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Analysis of an electrically actuated fractional model of viscoelastic microbeams

Seyed Masoud Sotoodeh Bahraini*[1], Mohammad Eghtesad[2], Mehrdad Farid[2] and Esmaeal Ghavanloo[2]

[1]Department of Mechanical Engineering, Yazd University, Yazd, 89195-741, Iran
[2]School of Mechanical Engineering, Shiraz University, Shiraz, Iran

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Abstract. The MEMS structures usually are made from silicon; consideration of the viscoelastic effect in microbeams duo to the phenomena of silicon creep is necessary. Application of the fractional model of microbeams made from viscoelastic materials is studied in this paper. Quasi-static and dynamical responses of an electrically actuated viscoelastic microbeam are investigated. For this purpose, a nonlinear finite element formulation of viscoelastic beams in combination with the fractional derivative constitutive equations is elucidated. The four-parameter fractional derivative model is used to describe the constitutive equations. The electric force acting on the microbeam is introduced and numerical methods for solving the nonlinear algebraic equation of quasi-static response and nonlinear equation of motion of dynamical response are described. The deflected configurations of a microbeam for different purely DC voltages and the tip displacement of the microbeam under a combined DC and AC voltages are presented. The validity of the present analysis is confirmed by comparing the results with those of the corresponding cases available in the literature.

Keywords: viscoelastic microbeam; fractional derivatives; finite element method; electrical actuation; AC and DC voltage

1. Introduction

The viscoelastic behavior is observed in a number of materials which are used in a wide range of applications such as disks in human spine, polymers, elastomers, automobile bumpers, metals and alloys at elevated temperatures, concrete, soils, road construction and building materials, biological tissues and Micro-Electro Mechanical Systems (MEMS) structures. The mechanics of viscoelastic media has been investigated by many researches in the last few decades. Ferry (1980) investigated linear viscoelasticity of amorphous polymers. Mainardi (2010) studied the connections among fractional calculus, linear viscoelasticity and wave motion. He presented how fractional calculus provides a suitable method for describing dynamical properties of the linear viscoelastic media including problems of wave propagation and diffusion. Marques and Creus (2012) developed a presentation of viscoelasticity theory oriented toward numerical applications.

*Corresponding author, Ph.D. Student, E-mail: sotoodeh@stu.yazd.ac.ir

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The effect of viscoelastic behavior of MEMS structures has been the focus of many researches in the last few decades (Bethe *et al.* 1990, Teh and Lin 1999, Elwenspoek and Jansen 2004, Tuck *et al.* 2005). The viscoelastic properties of nine silicone-, polysulfide-, and polyether- based impression materials were determined using creep tests by Goldberg (1974). He argued that during deformation the materials demonstrated linear viscoelastic behavior and permanent deformation in these materials is a result of lack of recovery of elastic deformation as well as viscous flow. Fu and Zhang (2009) studied static and dynamic responses of an electrically actuated microbeam including viscoelastic effect. On the basis of the Euler-Bernoulli model, nonlinear static and dynamic responses of a viscoelastic microbeam under two kinds of electric forces (a purely direct current (DC) and a combined current composed of a DC and an alternating current (AC)) were studied by Fu and Zhang (2009). A new beam model was developed for the viscoelastic microbeam based on a modified couple stress theory by Zhang and Fu (2012). Rezazadeh *et al.* (2012) studied parametric oscillation of an electrostatically actuated microbeam using variational iteration method. In that paper, a micro-beam suspended between two conductive micro-plates, subjected to a same actuation voltage is considered. Bayat *et al.* (2013) studied vibration of an electrostatically actuated microbeam by an analytical approach.

Since the stress state in a viscoelastic material at the current time depends on the stress and strain histories, the constitutive equation should include time. Describing the dynamic properties of viscoelastic materials with weak frequency dependence, using integer derivative operators, convolution integral or internal variables, requires a great number of high-order time derivatives acting on both stress and strain. By using fractional derivative operators instead of integer derivative operators in the constitutive equations of viscoelasticity, one can overcome this difficulty. It should be noted that different fractional models have been developed to describe the viscoelastic behavior. Furthermore, the fractional constitutive equations improve curve-fitting properties, especially when experimental data from long time intervals or spanning several frequency decades need to be fitted (Schmidt and Gaul 2002). Fractional derivatives were used for the description of viscoelastic materials and the results obtained were in good agreement with the experimental ones (Caputo and Mainardi 1971, Caputo 1974). Bagley and Torvik (1983) reported a physical justification for the concept of fractional derivatives in combination with viscoelasticity. The fractional derivative Zener model which is characterized by four-parameters has been introduced and used in (Rogers 1983, Bagley and Torvik 1986, Pritz 1996). Moreover, the fractional derivative models characterized by five parameters have been presented and used by Pritz (2003). In another study, the transient response of viscoelastic materials involving fractional integro-differential operators was studied by Padovan (1987) using finite element method. Hereditary integral fractional constitutive equations were presented by Koeller (1984) and their implementation into finite element formulation has been investigated by Enelund and Josefson (1997). Enelund *et al.* (1999) studied the finite element implementation of fractional derivative viscoelastic formulations using the concept of internal variables. Implementations of fractional constitutive equations into Boundary Element Method have been presented for the time and frequency domains (Gaul and Schanz 1999, Gaul 1999). Three-dimensional fractional constitutive equations based on the Grünwaldian formulation have been derived and their implementation into an elastic finite element code was demonstrated by Schmidt and Gaul (2002). A finite element formulation for transient dynamic analysis of viscoelastic sandwich beams was proposed by Galucio *et al.* (2004). An electromechanically coupled finite element model to handle active–passive damped multilayer sandwich beams, consisting of a viscoelastic core sandwiched between layered piezoelectric faces was proposed by Trindade *et al.* (2001). A finite element formulation for dynamic transient analysis of a damped adaptive sandwich beam composed of a viscoelastic core and elastic-piezoelectric laminated faces, using fractional derivative model to characterize its viscoelastic behavior in time domain was demonstrated by Galucio *et al.* (2001). Bahraini *et al.* (2013) investigated large deflection of viscoelastic beams using a fractional derivative model, and developed a finite element implementation for nonlinear analysis of viscoelastic fractional models using the storage of both strain and stress histories. Also, Bahraini *et al.* (2014) studied application of fractional-order $PI^{\lambda}D^{\mu}$ controllers for vibration suppression of viscoelastic beams.

