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Second order effects of external prestress on frequencies of simply supported beam by energy method

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Abstract. Based on the energy method considering the second order effects, the natural frequencies of externally prestressed simply supported beam and the compression softening effect of external prestress force were analyzed. It is concluded that the compression softening effect depends on the loss of external tendon eccentricity. As the number of deviators increases from zero to a large number, the compression softening effect of external prestress force decreases from the effect of axial compression to almost zero, which is consistent with the conclusion mathematically rigorously proven. The frequencies calculated by the energy method conform well to the frequencies by FEM which can simulate the frictionless slide between the external tendon and deviator, the accuracy of the energy method is validated. The calculation results show that the compression softening effect of external prestress force is negligible for the beam with 2 or more deviators due to slight loss of external tendon eccentricity. As the eccentricity and area of tendon increase, the first natural frequency of the simply supported beams noticeably increases, however the effect of the external tendon on other frequencies is negligible.

Keywords: external prestress; simply supported beam; natural frequency; energy method; dynamic analysis

1. Introduction

In recent years, interest in the maintenance, rehabilitation and strengthening of bridge has increased. The technique of bridge strengthening using prestressing with external tendons has been studied as a possible means of strengthening single-span girders, thus the knowledge of the flexural natural frequencies of externally prestressed simply supported beam is of vital importance. One question will be asked whether the prestress force affects the flexural natural frequencies of beams or not? Many researchers have studied the influence of prestress force on the flexural natural frequencies of beams, and there are differences in their conclusions. The natural frequency of a simply supported axially compressed beam is

$$\omega_i = \frac{i\pi}{l}\sqrt{\frac{1}{m}\left[EI\left(\frac{i\pi}{l}\right)^2 - N\right]}, i = 1,2,3,... \quad (1)$$

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where E and I are the elastic modulus and the moment of inertia of the beam, l is the length of the beam, m is the mass per unit length of the beam, i is the mode number and N is the axial compression. Eq. (1) reveals that by increasing the axial compression, the natural frequency decreases due to compression softening, the effect of N is called the effect of axial compression hereinafter.

Saiidi (1994) determined the natural frequencies of a prestressed concrete bridge using Eq. (1). The paper by Saiidi (1994) has been followed by three discussions. In the first discussion, Dallasta (1996) pointed out Saiidi (1994) approach to consider the prestress force as external axial force is incorrect, and indicated that the effect of prestress force on the beam natural frequencies is negligible based on a linear model. In the second discussion, Deak (1996) pointed out that prestress force does not reduce the natural frequencies, nevertheless, the view is not supported by any analytical nor mathematical proof. In the third discussion, Jain (1996) pointed out that because the tendon becomes an integral part of the system, tension in the tendon cannot be treated as an external force, and hence the prestress force being an internal force does not cause the effect of compression softening, does not affect the natural frequencies of beams. Jaiswal (2008) investigated the first flexural natural frequency of beams by finite element method (FEM), and pointed out that the effect of prestress force on the first natural frequency depends on bonded and unbonded nature of the tendon, and also on the eccentricity of tendon. For the beams with bonded tendon, the prestress force does not have any appreciable effect on the first natural frequency. For the beams with unbonded tendon, the first natural frequency significantly changes with the prestress force and eccentricity of the tendon. Jaiswal (2008) findings are not supported by any analytical nor mathematical proof. Kanaka (1986) have shown that the prestress force reduces the natural frequency of the lower modes, based on a Rayleigh-Ritz formulation that describes prestress force as an external axial compression only. Chan (2000) indicated that the natural frequencies of a prestressed bridge decrease as the prestress force increases due to the compression softening. Dallasta (1999) have presented a general formulation for the vibration of beams, prestressed by internal frictionless tendons using kinematic relations of small displacements for the concrete beam. The formulation for the beam does not include the effect of the compression, however they indicated that the natural frequencies decrease as the prestress force increases. Kerr (1976) studied experimentally and analytically the dynamic response of a prestressed beam. It was found that the magnitude of the prestress force for a tendon that passes through the centroid of the beam cross-section has no effect on the dynamic response of the beam. The analytical model uses only a linear formulation for the study of the dynamic response of the prestressed beam. It is limited to straight tendons that pass through the centroid of the beam. Simsek (2007) indicated that the deflections of the beams increase as the prestress force increases because of the compression softening, i.e., the prestress force decreases the beam stiffness and the natural frequencies. Wang (2011) took the additional potential energy of the compression due to the prestressed force into consideration, and concluded that the prestress forces reduce the low transverse natural frequencies of the bridge. Jiang (2010) transferred the eccentric unbonded prestress force to an axial compression and a couple, concluded the prestress force results in compression softening effect. Hamed (2006) rigorously derived the equations of motion for a prestressed beam and its associated boundary and continuity condition using the variational principle of virtual work following Hamilton's principle. The mathematical model is rigorous and general, and is valid for any kind of boundary and continuity conditions as well as any tendon layout. Based on the derived governing equation, it has been mathematically rigorously proven that the magnitude of prestress force does not affect the natural frequencies of bonded or unbonded prestressed beams as opposed to some research works.

