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Bending of a rectangular plate resting on a fractionalized Zener foundation

Cheng-Cheng Zhang[1a], Hong-Hu Zhu[*1], Bin Shi[1b] and Guo-Xiong Mei[2c]

[1]School of Earth Sciences and Engineering, Nanjing University, Nanjing, China
[2]College of Civil Engineering and Architecture, Guangxi University, Nanning, China

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Abstract. The long-term performance of plates resting on viscoelastic foundations is a major concern in the analysis of soil-structure interaction. As a powerful mathematical tool, fractional calculus may address these plate-on-foundation problems. In this paper, a fractionalized Zener model is proposed to study the time-dependent behavior of a uniformly loaded rectangular thin foundation plate. By use of the viscoelastic-elastic correspondence principle and the Laplace transforms, the analytical solutions were obtained in terms of the Mittag-Leffler function. Through the analysis of a numerical example, the calculated plate deflection, bending moment and foundation reaction were compared to those from ideal elastic and standard viscoelastic models. It is found that the upper and lower bound solutions of the plate response estimated by the proposed model can be determined using the elastic model. Based on a parametric study, the impacts of model parameters on the long-term performance of a foundation plate were systematically investigated. The results show that the two spring stiffnesses govern the upper and lower bound solutions of the plate response. By varying the values of the fractional differential order and the coefficient of viscosity, the time-dependent behavior of a foundation plate can be accurately captured. The fractional differential order seems to be dependent on the mechanical properties of the ground soil. A sandy foundation will have a small fractional differential order while in order to simulate the creeping of clay foundation, a larger fractional differential order value is needed. The fractionalized Zener model is capable of accounting for the primary and secondary consolidation processes of the foundation soil and can be used to predict the plate performance over many decades of time.

Keywords: viscoelastic foundation; plate-on-foundation; fractional calculus; rheological model; time-dependent behavior; Mittag-Leffler function

1. Introduction

The static and dynamic responses of loaded plates resting on foundations are common problems for pavements, airports, high-rise buildings and underground structures. The Winkler foundation model, which consists of elastic spring elements, has been frequently utilized to solve

∗Corresponding author, Associate Professor, E-mail: zhh@nju.edu.cn
[a]Post-Graduate Student, E-mail: zhangchengcheng@gmail.com
[b]Professor, E-mail: shibin@nju.edu.cn
[c]Professor, E-mail: meiguox@163.com

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these problems (e.g., Straughan 1990, Saha 1997, Matsunage 2000, Buczkowski and Torbacki 2001, Zhong and Zhang 2006). In the past few decades, various methods have been adopted to facilitate the analyses, such as the finite strip method (Huang and Thambiratnam 2002), the symplectic superposition method (Zhong *et al.* 2009, Li *et al.* 2013) and the integral transform method (Kim and Roesset 1998, Kim 2004). However, it is found in engineering practices that the foundation materials exhibit significant rheological behavior and the plate deflection under loading is highly time-dependent. In order to address this inadequacy, a quantity of research has been carried out to study the time-dependent behavior of plates resting on viscoelastic foundations (e.g., Sonoda and Kabayashi 1980, Nassar 1981, Zaman *et al*. 1991, Sun 2003, Chen *et al*. 2011). Among these studies, some classical rheological models, such as the Maxwell, Kelvin, Bingham, Merchant, Zener, and Burgers models, are usually employed. To acquire a better predicting result, some more advanced models have been applied. But the number of parameters will thereupon be increased, resulting in difficulties in determining these model parameters (Chen *et al.* 2006). Being capable of describing the properties of viscoelastic materials with a fairly small number of parameters, fractional calculus may be an appropriate tool to address this problem.

Fractional calculus is a branch of calculus dealing with the generalization of differentiation and integration operator to an arbitrary order. Fractional calculus can trace its history back to the 17th century, and have been applied in various fields to date. As far as we have been able to ascertain, Gemant (1936) first introduced fractional calculus to viscoelasticity and proposed the fractionalized models for viscoelastic materials. The constitutive equations of these models include fractional-order differential operators, instead of integer-order ones. Since then, extensive studies have been carried out in this field and fractional calculus encounters much success in describing the rheological property of viscoelastic and viscoplastic materials (Bagley and Torvik 1986, Mainardi 2012). As reported by Welch *et al*. (1999), fractionalized viscoelastic models have several advantages over conventional ones, such as fewer model parameters and amenable to analysis using Fourier and Laplace transforms. In general, most viscoelastic models involving fractional calculus that have been developed in the last few decades are based on the replacement of standard rheological elements by fractional derivative ones intermediate between pure solids and pure liquids. Schiessel *et al*. (1995) generalized a number of rheological models by replacing all the Hookean springs and Newtonian dashpots by fractional derivative elements. By replacing a Newtonian dashpot in the standard Kelvin model with a fractional derivative element, Zhu *et al*. (2012) proposed a fractionalized Kelvin-Voigt model to account for the time-dependent settlement of soil ground. Yin *et al*. (2007, 2012, 2013) employed a single fractional derivative element to describe the rheological properties of a variety of geo-materials under different loading conditions. To date, however, there are only a few researchers applying fractional calculus-based models to analyze structural problems and soil-structure interactions (Gusella and Terenzi, 1997, Atanackovic and Stankovic 2004, Dikmen 2005, Lewandowski *et al.* 2012, Sumelka 2014), and the plate-on-foundation problems in particular. Some preliminary works have been carried out by Zhu *et al.* (2011), but the adopted model is not the most frequently used for geo-materials and cannot consider the instantaneous response of a foundation plate subjected to external loading. Therefore, a more rational model should be developed to solve this problem.