In this paper, the fractional four-parameter model of viscoelastic materials is applied to obtain constitutive equations of microbeams. The electric force in microbeam is defined for transverse loading. For quasi-static response of the microbeam a purely DC voltage and for dynamical response combination of those are considered. Quasi-static and dynamical analyses are separated in two sections. In one section, based on Euler-Bernoulli assumption, first, the kinematics of the viscoelastic beam and its displacement field are presented. Then, using the nonlinear strain-displacement relations, a finite element formulation for quasi-static analysis is presented. Since the governing algebraic equations are nonlinear, Picard method is used to solve the equations. In the next section, based on Euler-Bernoulli assumption, a finite element formulation for transient dynamic analysis of viscoelastic microbeams using fractional derivative constitutive equations is developed. The equation of motion in the dynamical analysis is nonlinear, too; therefore, an iterative method which can be considered as a modification of Newmark method is described to solve the equation. Two examples of a viscoelastic microbeam under a purely DC voltage (Fu and Zhang 2009) and a combined DC and AC voltage (Rezazadeh *et al.* 2012) are chosen to validate our formulations for quasi-static and dynamical responses of microbeam, respectively. The deflected configurations of a microbeam for three different voltages and three different dielectric constants of the gap are presented. The effect of order of fractional derivative in quasi-static response of the cantilever viscoelastic microbeam, is investigated. Furthermore, the dynamic response of a microbeam under a combined DC and AC voltage is presented.

2. Fractional viscoelastic constitutive equations

There are different definitions of fractional derivatives (Oldham and Spanier 1974). For developing numerical algorithms, the Grünwald definition, can be implemented easily based on the generalization of the backward difference. This definition of fractional derivative has the following form (Podlubny 1999)

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \lim_{N\to\infty}\left[\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=0}^{N-1} A_{j+1} f(t - j\frac{t}{N})\right] \tag{1}$$

where, α is the order of fractional derivative and A_{j+1} is called the Grünwaldian coefficient which is defined in term of gamma function

$$A_{j+1} \equiv \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} \tag{2}$$

Fractional derivatives can be calculated by approximating the infinite sum in Eq. (1) by a finite sum, such that $N_l < N < \infty$,

$$\frac{d^\alpha f(t)}{dt^\alpha} \approx \left(\frac{t}{N}\right)^{-\alpha} \sum_{j=0}^{N_l} A_{j+1} f(t - j\frac{t}{N}) \tag{3}$$

The Grünwald coefficients are weighting functions which decrease with growing j. The weights A_{j+1} are assigned to the values $f(t\text{-}jt/N)$, so the effect of the values of function f for small j's, in calculating the fractional derivative is larger.

The constitutive equation of the four-parameter model (one-dimensional viscoelastic model) is

$$\sigma(t) + \tau^\alpha \frac{d^\alpha \sigma(t)}{dt^\alpha} = E_0 \varepsilon(t) + \tau^\alpha E_\infty \frac{d^\alpha \varepsilon(t)}{dt^\alpha} \tag{4}$$

where σ and ε are the stress and the strain, E_0 and E_∞ are the relaxed and non-relaxed elastic moduli, and τ is the relaxation time. By introducing the constants

$$a = \tau^\alpha,\ c = E_0 \text{ and } b = \tau^\alpha E_\infty \tag{5}$$

Eq. (4) can be rewritten as

$$\sigma(t) + a\frac{d^\alpha \sigma(t)}{dt^\alpha} = c\varepsilon(t) + b\frac{d^\alpha \varepsilon(t)}{dt^\alpha} \tag{6}$$

3. Electrical actuation

In this section two model of electrically actuated microbeams, a clamped-free and a clamped-clamped microbeam are studied. Consider a cantilever microbeam with length L, width b, thickness h, and gap g_0 as shown in Fig. 1.

The electric force applied on the microbeam can be defined as (Fu and Zhang 2009)

$$F(x,t) = \frac{\varepsilon_v bV^2}{2(g_0 - w)^2} \tag{7}$$

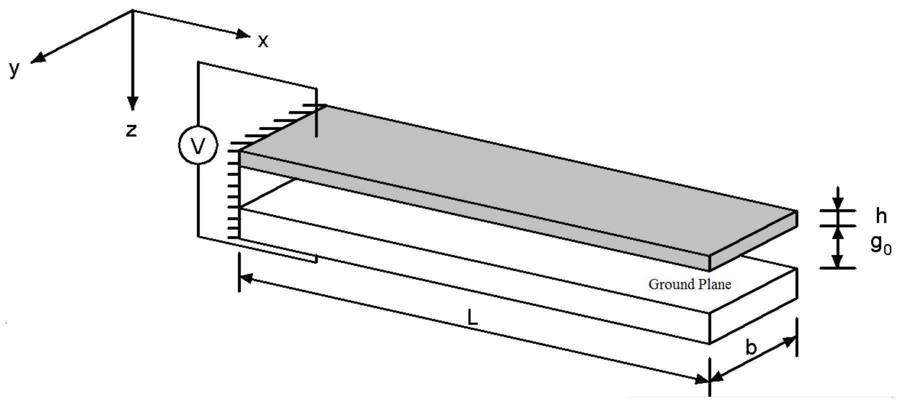


Fig. 1 Electrically actuated cantilever microbeam

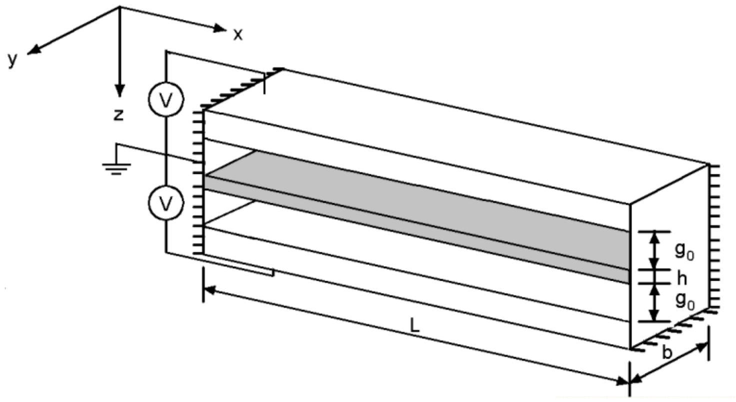


Fig. 2 Electrically actuated clamped-clamped microbeam

Model of a clamped-clamped microbeam is illustrated in Fig. 2 which is symmetrically located between two electrodes. The electric force applied on the microbeam can be defined as (Rezazadeh *et al.* 2012)

$$F(x,t) = \frac{\varepsilon_v bV^2}{2(g_0 - w)^2} - \frac{\varepsilon_v bV^2}{2(g_0 + w)^2} \tag{8}$$

where ε_v is the dielectric constant of the gap medium and V is the voltage difference between the microbeam and infinite ground plane. The voltage is composed of a DC polarization voltage and an AC voltage

$$V = V_D + V_A \cos\Omega t \tag{9}$$

where V_D is the DC voltage, V_A and Ω are the amplitude and frequency of the AC voltage, respectively. In this paper, for quasi-static response of the microbeam a purely DC voltage and for dynamical response combination of both voltages are considered.

