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Alternative approach for the derivation of an eigenvalue problem for a Bernoulli-Euler beam carrying a single in-span elastic rod with a tip-mounted mass

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Abstract. Many vibrating mechanical systems from the real life are modeled as combined dynamical systems consisting of beams to which spring-mass secondary systems are attached. In most of the publications on this topic, masses of the helical springs are neglected. In a paper (Cha *et al*. 2008) published recently, the eigencharacteristics of an arbitrary supported Bernoulli-Euler beam with multiple in-span helical spring-mass systems were determined via the solution of the established eigenvalue problem, where the springs were modeled as axially vibrating rods. In the present article, the authors used the assumed modes method in the usual sense and obtained the equations of motion from Lagrange Equations and arrived at a generalized eigenvalue problem after applying a Galerkin procedure. The aim of the present paper is simply to show that one can arrive at the corresponding generalized eigenvalue problem by following a quite different way, namely, by using the so-called "characteristic force" method. Further, parametric investigations are carried out for two representative types of supporting conditions of the bending beam.

Keywords: Bernoulli-Euler beams; spring-mass attachment; combined system; spring mass; characteristic force

1. Introduction

Examination of the existing literature shows that the free vibration problem of Bernoulli-Euler and Timoshenko beams restrained in various manners and carrying any number of attachments, i.e., point masses, spring-mass secondary systems, etc. has attracted the interest of many investigators. Some of the representative examples of a great number of publications on this subject are given and commented by Cha *et al*. (2008) therefore will be not referred to here. In the aforementioned study (Cha *et al.* 2008), the assumed modes method was used to derive an approximate but systematic formulation for the eigenfrequencies of an arbitrary supported beam with multiple spring–mass attachments. Recent studies on the free vibration problem of Bernoulli-Euler and Timoshenko beams with various attachments, published after Cha *et al*. (2008), will be outlined below briefly.

Lin (2009) successfully implemented NAM to determine the exact natural frequencies and

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mode shapes of the multispan Timoshenko beam carrying a number of various concentrated elements including point masses, rotary inertias, linear springs, rotational springs and spring-mass systems. The study (Yesilce *et al*. 2008) dealt with the dynamic analysis of a multi-span Timoshenko beam carrying multiple spring-mass systems and natural frequencies of the system were calculated by using the secant method. Yesilce (2010) investigated the free vibration analysis of Reddy-Bickford multi-span beams carrying multiple spring-mass systems. De Rosa *et al*. (2010) dealt with the dynamic analysis of a beam with exponentially varying cross section with an elastically support at left and a concentrated mass at right. Cha and Honda (2010) successfully implemented a characteristic force method to solve for the eigensolutions of any arbitrary supported linear structure carrying multiple lumped attachments. Wu and Chen (2012) presented an efficient technique to determine the forced vibrations response amplitudes of a multi-span beam carrying arbitrary concentrated elements using the steady response amplitudes of the system. In the study (Mei 2011), a wave-based approach was applied in analyzing bending vibrations of a uniform beam/rod with lumped and masses based on the advanced Timoshenko theory. Wang (2012) studied on the natural frequencies sensitivity analysis of a straight beam loaded with a lumped mass and he showed a closed-form solution of the frequency sensitivity of a beam-mass system. In the paper (Darabi *et al*. 2012), free vibrations of a beam-mass-spring system with different boundary conditions were analyzed both numerically and analytically. Wang and Wang (2012) studied the effect of an end mass, including a flexible base modeled by a rotational spring, for an exponentially tapered cantilever beam. Banerjee (2012) assemblied the dynamic stiffness matrix of a combined system which consists of a cantilever beam carrying a mass-spring attachment. The resulting eigenvalue problem was solved by applying the algorithm of Wittrick and Williams. Maximov (2014) obtained a simplified mathematical model for the dynamic effect, caused by a moving load on a Bernoulli-Euler beam supported with two rotational springs. In the study (Karaton 2014), two beam-column elements based on the Elasto-Fiber approach and Fiber&Bernoulli-Euler approach have been developed. To obtain the stiffness matrix, cubic Hermitian polynomials are used. For numerical application, seismic damage analyses for a 4-story frame and an 8-story reinforced concrete frame with soft-story are obtained to comparisons of reinforced concrete element according to both approaches.

The common aspect of nearly all of the publications mentioned above is that the masses of the attached helical springs of the spring-mass secondary systems, are not taken into account. The few exceptions in this context are the studies of Wu (2005), Gürgöze (2005), Gürgöze *et al.* (2006), Wu (2006), Wu and Hsu (2007), Cha *et al*. (2008).

Wu (2005) analyzed a beam carrying several two degree-of-freedom spring-mass systems, where the inertial effect of the helical springs are considered. Gürgöze (2005) analyzed the free vibrations of a cantilever beam carrying one tip-mounted helical spring-mass system with the spring-mass included. In a subsequent paper, Gürgöze *et al*. (2006) analyzed the free vibrations of a simply supported beam with a single, arbitrary located in-span helical spring-mass system using the same method. Wu (2006) studied the inertial effect of the helical springs of the absorber on suppressing the dynamic responses of the beam. Wu and Hsu (2007) presented two methods to obtain the vibration modes of a simply supported beam carrying multiple point masses and spring-mass systems in which the mass of each helical spring, is considered. In the recent paper (Cha *et al.* 2008) as stated previously, Cha *et al*. (2008) used the assumed modes method to obtain an approximate but systematic formulation for the eigenfrequencies of an arbitrary supported beam with multiple helical spring-mass attachments, where each helical spring was modeled as an axially vibrating elastic rod.

As is known, in the assumed modes method, the space-dependent trial functions must satisfy the geometric boundary conditions of the bare linear structure without any attachments. The assumed solution in the form of a series of space-dependent trial functions multiplied by the time dependent generalized coordinates is substituted into the expressions of the total kinetic and potential energies, after which equations of motion are obtained by means of the direct application of Lagrange's equations (Cha *et al*. 2008).

Cha *et al*. (2008) used in the first part of their article, the eigenfunctions of the bare beam and bare rod as the space-dependent trial functions in the assumed series solutions which they refered to as the "traditional approach". Then, in order to achieve higher convergence rates, they proposed a new scheme according to which they used a small number of the eigenfunctions of the bare rod plus a spatially linear-varying static mode. The numerical results i.e., natural frequencies and mode shapes stemming from both approaches were given in various tables showing the strength of their method. It is proper to note that both approaches led to the solution of a generalized eigenvalue problem.

Cha *et al*. (2008) used the assumed modes method in the usual sense and obtained the equations of motion from Lagrange's Equations and arrived at a generalized eigenvalue problem.

In the present study, a quite different method is used leading exactly to the same eigenvalue problem. To this end, the differential equations of motion of the vibrating system are written more or less directly by making use of the concept of the "characteristic force" (Cha and Honda 2010). Discretization of the system is then achieved by using the same series solutions as in Cha *et al.* (2008), giving in turn the same eigenvalue problem formulation.

On the other hand, the approach presented here, directly uses the expression for the vertical force that the longitudinally vibrating rod with tip mass exerts on the bending beam (characteristic force). This may be an important issue from the viewpoint of technical applications.

The only type of supporting conditions which was not considered in Cha *et al*. (2008), was that of the clamped-clamped beam. This system is investigated by the proposed method and it is shown that the agreement of the numerical results compared with those of a finite element method formulation is quite good. Further, on the representative example of the clamped-free beam, parametric investigations are carried out by changing the rod stiffness and the tip mass of the rod to study how they affect eigenfrequencies and mode shapes of the combined system.

It is appropriate to mention the fact that although in the system considered in Cha *et al*. (2008), several rod-mass systems are attached to the beam, in the present work, only one representative rod-mass attachment is considered to present the new methodology. The formulation obtained can easily be extended to the case of multiple rod-mass attachments.

2. Theory

This study deals with the natural vibration problem of an arbitrarily supported Bernoulli-Euler beam with an in-span helical spring-mass system. The helical spring is modeled as an axially vibrating rod. A representative mechanical system consisting of a clamped-free beam is shown in Fig. 1. However, it must be underlined that the formulation in this section is quite general and is applicable to a beam with any boundary conditions.

The representative system consists of a cantilevered Bernoulli-Euler beam to which an axially vibrating elastic rod with tip mass M is attached in-span. Axially vibrating rod with tip mass can be thought of as a realistic model of a conventional helical spring-mass system. It is assumed that this

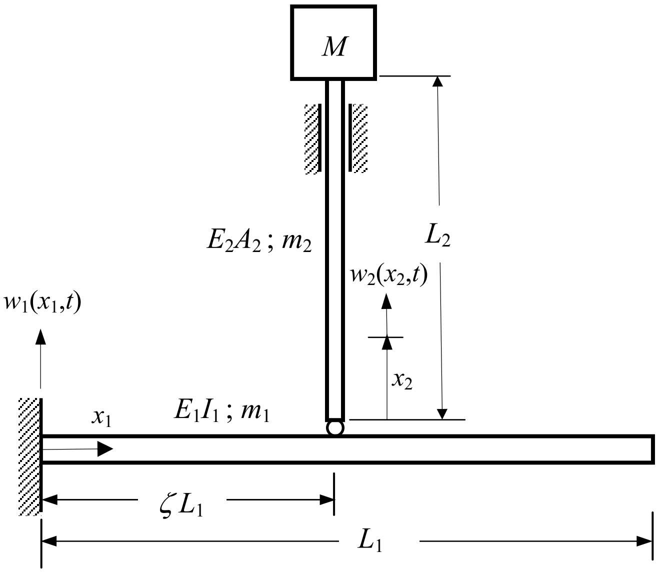


Fig. 1 A cantilevered beam carrying an in-span axially vibrating elastic rod with a tip mass

combined system vibrates only in the plane of the paper. The physical properties of the system are as follows: The length, mass per unit length and bending rigidity of the beam are L_1, m_1, E_1I_1 whereas the corresponding quantities and the axial rigidity of the rod are L_2, m_2, E_2A_2, respectively. It is to be noted that E_2A_2/L_2 corresponds to the spring constant of the conventional helical spring.

