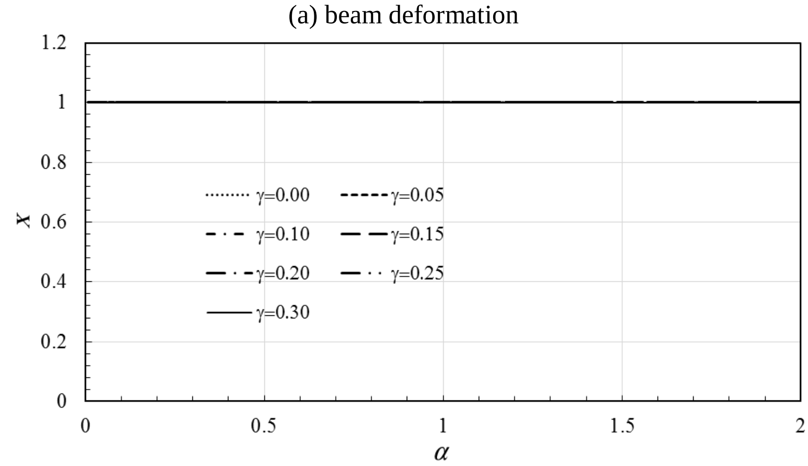


(b) flexural moment

Fig. 5 Continued

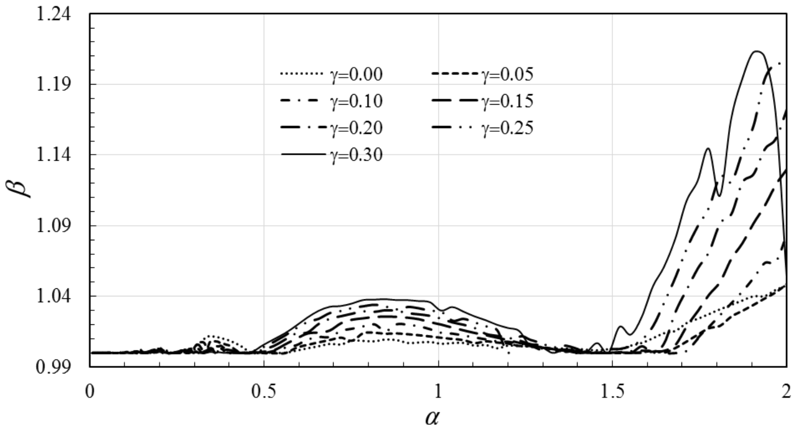


(a) beam deformation

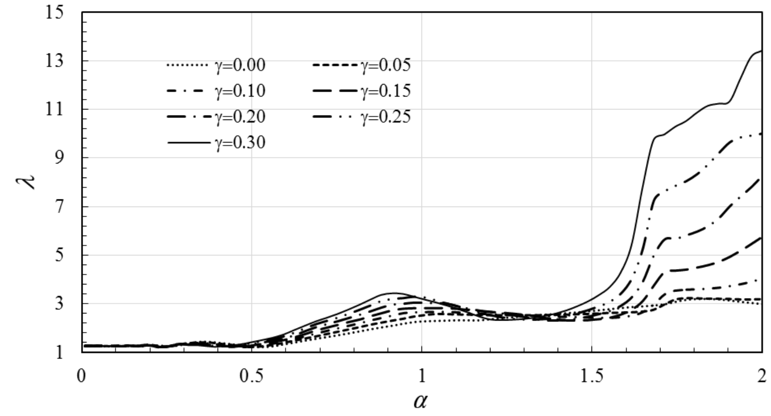


(b) flexural moment

Fig. 6 The point at which the absolute maximum dynamic response takes place for a simply supported-Clamped beam (S-C)

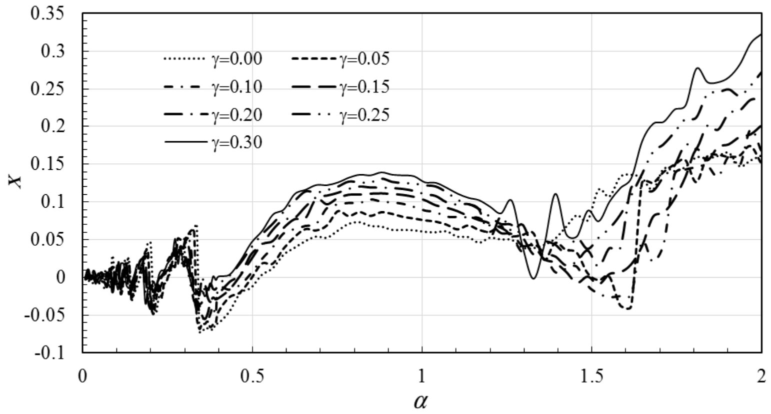


(a) beam deformation



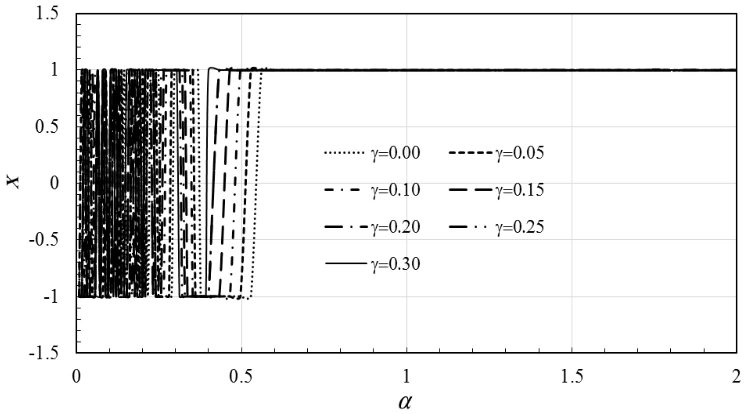
(b) flexural moment

Fig. 7 Absolute maximum dynamic response along beam length versus the maximum deformation at the mid-span for a Clamped-Clamped beam (C-C)



(a) beam deformation

Fig. 8 The point at which the absolute maximum dynamic response takes place for a Clamped-Clamped beam (C-C)



(b) flexural moment

Fig. 8 Continued

4. Conclusions

In this paper, one of the most significant concerns of the bridge engineers, i.e., determining the critical values of design parameters and their location is focused. To this end a robust and convenient numerical method based on OPSEM is utilized analyzing the dynamics of a beam to a moving inertial load. The maximum response of the beam at its mid-span is comprehensively compared with the absolute maximum response along the beam length. The presented results clarify that the overall maximum dynamic response of a beam subjected to a moving mass cannot be acceptably predicted by monitoring the mid-span for heavy moving loads with high velocities. It is shown that the value and the location of the absolute maximum response strongly depends on the weight and the motion velocity of the traveling inertial body. Regarding S-C beams the boundary conditions are not symmetric. The absolute maximum response for beam deformation so takes place in the left half of the beam, as it is expected increasing in mass leads to increase the absolute maximum response whereas β value decreases. Both the absolute and the mid-span maximum responses increase but this increment is higher for that of the mid-span and that is why the β value decreases. Besides, it is revealed that the difference between the maximum responses that the base beam experiences at its mid-span, could be highly different from the absolute maximum response in terms of the beam flexural moment rather than the beam deformation.

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