This paper aims at developing a fractionalized Zener model (FZM) to study the quasi-static plate-on-foundation problem. The analytical solutions are derived using the viscoelastic-elastic correspondence principle and the Laplace transforms. Based on the evaluation of the Mittag-Leffler function, the solutions of plate deflection, bending moment and foundation reaction are presented and compared with the results calculated from ideal elastic and standard viscoelastic models. The influences of key model parameters on the predicted plate performance are further examined through a parametric study.

2. Fractionalized zener model

2.1 Mathematical preliminaries

The *n*th derivate (*n* is a positive integer) of a suitable function $f(t)$, namely $D^n f(t)=d^n f(t)/dt^n$, is familiar to all who have some basic knowledge of calculus. When we replace *n* with a fraction or non-integer, the meaning of the above formula is extended, which is the origin of the concept of fractional calculus. Fractional calculus can be defined by different ways, such as the Grünwald-Letnikov definition, the Caputo definition, and the Riemann-Liouville definition. Because the fractional derivatives are mainly used during the formation of the plate-foundation interaction model, the last one, which is capable of facilitating the computation of fractional derivatives, is employed in this study.

Starting with Cauchy's integral formula, the Riemann–Liouville fractional integration of function $f(t)$ of order v is normally expressed as (Miller and Ross 1993, Nonnenmacher and Metzler 1995)

$$ {}_0D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} f(\xi) d\xi, \quad (\mathrm{Re}\, v > 0, t > 0) \tag{1} $$

where the subscripts 0 and t on D refer to the limits of the integration, and $\Gamma(v)$ denotes the gamma function with argument v. We note that the above integral corresponds to regular integration for v=1. Let $[\alpha]$ be the smallest integer that exceeds α, the Riemann-Liouville fractional derivate of order α is then defined by (Miller and Ross 1993)

$$ {}_0D_t^{\alpha} f(t) = {}_0D_t^{[\alpha]} \left[{}_0D_t^{-v} f(t) \right], \quad (\mathrm{Re}\, \alpha > 0, t > 0) \tag{2} $$

where $v=[\alpha]-\alpha>0$. In the sections to follow, the notations utilized in Eqs. (1) and (2) will be simplified by dropping the subscripts 0 and t.

2.2 The four-parameter fractionalized model

The standard viscoelastic models, such as the Kelvin and Maxwell models, are constructed by combining a class of basic elements like Hookean springs and Newtonian dashpots. We can write the constitutive relations between stress $\sigma(t)$ and strain $\varepsilon(t)$ of a spring and a dashpot in the form of differential operators

$$ \begin{cases} \sigma(t) = ED^0\varepsilon(t) \\ \sigma(t) = \eta D^1\varepsilon(t) \end{cases} \tag{3} $$

where E and η are modulus of elasticity and coefficient of viscosity, respectively. The spring and the dashpot can be used to simulate the stress-strain relation of an ideal solid (elasticity) and an ideal fluid (viscosity), respectively.

The fractionalized rheological models, however, are based on the utilization of the so-called

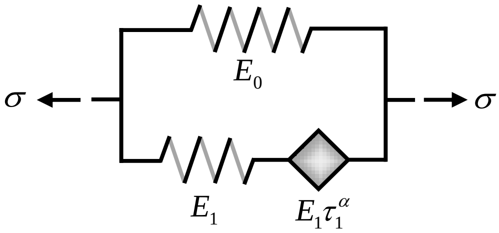


Fig. 1 Four-parameter fractionalized Zener model

"intermediate" model (Smit and de Vries 1970), or "spring-pot" model (Koeller 1984). The fractional derivative element is normally represented by a diamond (Bagley and Torvik 1979, Welch *et al.* 1999, Dikmen 2005), as shown in Fig. 1. Let $\tau=\eta/E$ be the creep time or relaxation time, the constitutive equation of this element is

$$\sigma(t) = E\tau^\alpha D^\alpha \varepsilon(t), \quad (0 \le \alpha \le 1) \tag{4}$$

where D^α denotes the fractional differentiation defined by Eq. (2). It is worth noting that for two limiting cases, namely α=0 and α=1, Eq. (4) reduces to the constitutive equations of a spring and a dashpot, respectively. As a result, this fractional derivative element has the capability of exhibiting the characteristics of both solids and fluids.

The standard Zener model (SZM) consists of a Maxwell model and a Hookean spring arranged in parallel. Schiessel *et al.* (1995) generalized this model by replacing these elements with fractional derivative elements, but the upgraded model involves nine parameters. A more common approach is to replace the dashpot with a fractional derivative element while the two springs remain unchanged, as shown in Fig. 1 (Mainardi and Spada 2011). Following the notations introduced above, the stress-strain relation of the FZM is

$$(D^\alpha + 1/\tau_1^{\ \alpha})\sigma(t) = (E_0 + E_1)(D^\alpha + 1/t_1^{\ \alpha})\varepsilon(t) \tag{5}$$

where $\tau_1 = \eta / E_1$, and $t_1 = \tau_1 \sqrt[\alpha]{1 + E_1 / E_0}$. It is noted that this FZM has four model parameters. When α=1, this model collapses to the SZM.