The planar bending displacements of the beam are denoted as $w_1(x_1,t)$, whereas the axial displacements of the vertical rod with tip mass M are denoted as $w_2(x_2,t)$, where $x_2=0$ corresponds to the attachment point of the rod to the beam. $w_2(x_2,t)$ is actually a "relative" displacement of the rod, with the matching condition $w_2(0,t)=0$. $w_1(x_1,t)$ and $w_2(x_2,t)$ are assumed to be small.

Making use of Gürgöze *et al.* (2008) and Cha and Honda (2010), the equations of motion of the vibrational system in Fig. 1 can be written as

$$E_1 I_1 w_1^{iv}(x_1,t) + m_1 \ddot{w}_1(x_1,t) = F(t)\delta\left(x_1 - \zeta L_1\right), \tag{1}$$

$$E_2 A_2 w_2''(x_2,t) - m_2 \ddot{w}_2(x_2,t) = m_2 \ddot{w}_1(\zeta L_1,t)\,. \tag{2}$$

Here, $\delta(x)$ denotes the well-known Dirac-delta function, whereas $F(t)$ represents the vertical force exerted by the axially vibrating elastic rod on the bending beam, which is referred to as the "characteristic force" (Cha and Honda 2010).

Using D'Alembert's principle, this force can be formulated as

$$F\left(t\right) = -m_2 \int_0^{L_2} \left[\ddot{w}_2(x_2,t) + \ddot{w}_1(\zeta L_1,t)\right] \mathrm{d}x_2 - M\left[\ddot{w}_1(\zeta L_1,t) + \ddot{w}_2(L_2,t)\right]. \tag{3}$$

It is to be emphasized that $\ddot{w}_2(x_2,t) + \ddot{w}_1(\zeta L_1,t)$ denotes the absolute acceleration at any point along the rod, and $\ddot{w}_1(\zeta L_1,t) + \ddot{w}_2(L_2,t)$ represents the absolute acceleration of the tip mass.

As in the assumed modes method, the lateral displacements of the beam can be expressed in the form of a finite series as

$$w_1(x_1,t)=\sum_{i=1}^{N}\phi_i(x_1)\eta_i(t) \tag{4}$$

where N represents the number of modes used in the series expansion of the bending displacement of the beam, $\phi_i(x_1)$ the eigenfunctions of the bare beam, and $\eta_i(t)$ are the generalized coordinates to be determined. In a similar way, the axial displacements of the rod can be represented in the form of another finite series as

$$w_2(x_2,t)=\sum_{r=1}^{P}\psi_r(x_2)a_r(t) \tag{5}$$

where P represents the number of modes used in the series expansion of the axial displacement of the rod, $\psi_r(x_2)$ the eigenfunctions of the bare rod without the tip mass, and $a_r(t)$ the corresponding generalized coordinates to be determined.

As the bare beam in Fig. 1 consists of a clamped-free beam, the functions $\phi_i(x_1)$ correspond to the eigenfunctions of a clamped-free Bernoulli-Euler beam which in normalized form (normalized with respect to the mass per unit length m_1 of the beam) are

$$\phi_i(x_1)=\frac{1}{\sqrt{m_1L_1}}\left[\cosh(\beta_ix_1)-\cos(\beta_ix_1)+\left(\frac{\sin(\bar{\beta}_i)-\sinh(\bar{\beta}_i)}{\cos(\bar{\beta}_i)+\cosh(\bar{\beta}_i)}\right)\left(\sinh(\beta_ix_1)-\sin(\beta_ix_1)\right)\right] \tag{6}$$

where the ith nondimensional eigenfrequency parameter $\bar{\beta}_i=\beta_iL_1$ satisfies the well-known transcendental equation

$$\cos(\bar{\beta}_i)\cosh(\bar{\beta}_i)+1=0\,. \tag{7}$$

Because the vertical rod is attached to the beam, the eigenfunctions $\psi_r(x_2)$ correspond to those of a fixed- free rod. The normalized form of these eigenfunctios are (Cha *et al.* 2008)

$$\psi_r(x_2)=\sqrt{\frac{2}{m_2L_2}}\sin\left[\frac{(2r-1)\pi}{2L_2}x_2\right],\qquad (r=1,2,\ldots). \tag{8}$$

For further considerations, it is in order to note here that $\psi_r(x_2)$ satisfy the following two geometric boundary conditions

$$\psi_r(0)=0,\qquad \psi_r'(L_2)=0 \tag{9}$$

where the prime represents a derivative with respect to x_2. It is a known fact that the eigenfunction expressions (6) and (8) satisfy the following orthonormality conditions

$$\int_0^{L_1}m_1\phi_r(x_1)\phi_s(x_1)\,dx_1=\delta_r^s,\qquad \int_0^{L_2}m_2\psi_r(x_2)\psi_s(x_2)\,dx_2=\delta_r^s \tag{10}$$

where δ_r^s denotes the Kronecker delta.

Before proceeding further, it is quite appropriate to give the expressions of the squared eigenfrequencies ω_i^2 of the bare, i.e., clamped-free beam

$$\omega_i^2 = \bar{\beta}_i^4 \frac{E_1 I_1}{m_1 L_1^4}, \qquad (i = 1, 2, \ldots). \tag{11}$$

The squared eigenfrequencies $\bar{\omega}_i^2$ of the bare, i.e., fixed-free rod are

$$\bar{\omega}_i^2 = \left[\frac{(2i-1)}{2}\pi\right]^2 \frac{E_2 A_2}{m_2 L_2^2}, \qquad (i = 1, 2, \ldots). \tag{12}$$

Substitution of the series solutions (4) and (5) into the differential Eqs. (1) and (2) where the functions $\eta_i(t)$ and $a_r(t)$ are assumed to be exponential in time

$$\begin{aligned} \eta_i(t) &= \bar{\eta}_i \mathrm{e}^{\lambda t}, \\ a_r(t) &= \bar{a}_r \mathrm{e}^{\lambda t}, \end{aligned} \tag{13}$$

and then carrying out the usual Galerkin-procedure leads to the following sets of equations for the unknowns $\bar{\eta}_i$ and $\bar{a}_r$

$$\left(\lambda^2 + \omega_i^2\right)\bar{\eta}_i + \left(m_2 L_2 + M\right)\lambda^2 f_i \left(\sum_{j=1}^{N} f_j \bar{\eta}_j\right) + \lambda^2 f_i \left(\sum_{r=1}^{P} \left(m_2 \bar{\alpha}_r + M\psi_r(L_2)\right)\bar{a}_r\right) = 0, \quad (i=1, 2, \ldots, N), \tag{14}$$

$$\left(\lambda^2 + \bar{\omega}_s^2\right)\bar{a}_s + \lambda^2 \left(\sum_{i=1}^{N} \left(m_2 \bar{\alpha}_s + M\psi_s(L_2)\right) f_i \bar{\eta}_i\right) + \lambda^2 \psi_s(L_2)\left(\sum_{r=1}^{P} \psi_r(L_2)\bar{a}_r\right) = 0, \quad (s=1,2,\ldots,P). \tag{15}$$

Here, the following abbreviations are used

$$f_i = \phi_i(\zeta L_1);$$

$$\omega_0^2 = \frac{E_1 I_1}{m_1 L_1^4}; \qquad \frac{\omega_i^2}{\omega_0^2} = \bar{\beta}_i^4,$$

$$(\bar{\beta}_1 = 1.875104, \ \bar{\beta}_2 = 4.694091, \ \bar{\beta}_3 = 7.854757, \ldots)$$

$$\bar{\alpha}_s = \int_0^{L_2} \psi_s(x_2)\, dx_2. \tag{16}$$

As is well known, the eigenfunctions $\psi_r(x_2)$ of the axially vibrating bare rod satisfy only the geometric boundary conditions (9), but not the dynamical boundary condition

$$M\left[\ddot{w}_1(\zeta L_1, t) + \ddot{w}_2(L_2, t)\right] + E_2 A_2 w_2'(L_2, t) = 0 \tag{17}$$

which is due to the tip mass M.

After the introduction of the series solutions (4) and (5) into the differential Eq. (2) and then applying the classical Galerkin-procedure, one is led to Eq. (15) which actually does not contain the terms which are multiplied by $\psi_s(L_2)$. The derivation of those specific terms enabling one to consider the dynamic boundary condition (17), is given below.

The mentioned terms of Eq. (15) are obtained from the first term $E_2A_2w_2''(x_2,t)$ of the differential Eq. (2).