4. Quasi-Static response

By using the Grünwald definition of the fractional derivative, Eq. (6) can be represented as

$$\sigma(t) + a\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=0}^{N_l} A_{j+1}\sigma(t - j\frac{t}{N}) = c\varepsilon(t) + b\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=0}^{N_l} A_{j+1}\varepsilon(t - j\frac{t}{N}) \tag{10}$$

Since A_1=1, Eq. (10) can be rewritten as

$$\begin{aligned} &\sigma(t) + a\left(\frac{t}{N}\right)^{-\alpha} \sigma(t) + a\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1}\sigma(t - j\frac{t}{N}) = \\ &c\varepsilon(t) + b\left(\frac{t}{N}\right)^{-\alpha} \varepsilon(t) + b\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1}\varepsilon(t - j\frac{t}{N}) \end{aligned} \tag{11}$$

which can be solved explicitly for $\sigma(t)$ as

$$\sigma(t)=\left(1+a\left(\frac{t}{N}\right)^{-\alpha}\right)^{-1}\times \left\{\left(c+b\left(\frac{t}{N}\right)^{-\alpha}\right)\varepsilon(t)+b\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=1}^{N_l}A_{j+1}\varepsilon(t-j\frac{t}{N})-a\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=1}^{N_l}A_{j+1}\sigma(t-j\frac{t}{N})\right\} \tag{12}$$

This relation can be simplified using the following abbreviations

$$R(t)=1+a\left(\frac{t}{N}\right)^{-\alpha},\ C^*(t)=c+b\left(\frac{t}{N}\right)^{-\alpha} \tag{13}$$

to obtain the following form

$$\sigma(t)=\left(R(t)\right)^{-1}\left(C^*(t)\ \varepsilon(t)+b\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=1}^{N_l}A_{j+1}\varepsilon(t-j\frac{t}{N})-a\left(\frac{t}{N}\right)^{-\alpha}\sum_{j=1}^{N_l}A_{j+1}\sigma(t-j\frac{t}{N})\right) \tag{14}$$

The stress and strain histories can be seen in the right hand sides of Eq. (14), this equation is used as the constitutive equation of viscoelastic beam in our nonlinear analysis.

4.1 Strain-Displacement relation

Based on Euler-Bernoulli assumption, the kinematics of the viscoelastic beam and its displacement field can be clearly illustrated as seen in Fig. 3.

The bending of beam with moderately large rotations but with small strains can be derived using the following displacement fields

$$u_1=u_0(x)-z\frac{\partial w_0}{\partial x},\ u_2=0\quad and\quad u_3=w_0(x) \tag{15}$$

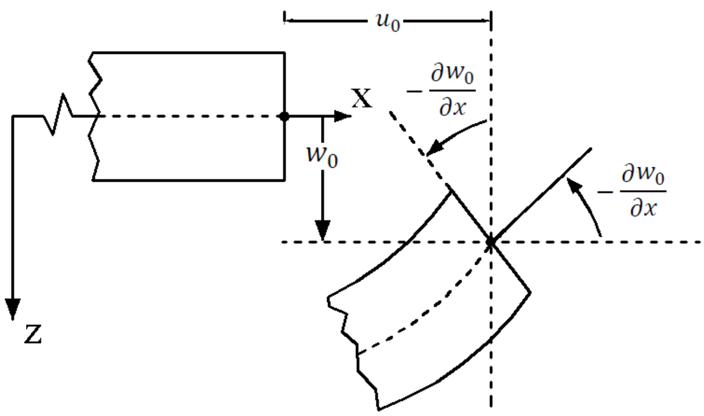


Fig. 3 Undeform and deform geometries of a viscoelastic beam

where (u_1, u_2, u_3) are the total displacements along the coordinate directions (x, y, z), (u_0, w_0) are the axial and lateral displacements of a point on the mid-plane of the undeformed beam. Using the nonlinear strain-displacement relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_j}\frac{\partial u_m}{\partial x_i}\right) \tag{16}$$

and omitting the high order terms but retaining only the square of du_3/dx (representing the rotation of the transverse normal line in the beam), we obtain

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 = \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right] - z\frac{d^2 w_0}{dx^2} \tag{17}$$

Other strains are zero. Note that the notations $x_1=x$, $x_2=y$, and $x_3=z$ are used. These strains are known as the von Kármán strains (Reddy, 2004).

4.2 Finite element formulation

This part presents a finite element formulation for the problem. Viscoelastic constitutive equations should be integrated into finite element formulations. The viscoelastic model used to describe the behavior of the beam is a four-parameter fractional derivative model. In order to implement the nonlinear viscoelastic model into the finite element formulation, the Grünwald definition of the fractional operator is employed. The generalized displacements

$$u^e = \begin{bmatrix} u_0 & w_0 \end{bmatrix}^T \tag{18}$$

are discretized using linear shape functions for u_0 (axial deformation) and cubic or Hermite shape functions for w_0 (lateral deformation). The generalized displacements are related to the elementary degrees-of-freedom vector

$$q^e = \begin{bmatrix} u_1 & w_1 & w_1' & u_2 & w_2 & w_2' \end{bmatrix}^T \tag{19}$$

by

$$u^e = H\, q^e \tag{20}$$

where the interpolation matrix H is defined as follows

$$H = \begin{bmatrix} H_1 & 0 & 0 & H_2 & 0 & 0 \\ 0 & H_3 & H_4 & 0 & H_5 & H_6 \end{bmatrix} \tag{21}$$

with

$$\begin{aligned} &H_1 = 1 - \frac{x}{L_e},\ H_3 = 1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3},\ H_5 = \frac{x^2}{L_e^2}\left(3 - \frac{2x}{L_e}\right) \\ &H_2 = \frac{x}{L_e},\ H_4 = x\left(1 - \frac{x}{L_e}\right)^2,\ H_6 = \frac{x^2}{L_e}\left(\frac{x}{L_e} - 1\right) \end{aligned} \tag{22}$$

By introducing H_x and H_z as

$$H_x = \begin{bmatrix} H_1 & 0 & 0 & H_2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad H_z == \begin{bmatrix} 0 & H_3 & H_4 & 0 & H_5 & H_6 \end{bmatrix} \tag{23}$$

the governing equations can be written as

$$\begin{aligned} \varepsilon_{xx}^e &= H_x' \, q^e + \frac{1}{2}\left(H_z' \, q^e\right)^T \left(H_z' \, q^e\right) - z \, H_z'' \, q^e \\ &= \left(H_x' + \frac{1}{2}\left(H_z' \, q^e\right)^T \left(H_z'\right) - z \, H_z'' \right) q^e \equiv H_\varepsilon q^e \end{aligned} \tag{24}$$

The variation of Eq. (24) can be written as

$$\begin{aligned} \delta\varepsilon_{xx}^e &= H_x' \delta q^e + \frac{1}{2}\left(H_z' q^e\right)^T \left(H_z' \delta q^e\right) + \frac{1}{2}\left(H_z' \delta q^e\right)^T \left(H_z' q^e\right) - z H_z'' q^e \\ &= \left(H_x' + \left(H_z' \, q^e\right)^T \left(H_z'\right) - z \, H_z'' \right) \delta q^e \equiv H_{\delta\varepsilon} \delta q^e \end{aligned} \tag{25}$$