Based on the fractional calculus, it is not difficult to derive the creep compliance and relaxation modulus of the FZM, which can be expressed by

$$J(t) = \frac{1}{E_0 + E_1}\left\{1 + \frac{E_1}{E_0}\left(1 - E_\alpha\left[-\left(\frac{t}{t_1}\right)^\alpha\right]\right)\right\} \tag{6}$$

$$G(t) = \left(E_0 + E_1\right)\left\{1 - \frac{E_1 / E_0}{1 + E_1 / E_0}\left(1 - E_\alpha\left[-\left(\frac{t}{\tau_1}\right)^\alpha\right]\right)\right\} \tag{7}$$

where E_α is the Mittag-Leffler function defined as

$$E_\alpha(t) = \sum_0^\infty \frac{t^n}{\Gamma(\alpha n + 1)} \tag{8}$$

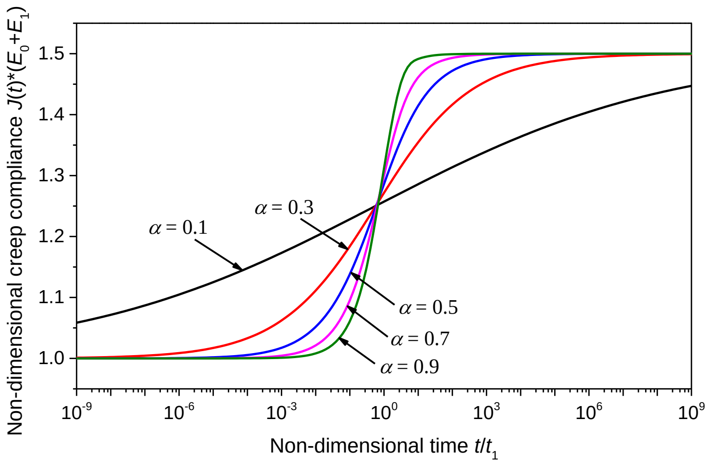


Fig. 2 Effect of fractional differential order on the non-dimensional creep compliance of the FZM

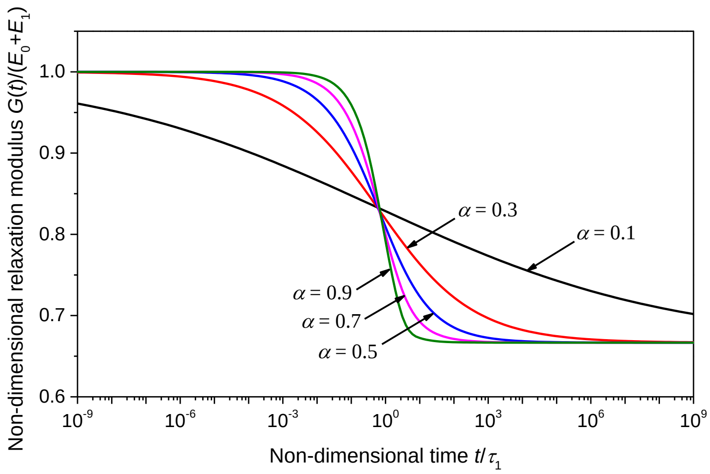


Fig. 3 Effect of fractional differential order on the non-dimensional relaxation modulus of the FZM

Figs. 2 and 3 illustrate the effects of the fractional differential order α on the non-dimensional creep compliance and relaxation modulus, respectively. For numerical calculation, it is convenient to let $E_0=2E_1$, and to introduce the non-dimensional times t/t_1 and t/τ_1. Referring to Figs. 2 and 3, it is seen that the FZM is able to describe the viscoelastic behavior of geo-materials for different time scales when α varies from 0 to 1. In addition, it is interesting that the parameters t_1 and τ_1 are two characteristic times. In the vicinity of t_1 or τ_1, a shift in the slope of the curves is observed.

3. Analytical solutions of soil-structure interaction using the FZM

Fig. 4 defines the coordinates x, y and z, and shows the case where a uniformly distributed load (UDL) of q_0 is applied on a rectangular thin plate simply supported along all four sides on a

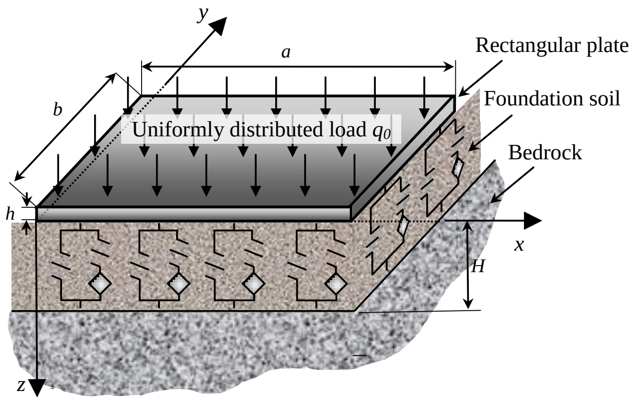


Fig. 4 Schematic illustration of a loaded rectangular thin plate on a fractionalized Zener foundation

fractionalized Zener foundation. The compressible soil foundation has an average thickness of *H*. The length, width, and thickness of the plate are *a*, *b*, and *h*, respectively. The governing equation for lateral deflection ($w(x, y)$) and bending moments (M_x, M_y) of the plate can be expressed by

$$D\nabla^2\nabla^2 w(x,y) + R(x,y) = q_0 \tag{9}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) \tag{10}$$

where $R(x, y)$ is the reaction of the foundation, and D is the bending rigidity of the plate given by

$$D = \frac{Eh^3}{12(1-\mu^2)} \tag{11}$$

where E and μ are Young's modulus and Poisson's ratio of the foundation plate, respectively.