After substituting the series solution (5) into Eq. (2) and carrying out the classical Galerkin-procedure, one obtains the term

$$E_2A_2\sum_{r=1}^{P}\left(\int_0^{L_2}\psi_r''(x_2)\psi_s(x_2)\,dx_2\right)a_r(t). \tag{18}$$

If integration by parts is applied to the definite integral in the brackets, one is led to

$$\int_0^{L_2}\psi_r''(x_2)\psi_s(x_2)\,dx_2=\psi_s(L_2)\psi_r'(L_2)-\int_0^{L_2}\psi_r'(x_2)\psi_s'(x_2)\,dx_2\,, \tag{19}$$

where $\psi_s(0)=0$ is used.

The second term on the right hand side of the above equation can be brought into the form:

$$\int_0^{L_2}\psi_r'(x_2)\psi_s'(x_2)\,dx_2=\frac{\bar{\omega}_r^{\,2}}{E_2A_2}\delta_r^s\,, \tag{20}$$

when the differential equation

$$E_2A_2\psi_r''(x_2)+\bar{\omega}_r^{\,2}m_2\psi_r(x_2)=0\,, \tag{21}$$

and the orthogonality properties of the eigenfunctions $\psi_r(x_2)$ are accounted for.

Now, if Eq. (19) and Eq. (20) are substituted into Eq. (18),

$$E_2A_2\sum_{r=1}^{P}\left(\int_0^{L_2}\psi_r''(x_2)\psi_s(x_2)\,dx_2\right)a_r(t)=\sum_{r=1}^{P}\big(E_2A_2\psi_s(L_2)\psi_r'(L_2)\big)a_r(t)-\sum_{r=1}^{P}\left(\bar{\omega}_r^{\,2}\delta_r^s\right)a_r(t)$$
$$(s=1,\ldots,P) \tag{22}$$

is obtained.

Herewith, via the dynamical boundary condition (17) and the series solutions (4) and (5); Eq. (22) and thus expression (18) can be written as

$$\begin{gathered}E_2A_2\sum_{r=1}^{P}\left(\int_0^{L_2}\psi_r''(x_2)\psi_s(x_2)\,dx_2\right)a_r(t)=\\ \psi_s(L_2)\left\{-M\left[\sum_{i=1}^{N}\phi_i(\eta L_1)\ddot{\eta}_i(t)+\sum_{r=1}^{P}\psi_r(L_2)\ddot{a}_r(t)\right]\right\}-\sum_{r=1}^{P}\left(\bar{\omega}_r^2\delta_r^s\right)a_r(t).\end{gathered} \tag{23}$$

If now in the above equation the assumed solutions (13) are substituted, it can easily be seen that the product term on the right hand side containing the braces is nothing else but the mentioned terms of Eq. (15).

It is to be noted that here and in Cha *et al*. (2008), the same series of solutions (4) and (5) are used for the solution via the assumed modes method. In Cha *et al*. (2008), the dynamic boundary condition is of no particular concern because it is automatically accounted for in the kinetic and potential energies (Meirovitch 1975). However, here, two additional terms have to be included by applying the Galerkin's method in order to consider the dynamic boundary condition.

Introducing further the definitions

$$\bar{\lambda} = \lambda/\omega_0 , \qquad \alpha_m = \frac{m_2 L_2}{m_1 L_1}, \qquad \alpha_M = \frac{M}{m_1 L_1}, \quad \alpha_k = \frac{E_2 A_2/L_2}{E_1 I_1/L_1^3}$$

$$\frac{\bar{\omega}_i^2}{\omega_0^2} = \frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}, \quad (\bar{\beta}_1^* = \pi/2, \ \bar{\beta}_2^* = 3\pi/2, \ldots)$$

$$\bar{z}_i = \frac{1}{2i-1}, \quad z_i = \frac{2\sqrt{2}}{\pi}\bar{z}_i$$

$$\bar{f}_i = f_i\sqrt{m_1 L_1} = \cosh\left(\bar{\beta}_i\eta\right) - \cos\left(\bar{\beta}_i\eta\right) + \left(\frac{\sin\left(\bar{\beta}_i\right) - \sinh\left(\bar{\beta}_i\right)}{\cos\left(\bar{\beta}_i\right) + \cosh\left(\bar{\beta}_i\right)}\right)\left(\sinh\left(\bar{\beta}_i\eta\right) - \sin\left(\bar{\beta}_i\eta\right)\right),$$

$$\bar{\psi}_r\left(L_2\right) = \psi_r\left(L_2\right)\Big/\sqrt{2/m_2 L_2} = \sin\left[\frac{(2r-1)\pi}{2}\right], \qquad \Omega_r = z_r\sqrt{\alpha_m} + \alpha_M\sqrt{\frac{2}{\alpha_m}}\bar{\psi}_r\left(L_2\right) \tag{24}$$

the two sets of equations given in (14) and (15) can be brought into the following simplified forms

$$\sum_{j=1}^{N}\left[\left(\alpha_m + \alpha_M\right)\bar{\lambda}^2\bar{f}_i\bar{f}_j + \left(\bar{\lambda}^2 + \bar{\beta}_i^4\right)\delta_{ij}\right]\bar{\eta}_j + \bar{\lambda}^2\bar{f}_i\left(\sum_{r=1}^{P}\Omega_r\bar{a}_r\right) = 0 \qquad (i=1,2,\ldots,N), \tag{25}$$

$$\left(\bar{\lambda}^2 + \frac{\alpha_k}{\alpha_m}\bar{\beta}_s^{*2}\right)\bar{a}_s + 2\frac{\alpha_M}{\alpha_m}\bar{\lambda}^2\bar{\psi}_s\left(L_2\right)\left(\sum_{r=1}^{P}\bar{\psi}_r\left(L_2\right)\bar{a}_r\right) + \bar{\lambda}^2\left[\frac{2}{\pi}\sqrt{2\alpha_m}\,\bar{z}_s + \sqrt{2}\frac{\alpha_M}{\sqrt{\alpha_m}}\bar{\psi}_s\left(L_2\right)\right]\left(\sum_{i=1}^{N}\bar{f}_i\bar{\eta}_i\right) = 0$$

$$(s=1,2,\ldots,P). \tag{26}$$

The $(N+P)$ Eqs. in (25) and (26) can be combined into the following matrix equation

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix}\begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{27}$$

Here, following submatrices and additional abbreviations are introduced

$$\bar{\mathbf{f}}_{\mathbf{i}_{Px1}} = \left[\bar{f}_i \quad \bar{f}_i \quad \cdots \quad \bar{f}_i\right]^T; \qquad \bar{\mathbf{F}}_{PxN} = \left[\bar{\mathbf{f}}_1 \quad \bar{\mathbf{f}}_2 \quad \cdots \quad \bar{\mathbf{f}}_{\mathbf{N}}\right] \qquad \bar{\mathbf{f}} = \left[\bar{f}_1 \quad \bar{f}_2 \quad \cdots \quad \bar{f}_N\right]^T$$

$$\boldsymbol{\Omega}_{\mathbf{r}_{Nx1}} = \left[\Omega_r \quad \Omega_r \quad \cdots \quad \Omega_r\right]^T; \qquad \boldsymbol{\Omega}_{NxP} = \left[\boldsymbol{\Omega}_{\mathbf{1}} \quad \boldsymbol{\Omega}_{\mathbf{2}} \quad \cdots \quad \boldsymbol{\Omega}_{\mathbf{P}}\right]$$

$$\bar{\boldsymbol{\eta}} = \left[\bar{\eta}_1 \quad \bar{\eta}_2 \quad \cdots \quad \bar{\eta}_N\right]^T; \qquad \bar{\mathbf{a}} = \left[\bar{a}_1 \quad \bar{a}_2 \quad \cdots \quad \bar{a}_P\right]^T,$$

$$[A] = \left(\alpha_m + \alpha_M\right)\bar{\lambda}^2\bar{\mathbf{f}}\,\bar{\mathbf{f}}^T + \mathbf{diag}\left(\bar{\lambda}^2 + \bar{\beta}_i^4\right),$$

$$[B] = \bar{\lambda}^2\mathbf{diag}\left(\bar{f}_i\right)\boldsymbol{\Omega},$$

$$[C] = \frac{2}{\pi}\bar{\lambda}^2\sqrt{2\alpha_m}\,\mathbf{diag}\left(\bar{z}_i\right)\bar{\mathbf{F}} + \bar{\lambda}^2\sqrt{2}\frac{\alpha_M}{\sqrt{\alpha_m}}\mathbf{diag}\left(\bar{k}_i\right)\bar{\mathbf{F}},$$

$$
\begin{aligned}
&[D] = \mathbf{diag}\left(\bar{\lambda}^2 + \frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}\right) + 2\frac{\alpha_M}{\alpha_m}\bar{\lambda}^2\bar{\mathbf{k}}\bar{\mathbf{k}}^T \\
&\bar{k}_i = \bar{\psi}_i(L_2), \quad \bar{\mathbf{k}} = \begin{bmatrix}\bar{k}_1 & \bar{k}_2 & \cdots & \bar{k}_P\end{bmatrix}^T .
\end{aligned} \tag{28}
$$

In order that the linear, homogeneous matrix Eq. (27) has nontrivial solutions, the determinant of its coefficient matrix should be zero, leading to the characteristic (frequency) equation

$$
\det\begin{bmatrix}[A] & [B] \\ [C] & [D]\end{bmatrix} = 0 . \tag{29}
$$

The $\bar{\lambda}$ values which are purely imaginary numbers in nature, making the above determinant equal to zero, give the nondimensionalized eigenfrequencies of the combined system in Fig. 1. However, it is a known fact that finding those characteristic values which make a determinant equal to zero, may be sometimes problematic from the numerical point of view. Hence, it is desirable to express the matrix Eq. (27) alternatively as an eigenvalue problem.