There exist three kinds of posttensioning prestress tendon, bonded tendon, unbonded tendon and external tendon. Bonded and unbonded tendons are placed inside the concrete beam. Bonded tendon is integrated with the surrounding concrete. Unbonded tendon can slide along the duct embedded in concrete, however the transverse displacement of unbonded tendon follows the displacement of the beam throughout the entire span, hence there is no loss of eccentricity. External tendon is placed outside the concrete beam, and is in contact only at the deviators and anchorages. Under load, external tendon is free to move relative to the concrete section in between the deviators and/or anchorages, which leads to loss of eccentricity. Hamed (2006) pointed out no effect of prestress force of bonded or unbonded tendon on the natural frequencies. Miyamoto (2000) dealt with the dynamic behavior of prestressed beam strengthened with external tendon. The analytical expression of the natural frequencies includes the external prestress force, the effect of the external prestress force is simply treated as the effect of axial compression. Simsek (2009) also treated the externally applied eccentric prestress force as an axial compression and a couple, the effect of external prestress force is equivalent to the effect of axial compression. The loss of external tendon eccentricity results in a reduction of flexural stiffness, consequentially results in the decrease of natural frequencies. The loss of eccentricity depends on the layout of external tendon, hence the influence of the external force on the natural frequencies is also related to the layout of external tendon, the effect of the external prestress force can not be simply treated as the effect of axial compression. To the best of the author’s knowledge, there is no reference that properly studies the effect of the external prestress force on the natural frequencies.

In the derivation of Miyamoto (2000) analytical expression of the natural frequencies, the key assumption is that the increment of tendon tension is proportional to the vibration displacement at midspan of beam, called proportional assumption hereinafter. In this study, it was indicated that there are some problems in the proportional assumption, in addition, the preceding problem of the effect of the external prestress force treated as the effect of the axial compression. Using the energy method considering the second order effects, the problems in the proportional assumption were solved, it was pointed out that although there is no effect of prestress force of bonded or unbonded tendon on the natural frequencies, the prestress force of external tendon does affect the natural frequencies, and the influence on the natural frequencies depends on the layout of the external tendon. Based on the new assumptions, the natural frequencies of the beam were obtained by the energy method considering the second order effects. Comparing with the method with proportional assumption, the energy method considering the second order effects gives more accurate calculation results which conform well to FEM results.

2. Natural frequency equation

In author’s view, there exists some problems in Miyamoto (2000) natural frequency formula, hence the derivation procedure is rewritten briefly as follow:

For free vibration, the governing partial differential equation is

$$EI\frac{\partial^4 y}{\partial x^4}+\frac{\partial^2 (P_{\mathrm{tc}}y)}{\partial x^2}-\frac{\partial^2 \Delta M_{\mathrm{p}}}{\partial x^2}+m\frac{\partial^2 y}{\partial t^2}=0 \tag{2}$$

where y is vibration displacement; $P_{\mathrm{tc}}=P_{\mathrm{tc}}^0+\Delta P_{\mathrm{tc}}$, P_{tc}, P_{tc}^0 and ΔP_{tc} are the horizontal components of prestress force P_t, initial prestress force P_t^0 and prestress force increment ΔP_t; ΔM_p

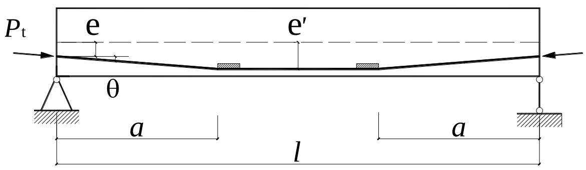

Fig. 1 Analysis model of vibration system

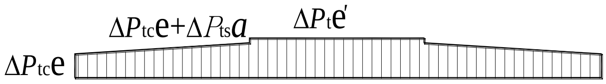

Fig. 2 Bending moment diagram ΔM_p

is the bending moment due to ΔP_t, ΔP_{tx} is the vertical component of ΔP_t, e and e' are the eccentricities, a is the position of deviators as shown in Fig. 1, the friction between the tendon and deviator is neglected. The effect of the prestress force and the effect of external tendon are represented by the terms $P_{tc}y$ and ΔM_p respectively in Eq. (2). The term $P_{tc}y$ implies the effect of the prestress force P_{tc} is treated as the effect of the axial compression, and the prestress force decreases the natural frequencies, which is not consistent with the Hamed (2006) conclusions.