By using the principle of virtual work

$$\int_V \left(\sigma_{xx} \delta\varepsilon_{xx}\right) dV = \delta W \tag{26}$$

and Eq. (14), we obtain

$$\begin{aligned} &\int_V \left(\delta\varepsilon_{xx}^T \sigma_{xx}\right) dV = \int_V \left(H_{\delta\varepsilon} \delta q^e\right)^T \left(R(t)\right)^{-1} \times \\ &\left(C^*(t) \; \varepsilon_{xx}(t) + b\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \varepsilon_{xx}(t - j\frac{t}{N}) - a\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \sigma_{xx}(t - j\frac{t}{N}) \right) dV \end{aligned} \tag{27}$$

Now by replacing Eqs. (24)-(25) into Eq. (27), we obtain

$$\begin{aligned} &\int_V \left(\delta\varepsilon_{xx}^T \sigma_{xx}\right) dV = \int_V \left(\left(H_x' + \left(H_z' \, q^e\right)^T \left(H_z'\right) - z \, H_z'' \right) \delta q^e \right)^T \times \\ &\qquad \left(R(t)\right)^{-1} \left(\; C^*(t) \left(\left(H_x' + \frac{1}{2}\left(H_z' \, q^e\right)^T \left(H_z'\right) - z \, H_z'' \right) q^e \right) \right. \\ &\qquad \left. + b\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \varepsilon_{xx}(t - j\frac{t}{N}) - a\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \sigma_{xx}(t - j\frac{t}{N}) \right) dV \end{aligned} \tag{28}$$

By substituting the strain and stress histories in the right hand sides of Eq. (28), the equilibrium equation will be in the following form

$$K^{*e}\left(q^e\right)q^e = F^{*e}\left(q^e\right) + F^e_{ext} \tag{29}$$

where the element stiffness matrix K^* and the element force vector F^* can be represented by

$$\begin{aligned} K^{*e}\left(q^e\right) &= \left(R(t)\right)^{-1} C^*(t) \\ &\times \int_V \left(H'_x + q^{eT} H'^{\,T}_z H'_z - z H''_z\right)^T \left(H'_x + \frac{1}{2}\left(H'_z q^e\right)^T \left(H'_z\right) - z H''_z\right) dV \end{aligned} \tag{30}$$

and

$$\begin{aligned} F^{*e}\left(q^e\right) &= \left(R(t)\right)^{-1} \int_V \left(H'_x + q^{eT} H'^{\,T}_z H'_z - z H''_z\right)^T \\ &\times \left(-b\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \varepsilon_{xx}\left(t - j\frac{t}{N}\right) + a\left(\frac{t}{N}\right)^{-\alpha} \sum_{j=1}^{N_l} A_{j+1} \sigma_{xx}\left(t - j\frac{t}{N}\right) \right) dV \end{aligned} \tag{31}$$

The Grünwaldian formulation of fractional derivatives requires the strain and stress histories at equivalent discrete times. The actual displacements and strains are established at the end of each time step. The constitutive Eq. (14) as a function of the actual strains and the strain and stress histories is substituted in Eq. (31).

4.3 Method of solution

Note that the Eq. (29) is nonlinear, since the element stiffness matrix K^* and the element force vector F^* are functions of the unknown vector q. The assembled nonlinear equations must be solved, after imposing boundary conditions, by a suitable method. Here, the Picard iterative method is used in which we seek an approximate solution to the nonlinear algebraic equations by linearization (Reddy 2006). The iterative method is outlined using a nonlinear matrix equation of the form

$$K_{xx}(q)q = F_{xx}(q) + F_{ext}(q) \tag{32}$$

where q is the vector of unknown nodal values.

5. Dynamical response

Let us introduce the internal variable $\bar{\varepsilon}$, as a strain function

$$\bar{\varepsilon} = \varepsilon - \frac{\sigma}{E_\infty} \tag{33}$$

such that the constitutive Eq. (6) can be rewritten as

$$\bar{\varepsilon} + \tau^\alpha \frac{d^\alpha \bar{\varepsilon}}{dt^\alpha} = \frac{E_\infty - E_0}{E_0}\varepsilon \tag{34}$$

Using the Grünwald approximation with $t/N = \Delta t$, we have:

$$\bar{\varepsilon}^{n+1} = (1-c)\frac{E_\infty - E_0}{E_0}\varepsilon^{n+1} - c\sum_{j=1}^{N_l} A_{j+1}\bar{\varepsilon}^{n+1-j} \tag{35}$$

where c is a dimensionless constant given as

$$c = \frac{\tau^\alpha}{\tau^\alpha + \Delta t^\alpha} \tag{36}$$

5.1 Strain-Displacement relation

Consider the kinematics of viscoelastic beam as shown in Fig. 2. The displacement field can be written as Eq. (15). Using Eq. (16) and omitting the high order terms, the strain-displacement relation can be written as:

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2} \tag{37}$$

5.2 Finite element formulation

This part presents a finite element formulation for transient dynamic analysis of viscoelastic microbeams using fractional derivative constitutive equations. The viscoelastic model used to describe the behavior of the core is a four-parameter fractional derivative model. In order to implement the viscoelastic model in the finite element formulation, the Grünwald definition of the fractional operator is employed. To solve the equations of motion, a direct time integration method based on the implicit Newmark scheme in conjunction with an iterative method, is used. One of the main characteristic of the proposed algorithm lies in the storage of displacement history only, reducing considerably the numerical efforts related to the non-locality of the fractional operators. The generalized displacements $u^e = [u_0 \quad w_0]^T$ are discretized with linear (axial displacement) and cubic (lateral deflection) shape functions. They are related to the elementary degrees-of-freedom vector $q^e = [u_1 \quad w_1 \quad w_1' \quad u_2 \quad w_2 \quad w_2']^T$ by $u^e = H\,q^e$, where the interpolation matrix H is defined in Eq. (21).