It can easily be shown that the corresponding generalized eigenvalue problem is as follows:

$$
\begin{bmatrix}[B'] & [0] \\ [0] & [E']\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{\eta}} \\ \bar{\mathbf{a}}\end{bmatrix} = \lambda^* \begin{bmatrix}[A'] & [C'] \\ [D'] & [D'']\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{\eta}} \\ \bar{\mathbf{a}}\end{bmatrix} \tag{30}
$$

with

$$
\begin{aligned}
&\lambda^* = -\bar{\lambda}^2 \\
&[A'] = (\alpha_m + \alpha_M)\bar{\mathbf{f}}\bar{\mathbf{f}}^T + [I]_{NxN} \\
&[B'] = \mathbf{diag}\left(\bar{\beta}_i^4\right) \\
&[C'] = \mathbf{diag}\left(\bar{f}_i\right)\boldsymbol{\Omega} \\
&[D'] = \left\{\frac{2}{\pi}\sqrt{2\alpha_m}\,\mathbf{diag}(\bar{z}_i) + \sqrt{2}\frac{\alpha_M}{\sqrt{\alpha_m}}\mathbf{diag}\left(\bar{k}_i\right)\right\}\bar{\mathbf{F}} \\
&[E'] = \mathbf{diag}\left(\frac{\alpha_k}{\alpha_m}\bar{\beta}_i^{*2}\right) \\
&[D''] = [I]_{PxP} + 2\frac{\alpha_M}{\alpha_m}\bar{\mathbf{k}}\bar{\mathbf{k}}^T .
\end{aligned} \tag{31}
$$

where $[C'] = [D']^T$.

Here, $[I]_{N\times N}$ denotes the $N\times N$ identity matrix, and [0] $N\times P$, or $P\times N$ zero matrix.

It can be shown that the submatrices given above are simply the nondimensional forms of the submatrices given in Eqs. (8) and (9) of Cha *et al.* (2008). Hence, it can be stated finally that: Exactly the same eigenvalue problem in Cha *et al.* (2008) is obtained here, tracing a quite different route.

Recall that up to now, the beam under investigation is assumed to be a clamped-free Bernoulli-Euler beam, as seen from Fig. 1. However, the formulations made in the present work are quite general such that other types of supporting conditions for the beam are applicable as well.

For the sake of the completeness, in the following part, the corresponding formulations for a simply-supported beam, a fixed-simply supported beam and a fixed-fixed supported beam are given. At this point, it is quite in order to refer to the work of Gonçalves *et al.* (2007). Numerical results for the clamped-free beam and the fixed-fixed beam are given in the next section.

For a simply supported uniform beam, the eigenfunctions, normalized (normalized with respect to the mass per unit length m_1 of the beam) are

$$\phi_i(x_1) = \sqrt{\frac{2}{m_1 L_1}} \sin\left(\frac{i\pi x_1}{L_1}\right) \tag{32}$$

and its squared eigenfrequencies are

$$\omega_i^2 = (i\pi)^4 \frac{E_1 I_1}{m_1 L_1^4}. \tag{33}$$

Hence, the corresponding definitions in (16) and (24) should be replaced by

$$f_i = \phi_i(\zeta L_1), \qquad \bar{f}_i = \sqrt{m_1 L_1} f_i = \sqrt{2}\sin(i\pi\eta)$$

$$\frac{\omega_i^2}{\omega_0^2} = (i\pi)^4 = \bar{\beta}_i^4, \qquad (\bar{\beta}_1 = \pi, \bar{\beta}_2 = 2\pi, \ldots). \tag{34}$$

For a fixed-simply supported uniform beam, the eigenfunctions, normalized (normalized with respect to the mass per unit length m_1 of the beam) are

$$\phi_i(x_1) = \frac{1}{\sqrt{m_1 L_1}}\left[\cos(\beta_i x_1) - \cosh(\beta_i x_1) + \left(\frac{\cosh(\bar{\beta}_i) - \cos(\bar{\beta}_i)}{\sin(\bar{\beta}_i) - \sinh(\bar{\beta}_i)}\right)\left(\sin(\beta_i x_1) - \sinh(\beta_i x_1)\right)\right] \tag{35}$$

where $\bar{\beta}_i = \beta_i L_1$ denotes ith root of the following transcendental equation

$$\sin(\bar{\beta}_i)\cosh(\bar{\beta}_i) - \cos(\bar{\beta}_i)\sinh(\bar{\beta}_i) = 0. \tag{36}$$

The ith squared eigenfrequency of the beam is

$$\omega_i^2 = \bar{\beta}_i^4 \frac{E_1 I_1}{m_1 L_1^4}. \tag{37}$$

Hence

$$f_i = \phi_i(\zeta L_1),$$

$$\bar{f}_i = f_i\sqrt{m_1 L_1} = \left[\cos(\eta\bar{\beta}_i) - \cosh(\eta\bar{\beta}_i) + \left(\frac{\cosh(\bar{\beta}_i) - \cos(\bar{\beta}_i)}{\sin(\bar{\beta}_i) - \sinh(\bar{\beta}_i)}\right)\left(\sin(\eta\bar{\beta}_i) - \sinh(\eta\beta_i)\right)\right] \tag{38}$$

$$\frac{\omega_i^2}{\omega_0^2} = \bar{\beta}_i^4, \quad \left(\bar{\beta}_1 = 3{,}926602, \quad \bar{\beta}_2 = 7{,}068583, \quad ..\right). \tag{39}$$

For a fixed-fixed (i.e., clamped-clamped) uniform beam, the eigenfunctions, normalized (with respect to the mass per unit length m_1 of the beam) are

$$\phi_i\left(x_1\right)=\frac{1}{\sqrt{m_1 L_1}}\left[\cosh\left(\beta_i x_1\right)-\cos\left(\beta_i x_1\right)+\left(\frac{\cos\left(\bar{\beta}_i\right)-\cosh\left(\bar{\beta}_i\right)}{\sinh\left(\bar{\beta}_i\right)-\sin\left(\bar{\beta}_i\right)}\right)\left(\sinh\left(\beta_i x_1\right)-\sin\left(\beta_i x_1\right)\right)\right] \tag{40}$$

where $\bar{\beta}_i=\beta_i L_1$ denotes the ith root of the following transcendental frequency equation:

$$\cos\left(\bar{\beta}_i\right)\cosh\left(\bar{\beta}_i\right)-1=0\,. \tag{41}$$

The ith squared eigenfrequency of the beam is the same as in Eq. (37).

Here, the corresponding definitions in (16) and (24) are to be replaced by:

$$f_i=\phi_i\left(\zeta L_1\right),$$

$$\bar{f}_i=f_i\sqrt{m_1 L_1}=\left[\cosh\left(\eta\bar{\beta}_i\right)-\cos\left(\eta\bar{\beta}_i\right)+\left(\frac{\cos\left(\bar{\beta}_i\right)-\cosh\left(\bar{\beta}_i\right)}{\sinh\left(\bar{\beta}_i\right)-\sin\left(\bar{\beta}_i\right)}\right)\left(\sinh\left(\eta\bar{\beta}_i\right)-\sin\left(\eta\bar{\beta}_i\right)\right)\right] \tag{42}$$

and

$$\frac{\omega_i^2}{\omega_0^2}=\bar{\beta}_i^4,\qquad\left(\bar{\beta}_1=4{,}730041,\quad\bar{\beta}_2=7{,}853205,\quad..\right) \tag{43}$$

3. Numerical results

This section is devoted first to the numerical application of some of the formulas established in the preceding section to the special case of a clamped-clamped beam which was not investigated in Cha *et al*. (2008). Secondly, on the example of a clamped-free beam, as depicted in Fig. 1, parametric investigations are carried out by varying the rod stiffness and the tip mass on the rod, to study their effects on the eigencharacteristics of the whole system.

All the numerical results given in this section were obtained by using MATLAB.

In Table 1, the nondimensional eigenfrequencies of the clamped-clamped Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass, are given. As mentioned above, this combined system is not dealt with in Cha *et al*. (2008). Following physical parameter values are chosen: α_m=0.1, α_M=2, α_k=48, ζ=0.37. It is in order to recall that the square roots of the eigenvalues λ_i^* obtained from the generalized eigenvalue problem (30), correspond to the nondimensional eigenfrequencies ω_i in Cha *et al.* (2008).

The values in the second column are the results of a conventional FEM-formulation with 100 elements for both the beam and the rod, N being the number of eigenfunctions used in series expansion (4) of the bending vibrations of the beam, used also in Cha *et al.* (2008). Recall that P here denotes the number of the eigenfunctions used in the series expansion (5) of the axial vibrations of the rod, corresponding to Nn in Cha *et al.* (2008). The values written in parenthesis below the eigenfrequencies denote the corresponding relative errors with respect to FEM-solution.