To solve Eq. (2), two relationships must be established, the relationship between $P_{tc}y$ and y, the relationship between ΔM_p and y. Miyamoto (2000) neglected the term $\Delta P_{tc}y$ in $P_{tc}y$, hence $P_{\mathrm{tc}}y \approx P_{\mathrm{tc}}^{0}y$ since the maximum value $y_{\max}<<e$, therefore $\Delta P_{tc}y<<\Delta P_{tc}e$ ($\Delta P_{tc}e$ is included in ΔM_p as shown in Fig. 2). In author's view, $\Delta P_{tc}y$ is negligibly small, not due to $y_{\max}<<e$, but due to the fact that ΔP_{tc} is proportional to y where y can be infinitesimal, $\Delta P_{tc}y$ becomes a high order infinitesimal, hence $\Delta P_{tc}y$ can be neglected in comparison with infinitesimal $P_{\mathrm{tc}}^{0}y$. Acknowledging that in some beams, where e=0, the $y_{\max}<<e$ is not a valid assumption. Therefore $P_{tc}y$ can be replaced by $P_{\mathrm{tc}}^{0}y$, then the relationship between $P_{tc}y$ and y is established. Miyamoto (2000) simplified ΔM_p as a uniform bending moment

$$\Delta M_{\mathrm{p}} = \Delta P_{\mathrm{t}}(e\cos\theta + a\sin\theta) \tag{3}$$

Miyamoto (2000) assumed that ΔP_t is proportional to midspan displacement $y(0.5l)$, combining the proportional assumption and Eq. (3) gives

$$\Delta P_{\mathrm{t}} = \eta y(0.5l) \tag{4}$$

$$\Delta M_{\mathrm{p}} = \eta(e\cos\theta + a\sin\theta)y(0.5) \tag{5}$$

where $\eta = \dfrac{24EI(e\cos\theta + a\sin\theta)}{l^2(\mu + 4\lambda\cos\theta)}$, $\mu = (4\cos\theta - 3)(e\cos\theta + a\sin\theta)^2$, $\lambda = I/A + \dfrac{EIl_{\mathrm{t}}}{E_{\mathrm{t}}A_{\mathrm{t}}l}$, A is area of the beam cross section, E_t, A_t and l_t are the elastic modulus, cross section area and length of tendon. Substituting Eq. (5) into Eq. (2) gives

$$EI\frac{\partial^4 y}{\partial x^4} + P_{\mathrm{tc}}^{0}\frac{\partial^2 y}{\partial x^2} - \eta(e\cos\theta + a\sin\theta)\frac{\partial^2 y(0.5l)}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} = 0 \tag{6}$$

$y(0.5l)$ in Eqs. (4), (5) and (6) is replaced by y in Miyamoto (2000) equations. Due to this replacement, Eq. (6) can be solved for natural frequencies

$$\omega_i = \frac{i\pi}{l}\sqrt{\frac{1}{m}\left[EI\left(\frac{i\pi}{l}\right)^2 - P_{\mathrm{tc}}^0 + \eta(e\cos\theta + a\sin\theta)\right]}, i = 1,2,3,... \tag{7}$$

2. Some problems in proportional assumption

In author's view, there are two problems in the proportional assumption between ΔP_t and $y(0.5l)$. The first problem is: the proportional assumption is rational for the first vibration mode, but for the antisymmetric (even number) vibration modes, $y(0.5l)=0$, the proportional assumption is not valid. The second problem is: in order to solve Eq. (6) conveniently, $y(0.5l)$ in Eqs. (4), (5) and (6) is replaced by $y(x)$, that is, ΔP_t is not proportional to $y(0.5l)$, but is proportional to $y(x)$. Such replacement is wrong because the prestress force increment ΔP_t is a constant along the whole tendon under the frictionless assumption between the tendon and deviator, but the value $y(x)$ varies along x, therefore, the proportional assumption between ΔP_t and function $y(x)$ is wrong. In addition, the effect of the initial prestress force P_{tc}^0 in Eq. (7) is simply treated as the effect of axial compression, which is not consistent with the Hamed (2006) conclusion.