The kinematic energy of viscoelastic beam can be written as

$$T = \frac{1}{2}\int_0^l \rho A(\dot{u}^2 + \dot{w}^2)dx \tag{38}$$

by using finite element discretization, the variation of kinetic energy will be obtained as following form

$$\delta T = \left(\delta q^e\right)^T \int_0^{L_e} \rho A H_\varepsilon^T H_\varepsilon dx\, \ddot{q}^e \tag{39}$$

The governing equations can be written as

$$\varepsilon_{xx}^{e} = H'_x\, q^e - z\, H''_z\, q^e = \left(H'_x - z\, H''_z\right) q^e \equiv H_\varepsilon q^e \tag{40}$$

the variation of Eq. (40) can be written as

$$\delta\varepsilon_{xx}^{e} = H'_x \delta q^e - z H''_z q^e = \left(H'_x - z\, H''_z\right)\delta q^e \equiv H_\varepsilon \delta q^e \tag{41}$$

The internal energy of viscoelastic beam by using finite element discretization can be written as

$$\delta U = \left(\delta q^e\right)^T \left\{\int_0^{L_e} E_\infty \left(A\, H_x'^T H'_x + I\, H_z''^T H''_z\right) dx q^e - \int_0^{L_e} E_\infty \left(A\, H_x'^T H'_x + I\, H_z''^T H''_z\right) dx \bar{q}^e\right\} \tag{42}$$

By using the principle of virtual work, the governing equation will be obtained as

$$M^e\, \ddot{q}_{n+1}^{e} + (K^e)\, q_{n+1}^{e} = F_{ext}^{e} + \bar{F}_{n+1}^{e} \tag{43}$$

The stiffness matrix K^e and the loading vector $\bar{F}_{n+1}^{e}$, arising from the viscoelastic behavior of the microbeam, are given by

$$K^e = \left[c\left(E_\infty - E_0\right) + E_0\right]\int_0^{L_e} \left(A H_x'^T H'_x + I\, H_z''^T H''_z\right) dx \tag{44}$$

$$\bar{F}^{\,e} = -c\, E_\infty \int_0^{L_e} \left(A\, H_x'^T H'_x + I\, H_z''^T H''_z\right) dx \sum_{j=1}^{N_l} A_{j+1}\, \bar{q}_{n+1-j}^{\,e} \tag{45}$$

the element mass matrix is given by

$$M^e = \int_0^{L_e} \rho A (H_x^T\, H_x + H_z^T\, H_z)\, dx \tag{46}$$

Also, “inelastic displacements” by using Eq. (40) will be obtained as

$$\bar{q}_{n+1}^{e} = (1-c)\frac{E_\infty - E_0}{E_0}\, q_{n+1}^{e} - c\sum_{j=1}^{N_l} A_{j+1}\, \bar{q}_{n+1-j}^{e} \tag{47}$$

5.3 Method of solution

Note that the element force matrix F_{ext} in the equation of motion, Eq. (43), is a function of the unknown value w. The assembled nonlinear equation of motion must be solved, after imposing boundary conditions, by a suitable method. Here, we describe an iterative method which can be considered as a modification of Newmark method in (Galucio *et al.* 2004). The iterative method is outlined using a nonlinear matrix equation of the form

$$M\ddot{q} + Kq = F_{ext}(q) + F \tag{48}$$

where q is the vector of unknown nodal values. The Newmark scheme for implementation in structural dynamics with some modifications involves the following algorithm

1. Initialize at $t=0$:

$$q_0, \dot{q}_0, \ddot{q}_0 = M^{-1}(F_0 - K\, q_0), \quad \bar{q}_0 = (1-c)\frac{E_\infty - E_0}{E_\infty}\, q_0$$

2. Enter time step loop, assuming that at t_n, the state is completely known:

$$q_n, \dot{q}_n, \ddot{q}_n, \overline{q}_n$$

Predict displacement and velocity

$$q_{n+1}^{pred} = q_n + \Delta t\, \dot{q}_n + (0.5 - \beta)\Delta t^2\, \ddot{q}_n$$

$$\dot{q}_{n+1}^{pred} = \dot{q}_n + (1-\gamma)\Delta t\, \ddot{q}_n$$

Calculate the modified loading

$$\overline{F}_{n+1} = -\frac{c\,E_0}{c\,(E_\infty - E_0) + E_0} K \sum_{j=1}^{N_l} A_{j+1}\, \overline{q}_{n+1-j}$$

Form residual

$$R_{n+1} = F_{n+1} - K\, q_{n+1}^{pred} \quad \text{where} \quad F_{n+1} = \overline{F}_{n+1} + F_{ext}$$

In this step since the vector F_{n+1} is unknown, the direct iteration method is based on the scheme

$$R_{r+1} = F_r\,(q_r^{pred}) - K\, q_{n+1}^{pred} \tag{49}$$

where q_r^{pred} denotes the solution at the rth iteration in the next step. Thus, in the direct iteration method, the coefficients F_i (i=1,…,n) are evaluated using the solution q_r^{pred} from the previous iteration, and the solution at the (r+1)th iteration is obtained by the Newmark method. At the beginning of the iteration (i.e., r=0), we assume a solution q_0^{pred} based on our qualitative understanding of the solution behavior. For example, $q_0^{pred} = 0$ would yield the linear solution of the problem at the end of the first iteration, q_1^{pred}. The iteration is continued until the difference between q_r^{pred} and q_{r+1}^{pred} is reduced to a preselected error tolerance (Reddy, 2006). The error criterion has the form

$$q_{r+1}^{pred} = (q_{r+1}^{pred} - q_r^{pred}) \times 0.1 + q_{r+1}^{pred} \tag{50}$$

Evaluate acceleration by solving the following linear system

$$(M + \beta\Delta t^2\, K)\,\ddot{q}_{n+1} = R_{n+1}$$

Correct displacement and velocity

$$q_{n+1} = q_{n+1}^{pred} + \beta\Delta t^2\, \ddot{q}_{n+1}$$

$$\dot{q}_{n+1} = \dot{q}_{n+1}^{pred} + \gamma\,\Delta t\, \ddot{q}_{n+1}$$

Evaluate the inelastic displacements history by

$$\overline{q}_{n+1}^{e} = (1-c)\frac{E_\infty - E_0}{E_0} q_{n+1}^{e} - c\sum_{j=1}^{N_l} A_{j+1}\, \overline{q}_{n+1-j}^{e}$$