For the clamped-clamped beam, the common forms of eigenfunctions, given by (34) permit evaluation of only the first 12 modes or so due to numerical issues (Cha 2013). Therefore, in case

Table 1 The nondimensional eigenfrequencies $\sqrt{\lambda_i^*}$ of the clamped-clamped Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. Nondimensional physical parameters of the combined system are taken as: α_m=0.1, α_M=2, α_k=48, ζ=0.37

$\sqrt{\lambda_i^*}$	FEM	N=10			N=100		
		P=10	P=100	P=200	P=10	P=100	P=200
1	4.409433	4.445268 (%8.13×10^{-1})	4.413158 (%8.45×10^{-2})	4.411393 (%4.44×10^{-2})	4.445067 (%8.08×10^{-1})	4.412962 (%8.01×10^{-2})	4.411197 (%4.00×10^{-2})
2	23.611707	23.666629 (%2.33×10^{-1})	23.617223 (%2.33×10^{-2})	23.614499 (%1.18×10^{-2})	23.666547 (%4.11×10^{-1})	23.617147 (%2.30×10^{-2})	23.614424 (%1.15×10^{-2})
3	58.809347	59.051877 (%4.12×10^{-1})	58.835293 (%4.41×10^{-2})	58.822663 (%2.26×10^{-2})	59.050962 (%4.11×10^{-1})	58.834254 (%4.24×10^{-2})	58.821616 (%2.09×10^{-2})
4	73.597601	74.803537 (%1.64×10^{0})	73.717210 (%1.63×10^{-1})	73.658964 (%8.34×10^{-2})	74.797264 (%1.63×10^{0})	73.711341 (%1.55×10^{-1})	73.653117 (%7.54×10^{-2})
5	120.868476	120.875343 (%5.68×10^{-3})	120.869168 (%5.72×10^{-4})	120.868803 (%2.70×10^{-4})	120.875330 (%5.67×10^{-3})	120.869149 (%5.57×10^{-4})	120.868783 (%2.54×10^{-4})
6	138.480971	141.269892 (%2.01×10^{0})	138.746037 (%1.91×10^{-1})	138.609411 (%9.27×10^{-2})	141.255027 (%2.00×10^{0})	138.731673 (%1.81×10^{-1})	138.595076 (%8.24×10^{-2})
7	196.192216	197.131589 (%4.79×10^{-1})	196.286721 (%4.82×10^{-2})	196.232681 (%2.06×10^{-2})	197.128750 (%4.77×10^{-1})	196.282307 (%4.59×10^{-2})	196.228164 (%1.83×10^{-2})
8	211.140764	214.615049 (%1.65×10^{0})	211.418246 (%1.31×10^{-1})	211.257168 (%5.51×10^{-2})	214.596011 (%1.64×10^{0})	211.401829 (%1.24×10^{-1})	211.240899 (%4.74×10^{-2})
9	275.843529	281.264754 (%1.97×10^{0})	276.232554 (%1.41×10^{-1})	275.965409 (%4.42×10^{-2})	281.236163 (%1.95×10^{0})	276.204084 (%1.31×10^{-1})	275.936973 (%3.39×10^{-2})
10	299.034338	299.403864 (%1.24×10^{-1})	299.053940 (%6.56×10^{-3})	299.038503 (%1.39×10^{-3})	299.402707 (%1.23×10^{-1})	299.053532 (%6.42×10^{-3})	299.038120 (%1.26×10^{-3})

of N=100, the following alternative forms of eigenfunctions of a clamped-clamped Bernoulli-Euler beam are used in the numerical evolutions (Gonçalves *et al.* 2007)

$$\phi_i(x_1) = \frac{1}{\sqrt{m_1 L_1}}\left[\mathrm{e}^{-\bar{\beta}_i \frac{x_1}{L_1}} - \cos\left(\bar{\beta}_i \frac{x_1}{L_1}\right) + (1+\nu)\sin\left(\bar{\beta}_i \frac{x_1}{L_{1\,i}}\right) - \nu\sinh\left(\bar{\beta}_i \frac{x_1}{L_{1\,i}}\right)\right] \tag{44}$$

with

$$\nu = \left(\frac{\mathrm{e}^{-\bar{\beta}_i} - \cos\left(\bar{\beta}_i\right) + \sin\left(\bar{\beta}_i\right)}{\sinh\left(\bar{\beta}_i\right) - \sin\left(\bar{\beta}_i\right)}\right). \tag{45}$$

It is shown that the above alternative forms allow the evaluation of high-order modes without numerical problems.

It is seen that for a fixed N, the “best” results i.e., the smallest errors are obtained for P=200. The overall “best” results are obtained for N=100 and P=200. Further, it is interesting to see that the errors of the 10th eigenfrequency are much more smaller than those of the fundamental eigenfrequency. Furthermore, the investigation of Table 1 allows the following important observation that P, the number of the eigenfunctions of the axial vibrations of the rod, is more

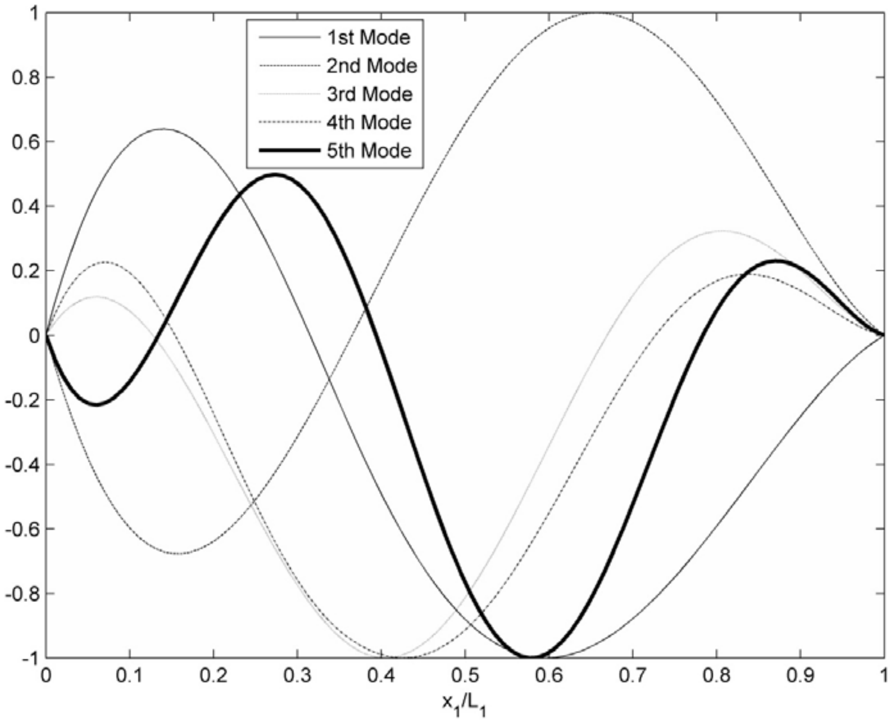


Fig. 2 The first five mode shapes of a clamped-clamped Bernoulli-Euler beam with an in-span attached axially vibrating rod with a tip-mounted mass. The physical parameters are the same as those of Table 1

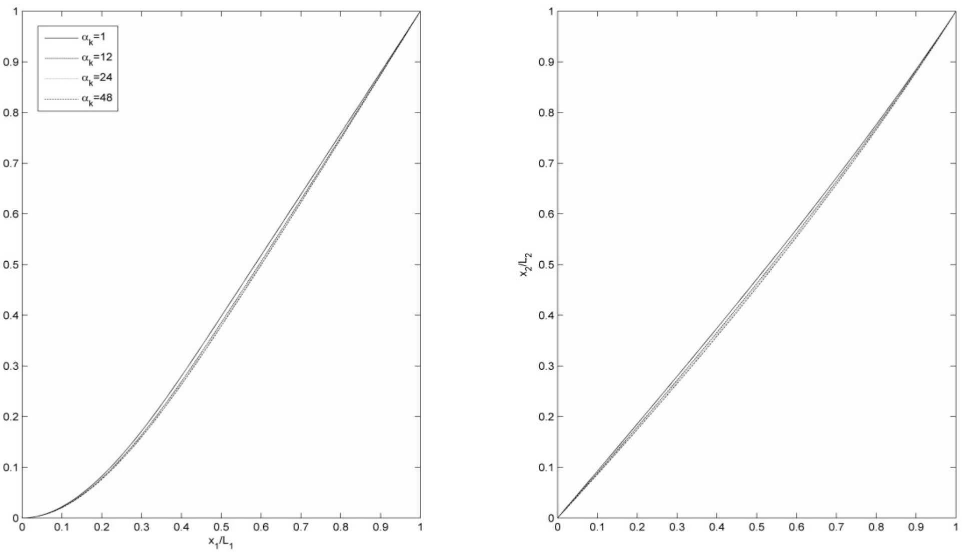
effective than N, the number of eigenfunctions of the bending vibrations of the beam, on the convergence of the numerical solutions. Further, when the number N increases from 10 to 100 for a constant P-value, the errors in eigenfrequencies are somewhat decreased. However, it is worth noting that the decreases in the errors are more apparent when P increases for a constant N-value.

To sum up, the eigenfrequencies of the combined system investigated above, can be obtained via the eigenvalue problem formulation re-established here, with a very satisfactory accuracy. The corresponding eigenmodes of the clamped-clamped beam with an in-span attached axially vibrating rod with tip-mounted mass are depicted in Fig. 2.