3.1 Elastic solution

First, the elastic solution of this plate-on-foundation problem will be given. For the plate resting on a Winkler-type foundation consisting of elastic springs with stiffness of $k = E/H$, the foundation reaction can be expressed by

$$R(x,y) = kw(x,y) \tag{12}$$

Note that here we consider the average strain of the foundation soil. Now the deflection of the plate can be derived from Eq. (9) as (Timoshenko and Woinowsky-Krieger 1959)

$$w(x,y) = \frac{16q_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{b}}{(2m-1)(2n-1)\left\{\pi^4 D\left[\left(\frac{2m-1}{a}\right)^2+\left(\frac{2n-1}{b}\right)^2\right]^2+k\right\}} \tag{13}$$

The foundation reaction and the bending moments of the plate can be readily calculated using Eq. (12) and Eq. (10), respectively. Taking the Laplace transforms of Eqs. (9), (12), and (13), we obtain the governing equation and the resulting deflection expressed in the "*s*" domain

$$D\nabla^2\nabla^2\overline{w}(x,y,s)+k\overline{w}(x,y,s)=\frac{q_0}{s} \tag{14}$$

$$\overline{w}(x,y,s)=\frac{16q_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{b}}{(2m-1)(2n-1)\left\{\pi^4 D\left[\left(\frac{2m-1}{a}\right)^2+\left(\frac{2n-1}{b}\right)^2\right]^2+k\right\}s} \tag{15}$$

3.2 Viscoelastic solution

If the load shown in Fig. 4 is applied quasi-statically, it can be represented in the form of

$$q(t)=q_0H(t) \tag{16}$$

where $H(t)$ is the Heaviside step function. Then the deflection $w(x, y, t)$ of the plate is given by

$$D\nabla^2\nabla^2 w(x,y,t)+R(x,y,t)=q(t) \tag{17}$$

Assuming that the reaction of the foundation $R(x, y, t)$ is governed by the fractionalized Zener viscoelastic equation, it satisfies

$$(k_0+k_1)(D+1/t_1^{\alpha})w(x,y,t)=(D+1/\tau_1^{\alpha})R(x,y,t) \tag{18}$$

$\tau_1=\eta^*/k_1$

$t_1=\tau_1\sqrt[\alpha]{1+k_1/k_0}$

where k_0, k_1 and η^* are coefficients of the foundation.

The Laplace transforms of Eqs. (16)-(18) are

$$D\nabla^2\nabla^2\overline{w}(x,y,s)+\overline{k}(s)\overline{w}(x,y,s)=\frac{q_0}{s} \tag{19}$$

$$\overline{k}(s)=(k_0+k_1)\frac{(s^{\alpha}+1/t_1^{\alpha})}{(s^{\alpha}+1/\tau_1^{\alpha})} \tag{20}$$

In the theory of viscoelasticity, the correspondence principle (or elastic-viscoelastic analogy) was first proposed by Christensen (1982). It is stated that the Laplace- or Fourier-transformed elastic and viscoelastic equations are equivalent providing that the boundary conditions and the geometry are the same. In view of this principle, the viscoelastic solution can be obtained by replacing k with $\overline{k}(s)$ in Eq. (15), i.e.

$$\overline{w}(x,y,s)=\frac{16q_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{b}}{(2m-1)(2n-1)\left\{\pi^4 D\left[\left(\frac{2m-1}{a}\right)^2+\left(\frac{2n-1}{b}\right)^2\right]^2+(k_0+k_1)\frac{(s^{\alpha}+1/t_1^{\alpha})}{(s^{\alpha}+1/\tau_1^{\alpha})}\right\}s} \tag{21}$$

Taking the inverse Laplace transform of Eq. (21), we can express the time-dependent plate deflection as

$$w(x,y,t)=\frac{16q_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{b}}{(2m-1)(2n-1)\left(f+k_0+k_1\right)}\left[\frac{T}{\tau_1^{\alpha}}+\left(1-\frac{T}{\tau_1^{\alpha}}\right)E_{\alpha}\left(-\frac{t^{\alpha}}{T}\right)\right] \tag{22}$$

$$f=\pi^4 D\left[\left(\frac{2m-1}{a}\right)^2+\left(\frac{2n-1}{b}\right)^2\right]^2$$

$$T=\frac{f+k_0+k_1}{f/\tau_1^{\alpha}+(k_0+k_1)/t_1^{\alpha}}$$

Similarly, the foundation reaction is readily obtained as

$$R(x,y,t)=\frac{16q_0(k_0+k_1)}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{(2m-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{b}}{(2m-1)(2n-1)\left(f+k_0+k_1\right)}\left[\frac{T}{t_1^{\alpha}}+\left(1-\frac{T}{t_1^{\alpha}}\right)E_{\alpha}\left(-\frac{t^{\alpha}}{T}\right)\right] \tag{23}$$

and the bending moments are

$$\begin{aligned}M_x&=D\left[\left(\frac{2m-1}{a}\pi\right)^2+\mu\left(\frac{2n-1}{b}\pi\right)^2\right]w(x,y,t),\\ M_y&=D\left[\mu\left(\frac{2m-1}{a}\pi\right)^2+\left(\frac{2n-1}{b}\pi\right)^2\right]w(x,y,t)\end{aligned} \tag{24}$$

Eqs. (22)-(24) are the analytical solutions of plate deflection, foundation reaction, and bending moments for a loaded rectangular thin plate resting on a fractionalized Zener foundation. It is obvious that when the fractional differential order α=1, the Mittag-Leffler function reduces to e^t, and Eqs. (22)-(24) then turn into the results derived from the SZM.