3. Update time step and return to 2.

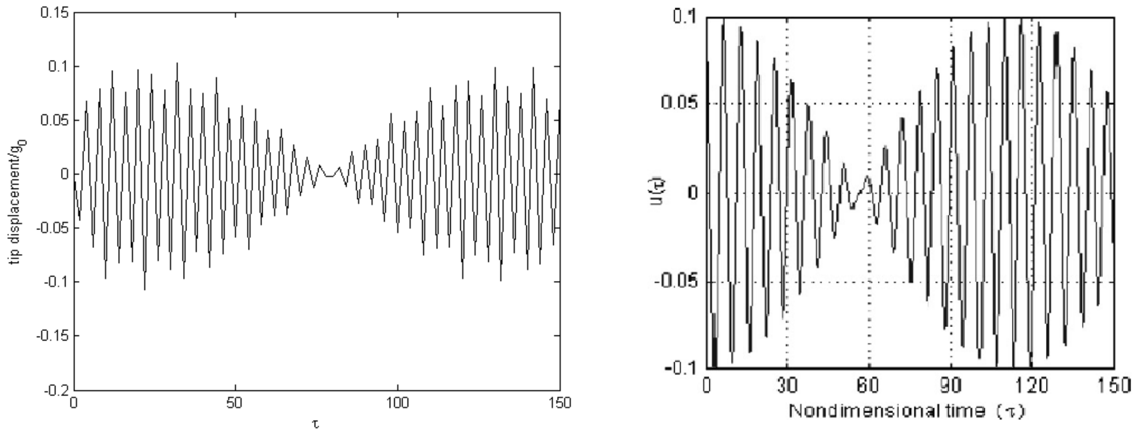
6. Validation

An example of a viscoelastic microbeam under a purely DC voltage (Fu and Zhang, 2009) was chosen to validate our formulation for a quasi-static response of microbeam. The length, width and thickness of the microbeam are L=80 μm, b=10 μm, h=1 μm, respectively, and g_0=3 μm. The beam is modeled by 10 elements to achieve the convergency; the material properties are taken as E_0=160.8 GPa, E_∞=64.32 GPa, ε_v=8.854×10^{-12} C^{2}N^{-1}m^{-2}, ρ=2231 kg/m^3 and υ=0.22. Moreover, the DC voltage is 37.6 volts. It should be noted that when a wide microbeam ($b \geq 5h$) is considered, the relaxation function should be changed into $E(t)/(1-\upsilon^2)$, where υ denotes the Poisson's ratio which is assumed to be time independent (Fu and Zhang, 2009). The tip displacement at t=4000 T for α=0.5 is obtained 0.42, which is in good agreement with the tip displacement (0.4) in the example presented in (Fu and Zhang 2009). $\left(T=\sqrt{\rho bhl^4/E_0 I}\right)$

A clamped-clamped microbeam symmetrically located between two electrodes is chosen from (Rezazadeh *et al.* 2012) to validate the dynamical response of microbeam. The Newmark parameters β=1/4 and γ=1/2 are chosen in order to obtain an unconditionally stable and second order accurate scheme (Galucio *et al.* 2005). The length, width and thickness of the microbeam are L=350 μm, b=100 μm, h=3 μm, respectively, and g_0=1 μm. The material properties are taken as E_0=169.6 GPa, E_∞=4E_0, ε_v=8.854×10^{-12} C^{2}N^{-1}m^{-2}, ρ=2231 kg/m^3 and υ=0.06. The beam is modeled by 6 elements to achieve the convergency. The DC and AC voltages are 25.58 and 5.66 volts, respectively. The tip displacement at t=150 T for α=0.5, ω/T=10 and N_l=75 is presented in Fig. 4, which is in good agreement with the example presented in (Rezazadeh *et al.* 2012).

7. Simulation results

The deflected configurations of a cantilever microbeam for three different voltages are plotted in Fig. 5. The material properties are a=200, b=1.3518×10^{14} N/m^2, c=1.6898×10^{11} N/m^2, α=0.5, and the beam is under three purely DC voltages 10, 15 and 20 volts.



(a) fractional viscoelastic model (b) (Rezazadeh *et al.* 2012)

Fig. 4 Comparison of dynamic response of a microbeam

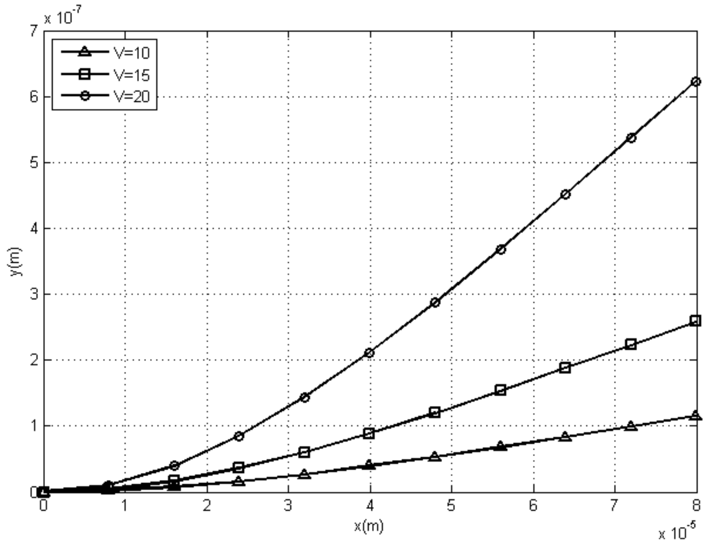


Fig. 5 Geometrical configurations of a cantilever microbeam for three different voltages

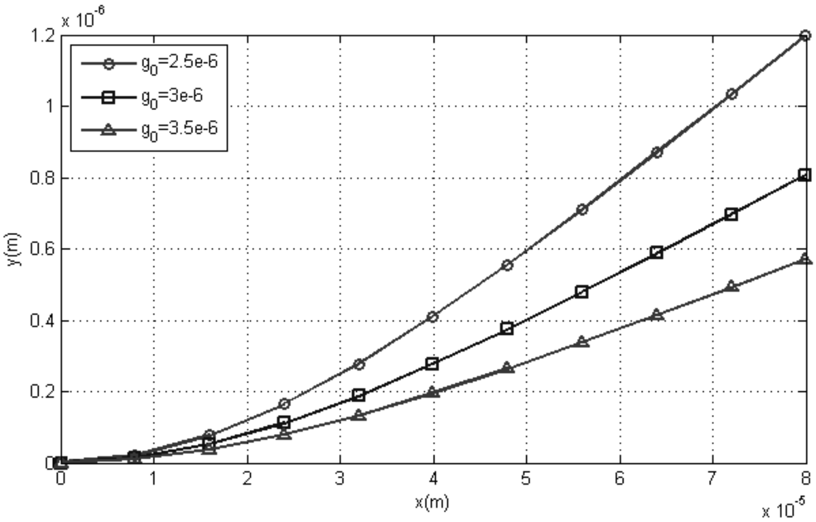


Fig. 6 Geometrical configurations of a cantilever microbeam at different values of the gap

It can be seen that with growing the DC voltage of microbeam, the transverse deflection increases.

The deflected configurations of a microbeam for three different values of the gap are presented in Fig. 6. The DC voltage is chosen as 10 volts. The effect of order of fractional derivative in quasi-static response of the cantilever viscoelastic microbeam, is compared in Fig. 7.

It is observed from Fig. 6 that with growing the gap, microbeam deflection decreases. The values of gap are 2, 2.5 and 3 μm. Deflection of a viscoelastic microbeam subjected to the electrical force is presented in Fig. 7. The orders of fractional derivatives are 0.3, 0. 4 and 0.5. It can be seen that with growing the order of fractional derivative, microbeam deflection increases.