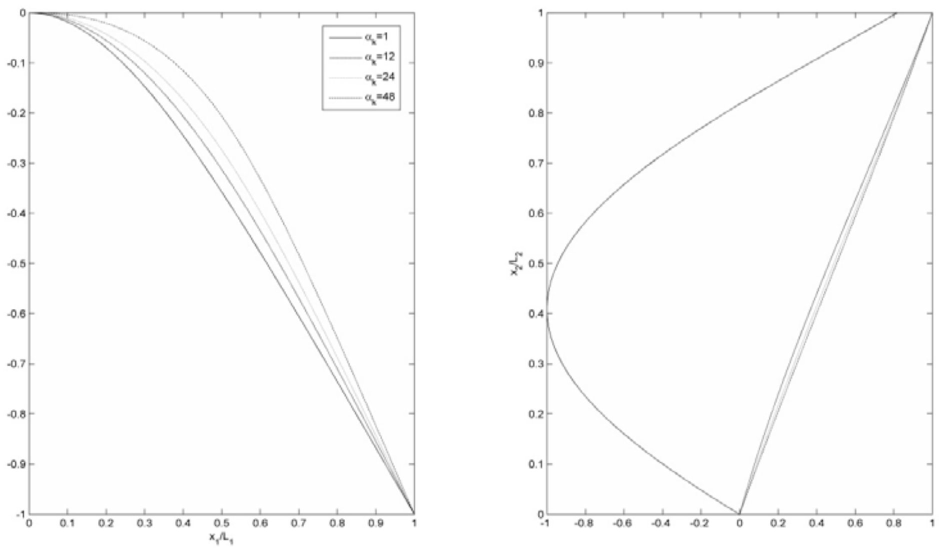
Further, results of the parametric investigation obtained by varying the stiffness of the rod and its tip mass on the eigencharacteristics of the combined system, consisting of the cantilevered beam and the rod with tip mass, as depicted in Fig. 1, are given in form of graphs in Figs. 3-6. All numerical results below are based on the assumption of N=100 and P=200.

The first five mode shapes of the cantilever beam with axially vibrating rod corresponding to reference parameters (α_m=1, α_M=2, α_k=1, ζ=0.5) are presented in Fig. 7. It is seen that the first four modes of cantilevered beam are similar to the first bending mode of a cantilevered bare beam. The fifth mode, however, is similar to the second bending mode of a cantilevered bare beam. The mode shapes of axially vibrating rod, on the other hand, appear similar to the first, second and third modes of a bare rod.

Further, the effects of the variation of the α_k parameter on the mode shapes are investigated. The individual mode shapes of the combined system corresponding to different α_k values are calculated and presented in Fig. 3. It is obvious that the first modes of the combined system corresponding to different α_k parameters do not show any significant variation. However, the results presented in Figs. 3(b)-3(e) show that the mode shapes for higher modes exhibit strong dependency on α_k parameter.

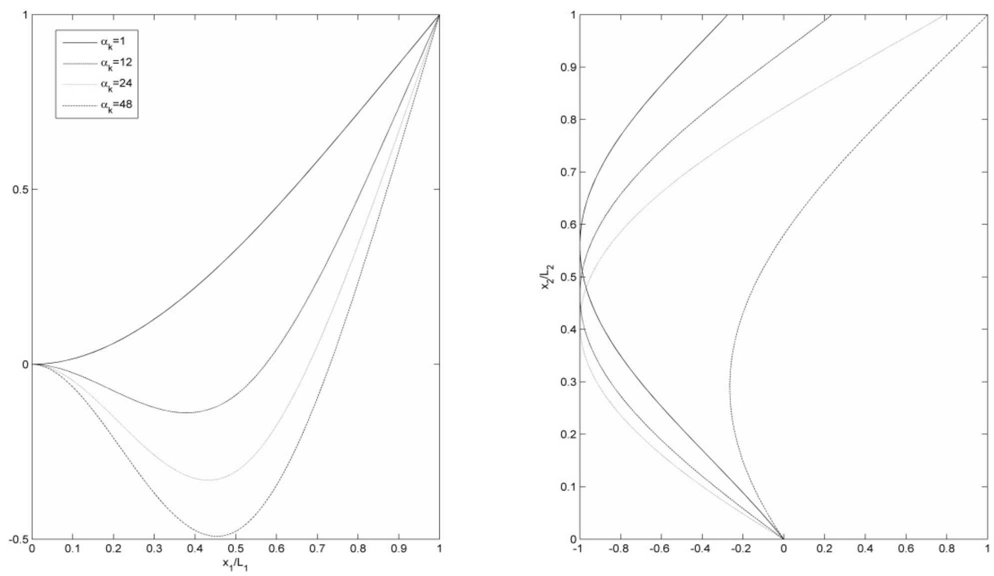


(a)

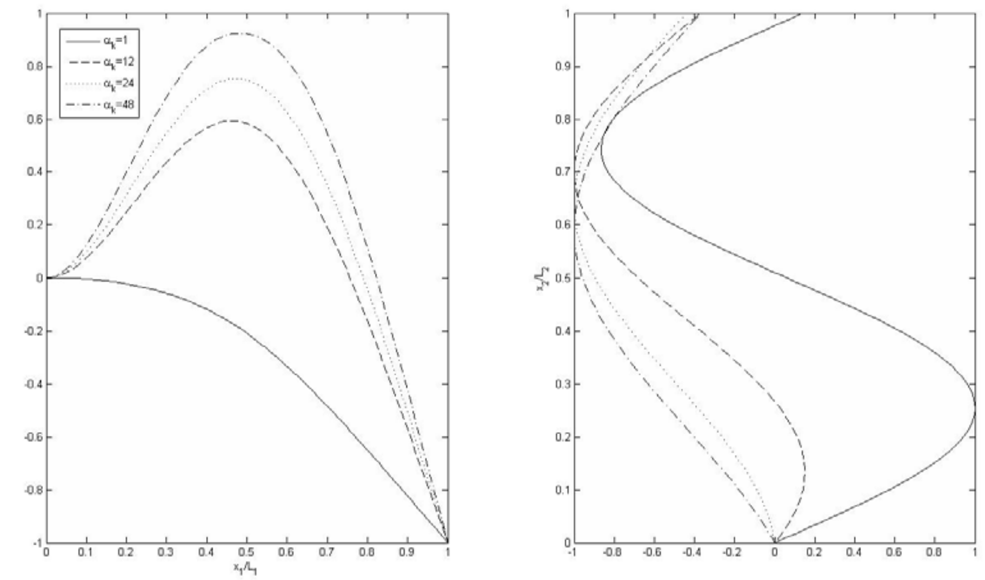


(b)

Fig. 3 The individual mode shapes of the combined system corresponding to different rod stiffness parameter α_k. Graphs on the left and right correspond to the mode shapes of the beam and the rod, respectively. (α_m=1, α_M=2, ζ=0.5) (a) First modes, (b) Second modes, (c) Third modes, (d) Fourth modes, (e) Fifth modes



(c)

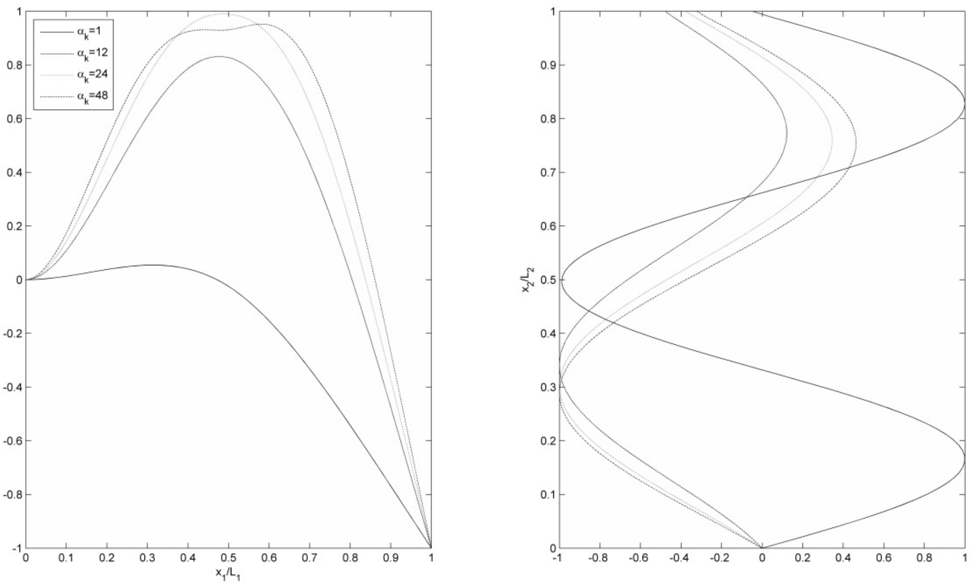


(d)

Fig. 3 Continued

In Fig. 4, the eigenfrequencies of the combined system are also plotted against α_k, as expected, increasing the stiffness of the rod, results in increasing natural frequencies. It is noted that the natural frequencies of the higher modes are affected more than those of the lower modes.