4. Result analysis of a numerical example

4.1 Properties of the FZM in comparison with classical models

A numerical example is established to analyze the time-dependent performance of a rectangular thin plate resting on a fractionalized Zener viscoelastic foundation subjected to a UDL of 100 kPa. Table 1 presents the corresponding values of related parameters used in this example. The calculated distributions of plate deflections, bending moments and foundation reactions along the longitudinal dimension, are presented in Figs. 5-7. In the calculation process, the asymptotic approximation of the Mittag-Leffler function is obtained using the simple algorithm proposed by Welch *et al.* (1999). First, a variable x_{crit} that is related to fractional differential order was sought. Then the Mittag-Leffler function was evaluated according to an empirical criterion for selection between the exact series and the asymptotic representation.

The plate deflections resulting from the FZM on t=0d, 100d, 500d, and 1500d, are shown in Fig. 5, together with those from the elastic model (EM) and the SZM. When t=0d, the calculated deflections from the SZM and FZM correspond to the EM solution given in Eq. (13) where the

Table 1 Properties of the plate and the foundation used in the numerical example

Plate	
Length a (m)	10
Width b (m)	10
Height h (m)	0.5
Bending rigidity D (MPa·m^3)	100
Foundation	
Spring stiffness k_0 (MPa·m^{-1})	5
Spring stiffness k_1 (MPa·m^{-1})	8
Coefficient of viscosity η^* (MPa·d·m^{-1})	2000
Fractional differential order α	0.5

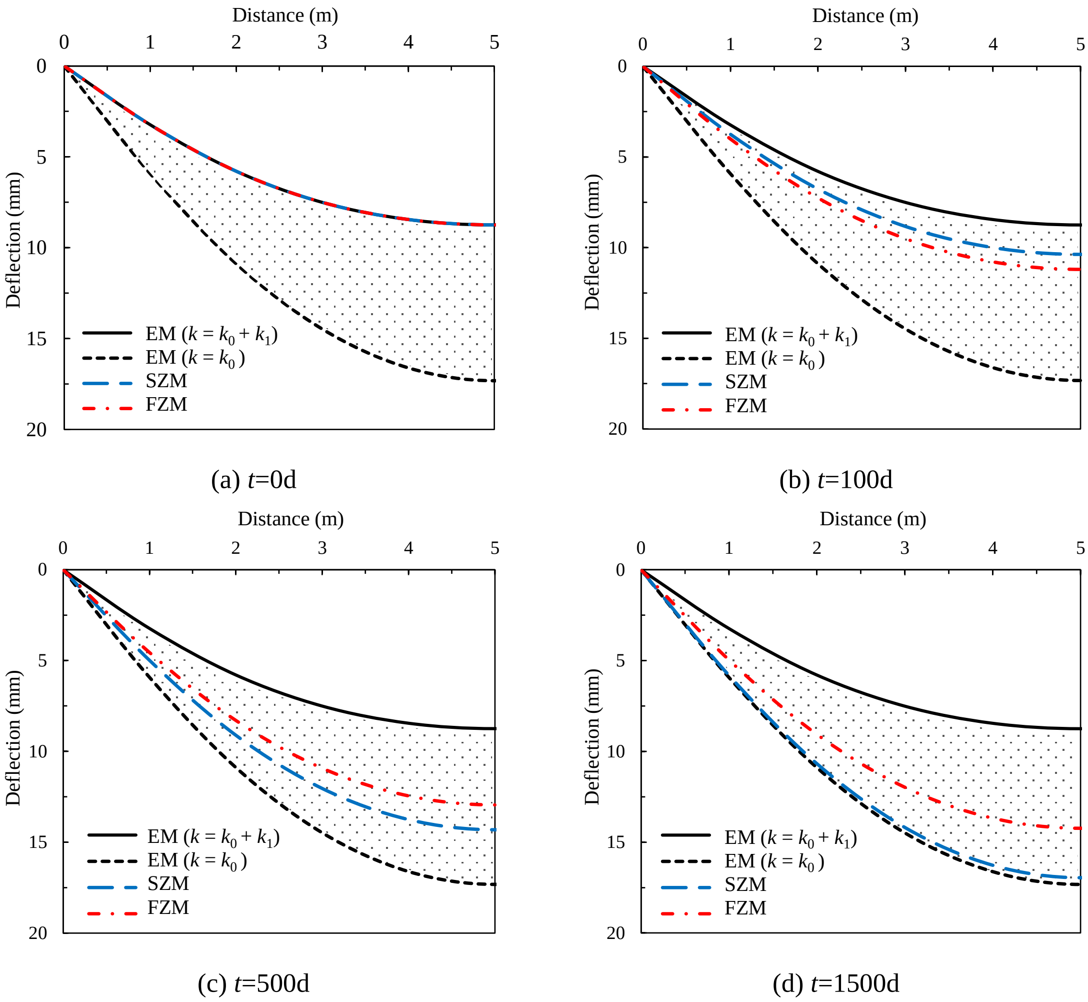
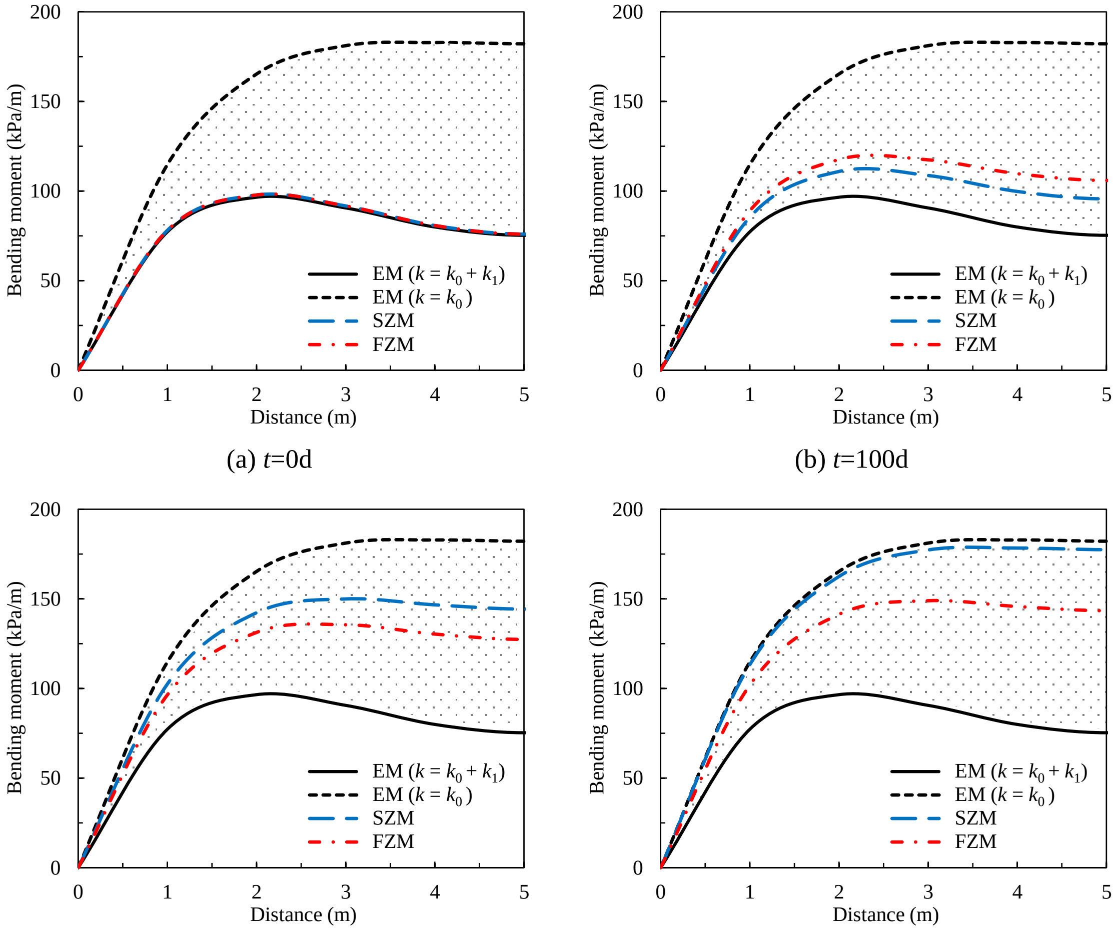