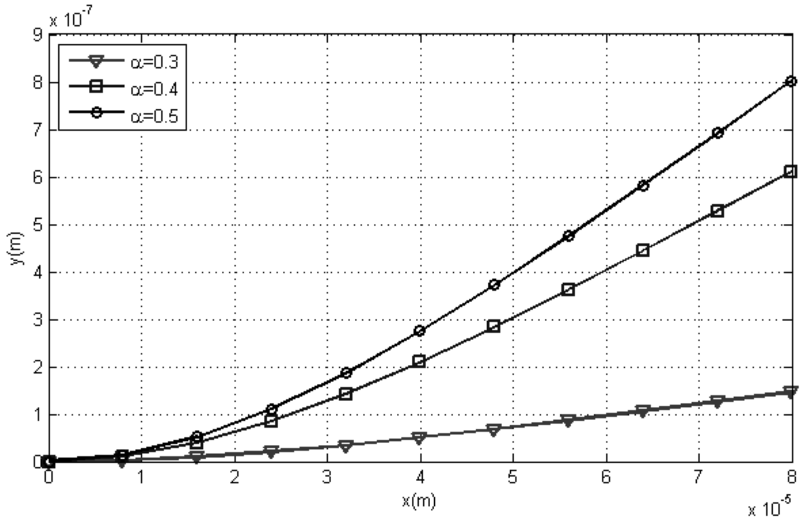


Fig. 7 Geometrical configurations of a cantilever microbeam at different orders of fractional derivative

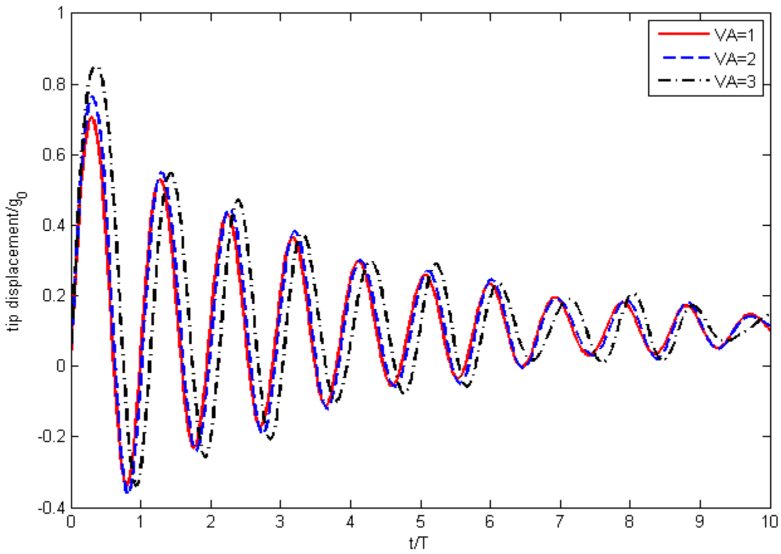


Fig. 8 Dynamic response of a clamped-free microbeam under a combined DC and AC voltage for three different AC voltages

The dynamic response of a clamped-free microbeam for N_l=400, V_D=35, and different AC voltages V_A=1 2, and 3, is plotted in Fig. 8. The material properties are E_0=35 GPa, E_∞=166 GPa, ε_v=8.854×10^{-12} C^{2}N^{-1}m^{-2}, ρ=2231 kg/m^3, υ=0.22, ω/T=10 and α=0.5. The length, width and thickness of the microbeam are L=80 μm, b=10 μm, h=1 μm, respectively and g_0=3 μm. The Newmark parameters are β=1/4 and γ=1/2. The dynamic response of a clamped-free microbeam for three different values of the gap is presented in Fig. 9. The AC voltage amplitude is chosen as 2 volts. The effect of order of fractional derivative on dynamic response of the clamped-free microbeam, is illustrated in Fig. 10.

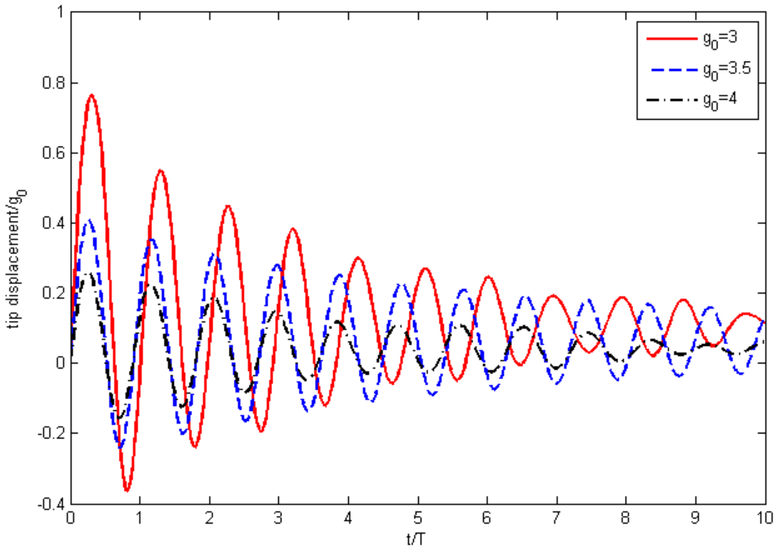


Fig. 9 Dynamic response of a clamped-free microbeam under a combined DC and AC voltage for three different values of the gap

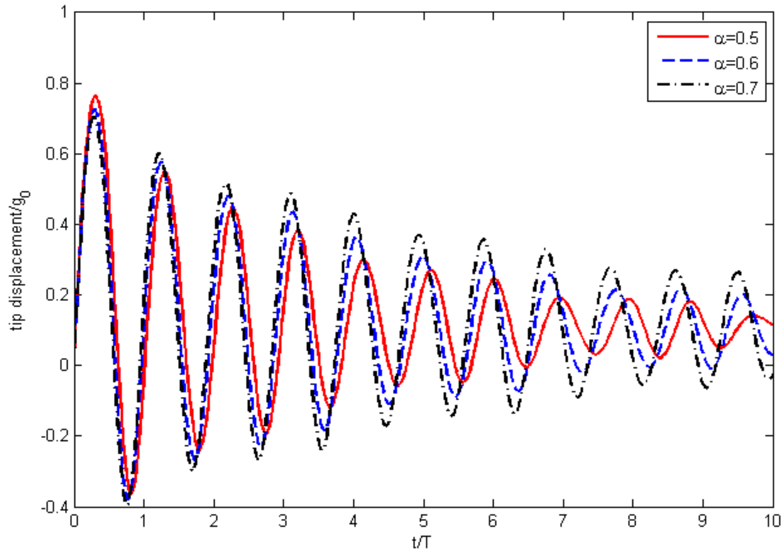


Fig. 10 Dynamic response of a clamped-free microbeam under a combined DC and AC voltage for three different orders of the fractional derivative

The dynamic response of middle point of a clamped-clamped microbeam for N_l=3000, V_D=35, and different voltages V_A=2, 4 and 6 is plotted in Fig. 11. The material properties are E_0=35 GPa, E_∞=166 GPa, ε_v=8.854×10^{-12} C^{2}N^{-1}m^{-2}, ρ=2231 kg/m^3, υ=0.22, ω/T=10 and α=0.5. The length, width and thickness of the microbeam are L=80 μm, b=10 μm, h=1 μm, respectively and g_0=3 μm. The Newmark parameters are β=1/4 and γ=1/2. The dynamic response of a clamped-clamped microbeam for three different values of the gap are presented in Fig. 12. The AC voltage