Having investigated the effects of the variation of the rod stiffness on the eigencharacteristics



(e)

Fig. 3 Continued

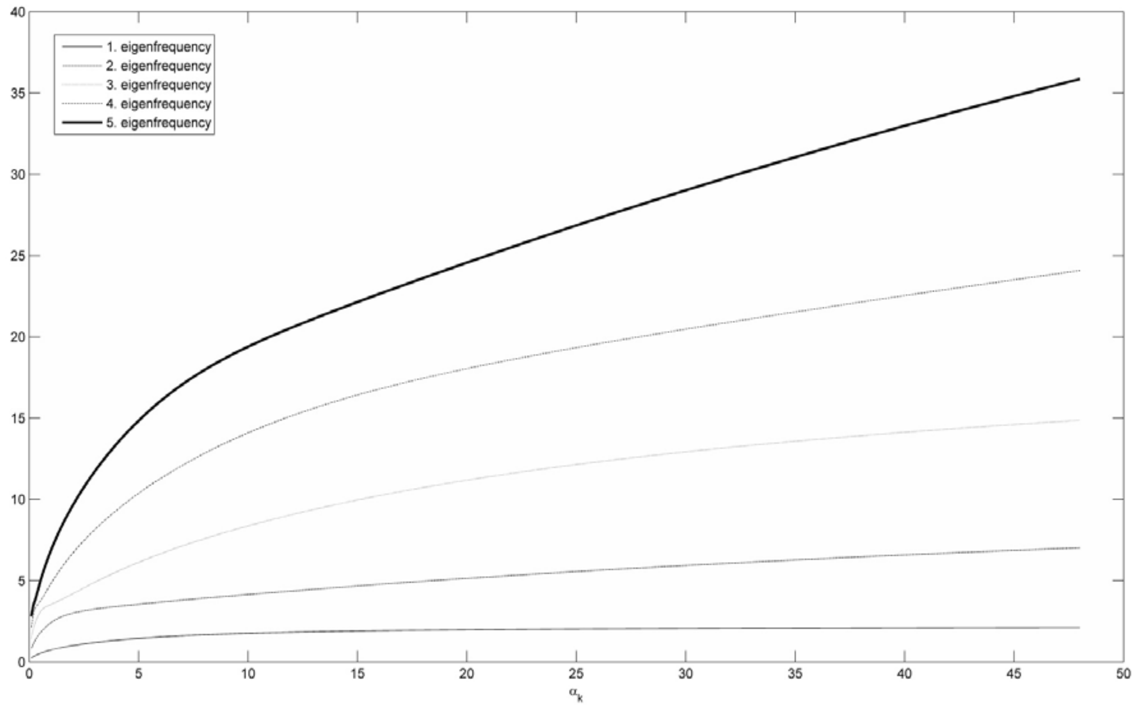
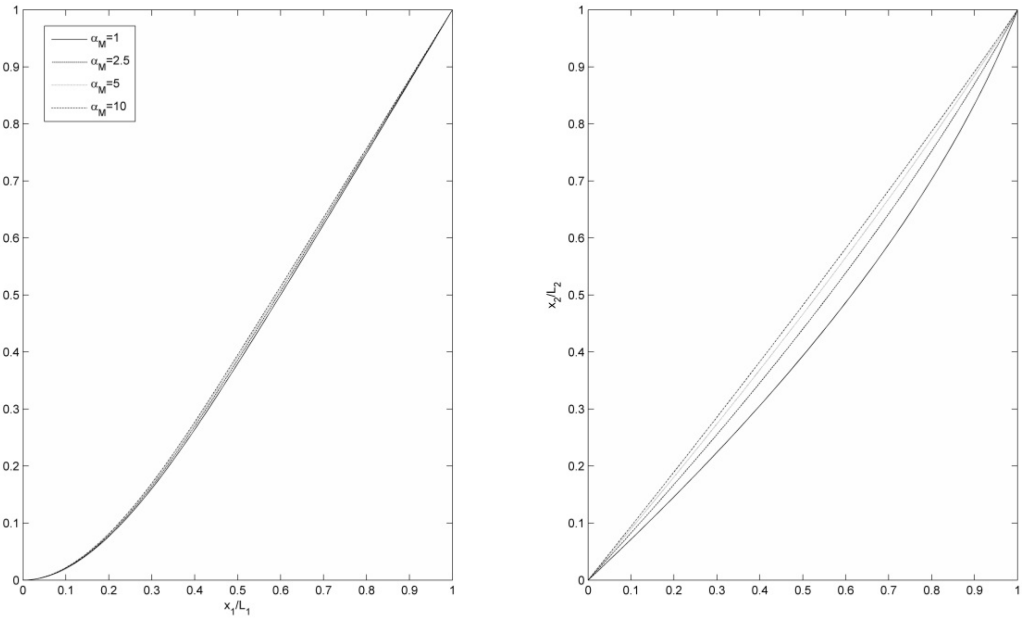
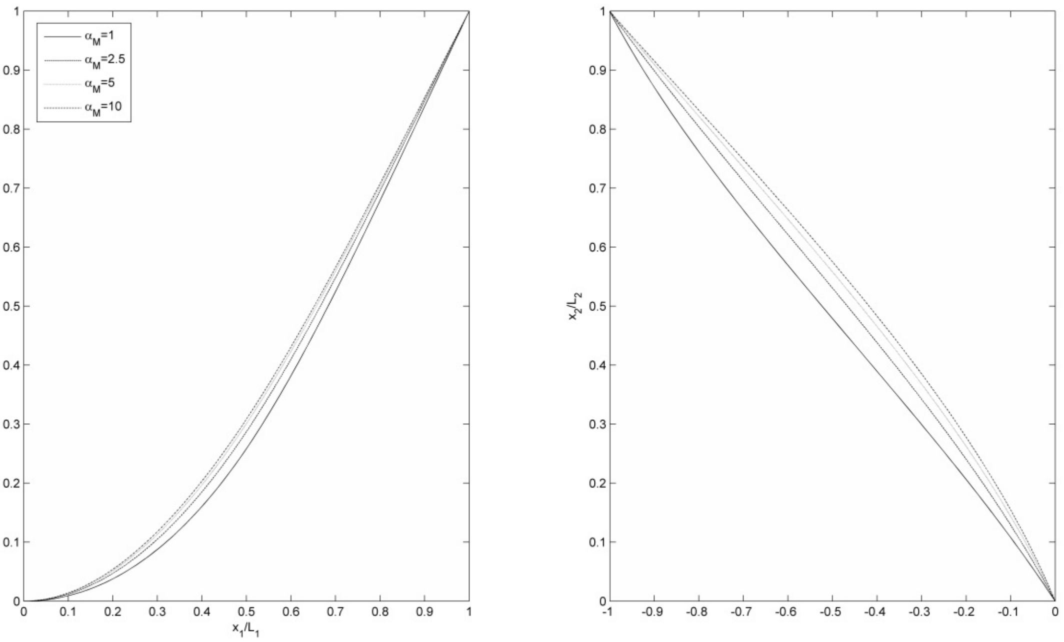


Fig. 4 The variation of the eigenfrequencies of the combined system as function of α_k (α_m=1, α_M=2, ζ=0.5)

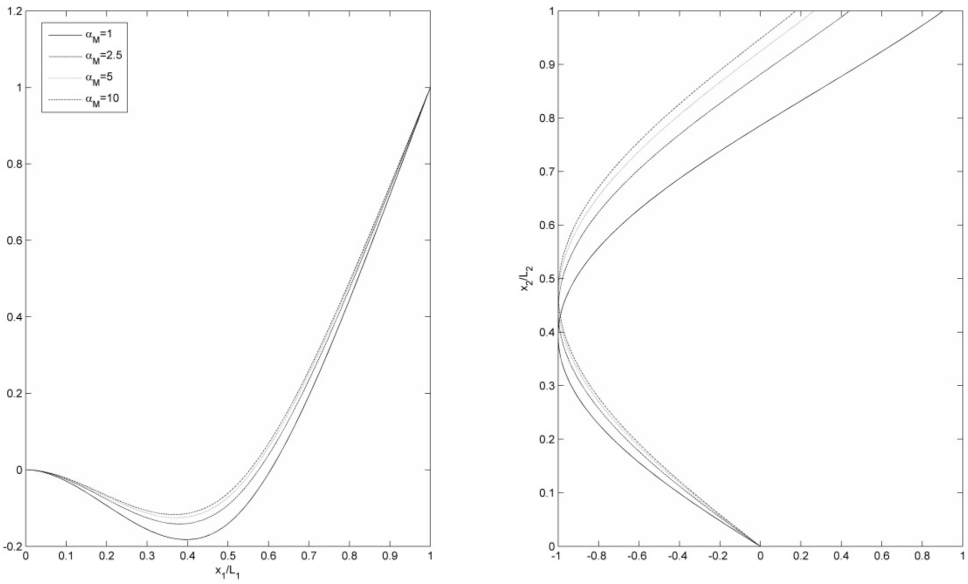


(a)

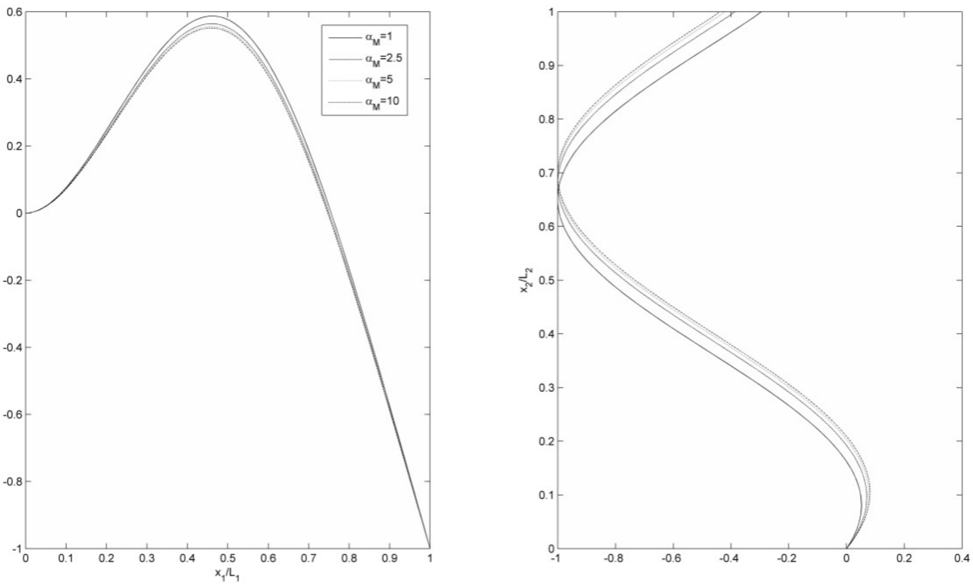


(b)