Fig. 5 Comparison of plate deflections calculated using elastic, standard Zener and fractionalized Zener models



(a) t=0d (b) t=100d

(c) t=500d (d) t=1500d

Fig. 6 Comparison of bending moments of the plate calculated using elastic, standard Zener and fractionalized Zener models

modulus k is replaced by k_0+k_1. In comparison with the FZM results, those calculated from the SZM are smaller at first (t=100d) but then develop quickly (t=500d), and eventually tend to be stable (t=1500d). With time lapses, the deflections calculated from the SZM and FZM tend towards the EM solution as well where k in Eq. (13) is replaced by k_0. Using the current parameters, it approximately takes 1500 d for the deflections from the SZM to be stable while for those using the FZM, it will be much longer. Similar trends have been obtained for the bending moments of the plate and foundation reactions, as shown in Figs. 6 and 7.

4.2 Parametric study of the FZM

In the proposed FZM, there are four model parameters, i.e. spring stiffness k_0 and k_1, coefficient of viscosity η^*, and fractional differential order α. As stated previously, we know that when t=0 and t=∞, the plate deflections and foundation reactions can be calculated from the elastic model as

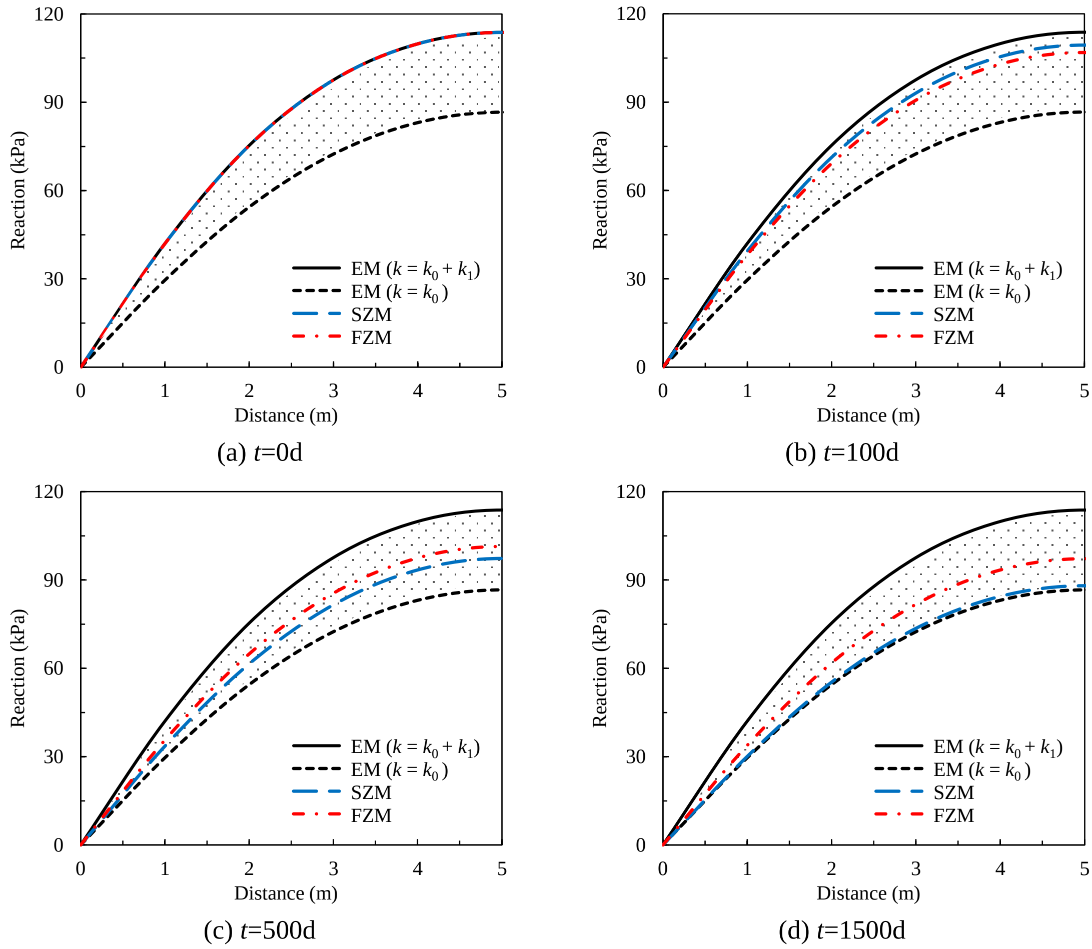


Fig. 7 Comparison of foundation reactions calculated using elastic, standard Zener and fractionalized Zener models

long as the modulus k is respectively replaced by k_0+k_1 and k_0, respectively. That is to say that the upper and lower bound solutions of the FZM are dependent on the spring stiffness k_0 and k_1.

Figs. 8 and 9 show the influences of the coefficient of viscosity η^* and the fractional differential order α on the time-dependent deflection at the plate center. It can be seen from Fig. 8 that the effect of η^* of the FZM on the plate deflection is quite similar to that of the SZM. η^* controls the rate of deflecting but has no effect on initial and overall amounts of the plate deflection. With the increase of η^*, it takes a longer time to reach the ultimate deflection. The influence of α on the plate deflection is depicted in Fig. 9. When α=0, the resulting deflection is not time-dependent and its value is equal to that calculated from the EM. In addition, the deflection-time curves in Fig. 9 can be divided into two stages: In Stage I, the deflection decreases with the increase of α; In Stage II, the deflection increases with an increasing α value. When α is relatively small, the deflection is large at first but then accumulates slowly as time elapses. In contrast, for a larger α value, the initial deflecting rate is smaller but it develops sharply over time. The deflection calculated from the SZM (i.e., α=1 in the FZM) falls into the latter case.

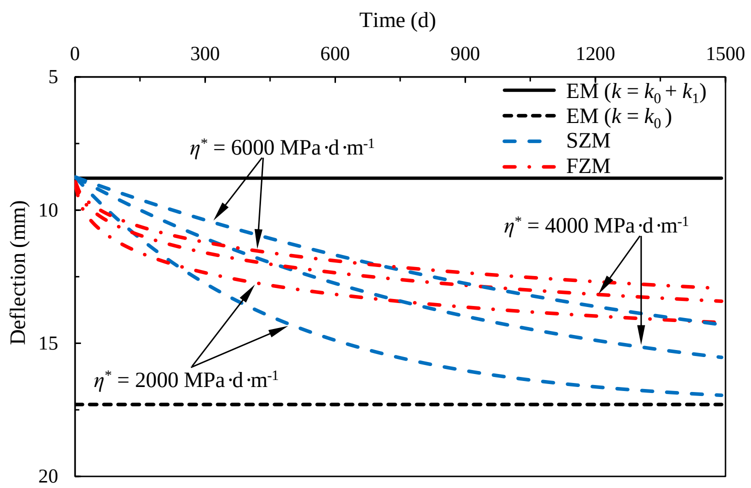


Fig. 8 Influence of coefficient of viscosity η^* on deflection-time curves of the plate center

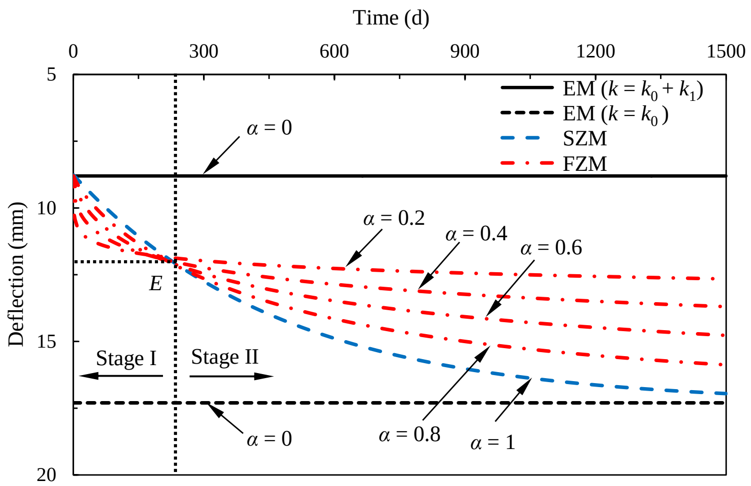


Fig. 9 Influence of fractional differential order α on deflection-time curves of the plate center