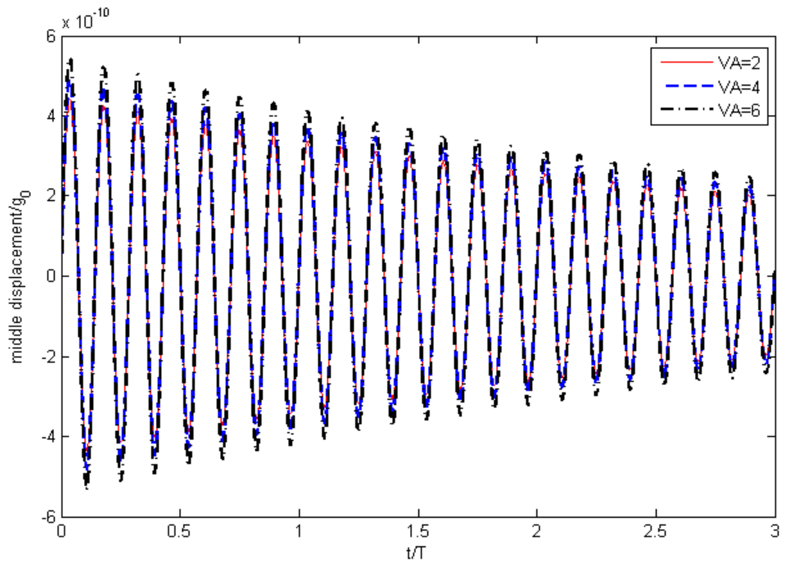


Fig. 11 Dynamic response of a clamped-clamped microbeam under a combined DC and AC voltage for three different AC voltages

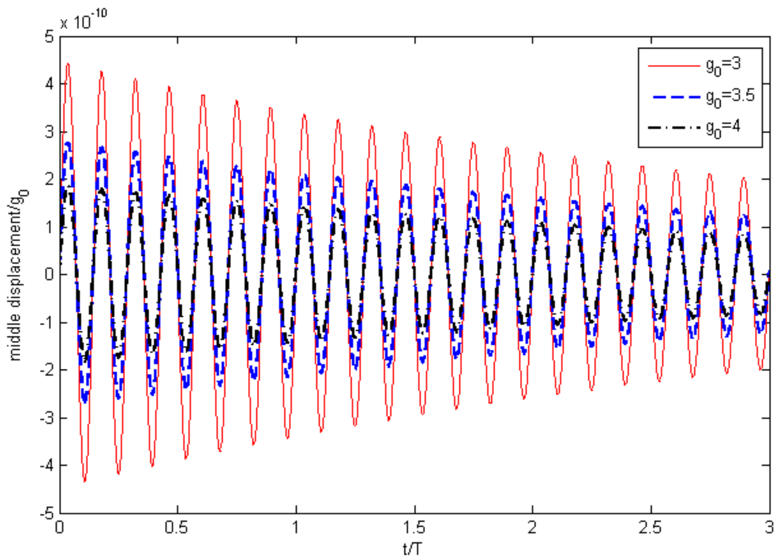


Fig. 12 Dynamic response of a clamped-clamped microbeam under a combined DC and AC voltage for three different values of the gap

amplitude is chosen as 2 volts. The effect of order of fractional derivative on dynamic response of a clamped-clamped microbeam, is illustrated in Fig. 13. The effect of DC and AC voltages on transverse deflection and the effect of vibration at the end of clamped-free microbeam, respectively, can be observed in Figs. 8-10. Additionally, Figs. 8-13 show that due to viscoelastic behavior of the microbeam, the vibration of microbeam is damped after a while. From Figs. 11 and 12, it can be observed that the amplitude of the vibrations increases with raising the AC

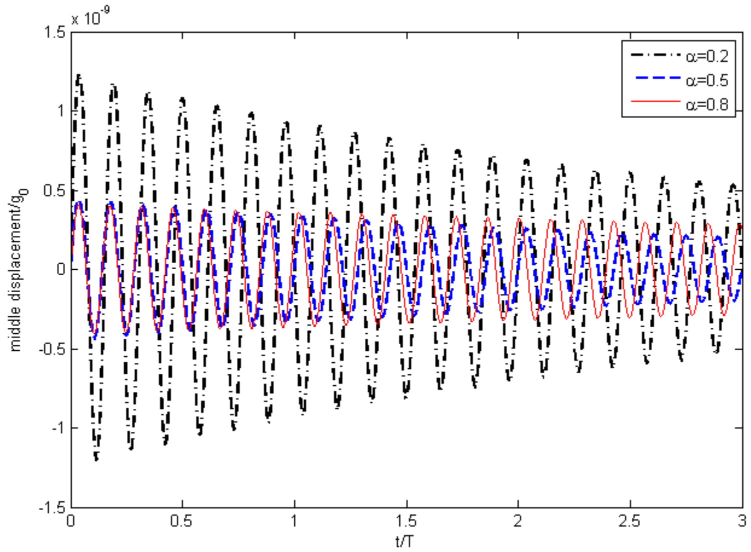


Fig. 13 Dynamic response of a clamped-clamped microbeam under a combined DC and AC voltage for three different orders of the fractional derivative

voltage amplitude, while it decreases by adding to the gap span. Moreover, changing the AC voltage and the gap, do not cause any change in the period of the vibrations. Furthermore, the effect of variation in the order of fractional derivative can be seen in both amplitude and frequency of vibrations in Fig. 13.

8. Conclusions

A finite element formulation of viscoelastic microbeams using fractional derivative constitutive equations has been developed. The four-parameter fractional derivative model has been used to describe the constitutive equation. The electrical force acting on the microbeam was defined. The kinematics of the viscoelastic microbeam in the quasi-static and dynamical analysis based on Euler-Bernoulli assumption has been shown. To solve the nonlinear algebraic equations in quasi-static analysis, the Picard iterative method has been used. An iterative method which can be considered as a modification of Newmark method was described for solving the equation of motion in dynamical analysis. The validation was provided good agreement with two examples which have been chosen to validate the described formulation. The deflected configurations of a microbeam for three different voltages and three different values of the gap were presented. The effect of the increase in the order of fractional derivative in quasi-static response of the cantilever viscoelastic microbeam, was presented. In addition, the dynamic response of clamped-free and clamped-clamped microbeams for three different voltages and three different values of the gap were demonstrated. The effects of the increase in the order of fractional derivative on dynamic response of the clamped-free and clamped-clamped viscoelastic microbeams, were studied. The effects of DC and AC voltages in transverse deflection, vibration at the end and middle points of the microbeams were also investigated. Finally, we think the results presented in this investigation would be helpful for both engineering applications and scientific studies.

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