Fig. 5 The individual mode shapes of the combined system corresponding to different tip mass parameter α_M. Graphs on the left and right correspond to the mode shapes of the beam and the rod, respectively. (α_m=2, α_k =24, ζ=0.5) (a) First modes, (b) Second modes, (c) Third modes, (d) Fourth modes, (e) Fifth modes



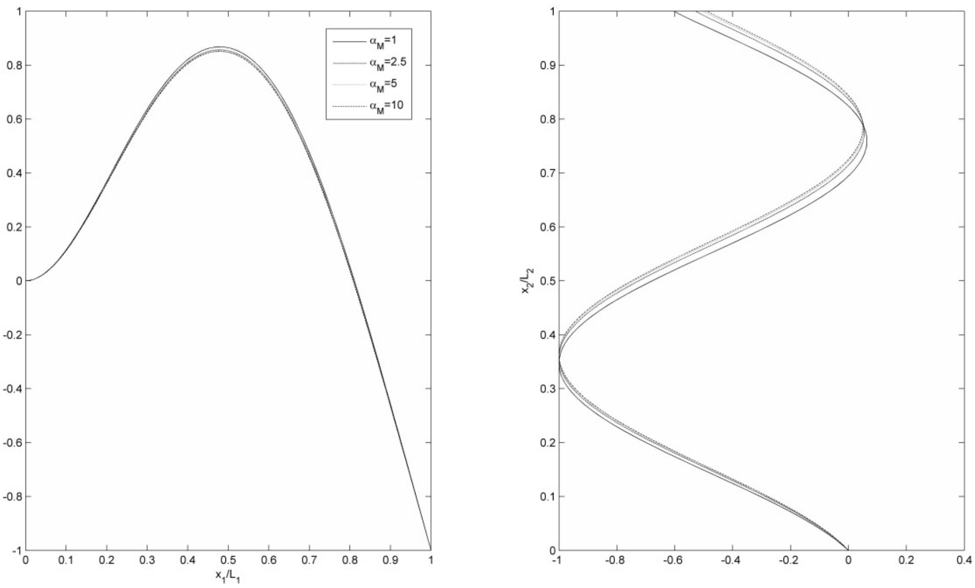
(c)



(d)

Fig. 5 Continued

of the combined system, it is quite in order also to investigate the effects of the tip mass parameter α_M on the mode shapes and eigenfrequencies. The individual mode shapes of the combined system corresponding to different α_M -values are calculated and presented in Fig. 5 with the following nondimensional parameter values α_k=24, α_m=2, ζ=0.5. Considering all the mode shapes in Fig. 5(a) to 5(e), it is seen that the mode shapes do not change significantly as a function of the tip mass parameter α_M. However, it is worth stating that third modes of the rod are somewhat affected, in



(e)
Fig. 5 Continued

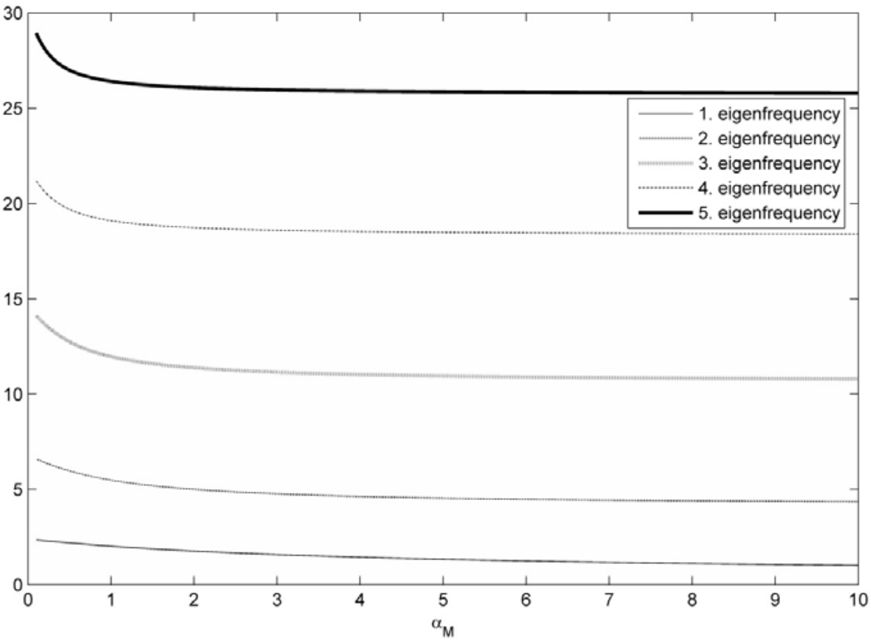
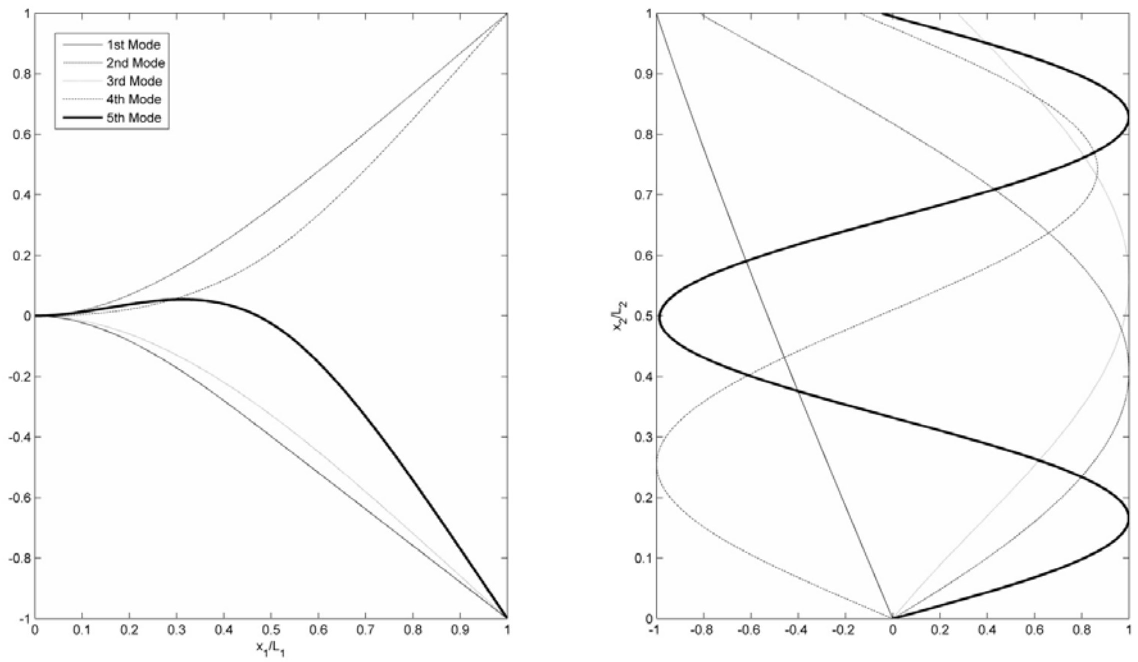


Fig. 6 The variation of the eigenfrequencies of the combined system as function of α_M (α_m=2, α_k=24, ζ=0.5)

the sense that the nodal points are shifted towards tip mass of the rod. In addition, the eigenfrequencies of the system for the first five modes are plotted against α_M in Fig. 6. Although, there is a gradual decrease in natural frequencies as the tip mass parameter α_M increases, the



(a) (b)

Fig. 7 The first five mode shapes of the combined system: (a) beam, (b) rod. (α_m=1, α_M=2, α_k=1, ζ=0.5)

change in natural frequencies is not significant when α_M is greater than approximately 2. The reason for this trend is that natural frequencies of the system approach to those of the system, when the mass M is approaching to ∞. This situation corresponds to the case when the free end of the rod is fixed.

4. Conclusions

There are a large number of publications in the technical literature on vibrations of combined dynamical systems consisting of beams to which helical spring-mass secondary systems are attached. In the majority of these studies, the helical springs are assumed to be massless. In a paper recently published, the eigencharacteristics of an arbitrary supported Bernoulli-Euler beam with several in-span helical spring-mass systems were obtained on the basis of an eigenvalue problem. In that work, the helical springs, were modeled as axially vibrating rods. The authors used the assumed modes method and obtained the equations of motion of the system from Lagrange's Equations and established the corresponding generalized eigenvalue problem after applying a Galerkin procedure.

In the present study, the equations of motion of the corresponding vibrating system are written directly by making use of the "characteristic force" concept. Discretization of the system is then achieved by using the same series solutions as in the paper mentioned. Afterwards, it is shown that the approach used here, which is quite different from that of the mentioned study leads actually to the same generalized eigenvalue problem. Then, parametric investigations are carried out for two representative types of supporting conditions of the bending beam.

It should be noted that the generalized eigenvalue problem re-established here can easily be generalized to the case of multiple spring-mass attachments.

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