4.3 Discussions

From the view of foundation settlement, the total settlement estimated by the FZM is comprised of three components, as shown in Fig. 10. In the classical soil mechanism, foundation settlement consists of elastic settlement (S_e), primary consolidation settlement (S_{pc}) and secondary consolidation settlement (S_{sc}), i.e.

$$S = S_e + S_{pc} + S_{sc} \tag{25}$$

The primary and secondary consolidation settlements of a soil layer with thickness H are (Das 2010)

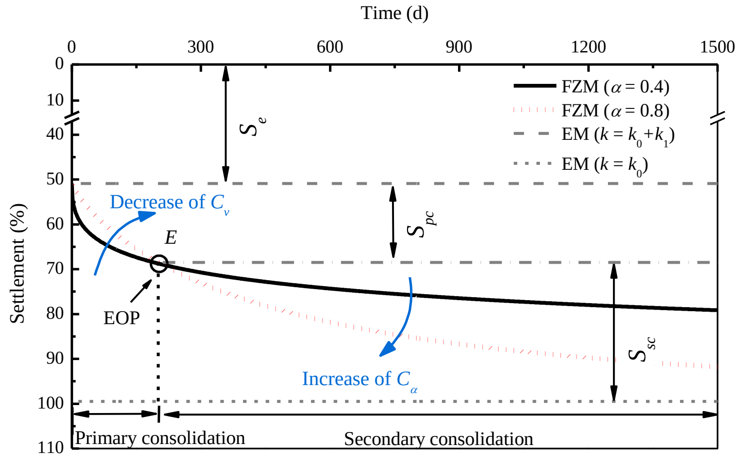


Fig. 10 Three components of the total ground settlement predicted by the FZM

$$\begin{cases} S_{pc} = \dfrac{C_c}{1+e_o} H \log\left(\dfrac{\sigma_0{}' + \Delta\sigma'}{\sigma_0{}'}\right) \\ S_{sc} = \dfrac{C_\alpha}{1+e_p} H \log\left(\dfrac{t_2}{t_1}\right) \end{cases} \tag{26}$$

where C_c and C_α are compression index and secondary consolidation index, respectively; e_o and e_p are the initial void ratio and the void ratio at the end of primary (EOP) consolidation, respectively; σ_0' and $\Delta\sigma'$ are the initial effective overburden pressure and the increase in the effective pressure, respectively; t_1 and t_2 are times.

It appears that the FZM can be used to describe such regulation as defined by Eqs. (25) and (26). The characteristic point "*E*" in Fig. 10 that divides the two time-dependent stages may be supposed to be the completion of the consolidation process, i.e., EOP. It is clear that the amount of the primary consolidation settlement is not affected by the fractional differential order α. As a result, α is not related to compression index C_c. However, this parameter is capable of describing how quick the foundation soil is consolidated under surcharge loads. More specifically, when α is relatively small (α=0.4), the development of primary consolidation settlement is quite quick with a much slower secondary consolidation process, and vice versa. The increase of α represents a smaller consolidation coefficient C_v and a larger secondary consolidation index C_α. Therefore, the α value is supposed to be related to the property of ground soil. A sandy foundation will have a small fractional differential order, while in order to simulate the creeping of clay foundations, a larger fractional differential order value is needed.

Another distinct benefit of the FZM is that there will be a limit of secondary compression when load duration is infinite. Comparatively, according to the conventional solution Eq. (26), when $t_2 \to \infty$, S_{sc} tends to infinity, which is physically impossible. Regarding this problem, the FZM prediction is more rational than the conventional one.

5. Conclusions

In this paper, a fractionalized model is upgraded based on the Zener model to describe the time-dependent performance of a loaded rectangular thin plate resting on a viscoelastic foundation. Analytical solutions of plate deflection, bending moment and foundation reaction are derived in terms of the Mittag-Leffler function. The following conclusions are drawn in this study:

• Through the comparison between the results calculated from the FZM and those calculated from the SZM and EM, it is found that the upper and lower bound solutions of the plate deflection, bending moment and foundation reaction of the FZM can be obtained from the EM.

• Together with the coefficient of viscosity η^*, the introduction of fractional differential order α provides a powerful method in describing the long-term performance of a foundation plate with a fairly small number of parameters.

• It is demonstrated in the parametric study that the FZM can provide a wide range of results with only four parameters. A small fractional differential order corresponds to a plate resting on a sandy foundation characterized by a larger initial deflection and a smooth deflection-time curve in the later period, while for a clay foundation with a smaller permeability coefficient, the value of fractional differential order should be increased. The fractionalized rheological model can simulate the complex characteristics of soil-structure interaction problems elegantly.

Despite the qualitative relationship between the model parameters and the plate response as stated above, we note that the physical meaning of the fractional differential order still remains to be investigated so as to gain a deeper understanding of the potential of fractional calculus in solving plate-foundation interaction problems. More laboratory and field works are required to be undertaken for verifying the proposed fractionalized viscoelastic model.

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