3. Energy method to analyze the natural frequencies

The energy method considering the second order effects is adopted to solve the problems existed in Miyamoto (2000) solution, and to find the natural frequencies of the beam shown in Fig. 3. The assumptions in the energy method are as follows: 1) the ith vibration mode $y_i=A_i\sin(i\pi x/l)\sin\omega_i t$, i=1,2,3…; 2) the beam is straight after applying external prestress force and before vibrating; 3) the axial deformation of the beam due to axial force is neglected.

The velocity of the beam is $\dot{y}_i=A_i\sin(i\pi x/l)\omega_i\cos\omega_i t$, i=1,2,3…, when $\dot{y}_i$ reaches the maximum or $y_i=0$, the kinetic energy of the beam is

$$K_{\mathrm{b0}} = \frac{1}{2}\int_0^l m\dot{y}_i^2 dx = \frac{1}{4}A_i^2\omega_i^2 lm \tag{8}$$

The strain energy of the beam is

$$U_{\mathrm{b0}} = \frac{1}{2EI}\int_0^l \left(P_{\mathrm{t}}^0 m_{\mathrm{t}}\right)^2 dx \tag{9}$$

where $P_t^0 m_t$ is the bending moment due to the initial prestress force P_t^0, m_t is the bending moment diagram due to the unit force of external tendon shown in Fig. 4, $m_{ij}=\cos\theta_i e_j$, angle θ_i and eccentricity e_j of external tendon are shown in Fig. 3. The strain energy of external tendon is

$$U_{\mathrm{t0}} = \frac{\left(P_{\mathrm{t}}^0\right)^2 l_{\mathrm{t}}}{2E_{\mathrm{t}}A_{\mathrm{t}}} \tag{10}$$

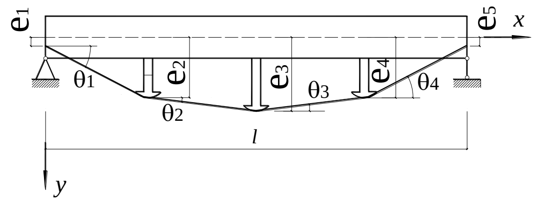

Fig. 3 Externally prestressed simply supported beam

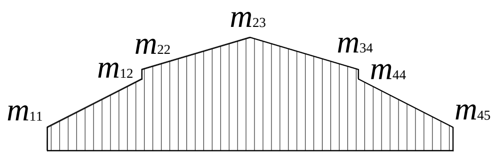

Fig. 4 Bending moment diagram due to the unit force of external tendon

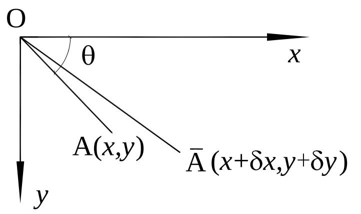

Fig. 5 Deformation of line OA

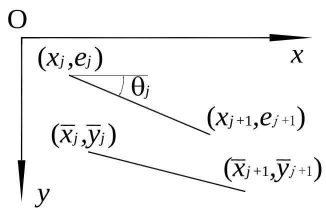

Fig. 6 Deformation of external tendon segment

When y_i reaches the maximum or $\dot{y}_i$=0, the bending moment of beam is: $M_{b1}=-EIy''_i-P_t^{o}m_t$. The strain energy of the beam is

$$U_{\mathrm{b1}}=\frac{1}{2EI}\int_0^l M_{\mathrm{b1}}^2 dx=\frac{EI}{4}A_i^2\left(\frac{i\pi}{l}\right)^4 l+\int_0^l y_i'' P_{\mathrm{t}}^0 m_{\mathrm{t}} dx+U_{\mathrm{b0}} \tag{11}$$

As for calculating the deformation of the external tendon, the deformation of line OA in Fig. 5 is investigated firstly. The deformations of line OA in x and y axes are δx and δy, taking the terms with the second differentiation into consideration, the deformation of line OA is

$$\mathrm{O\overline{A}}-\mathrm{OA}=\sin\theta\delta y+\cos\theta\delta x+\frac{(\sin\theta)^2}{2l_{\mathrm{A}}}(\delta x)^2+\frac{(\cos\theta)^2}{2l_{\mathrm{A}}}(\delta y)^2-\frac{\sin\theta\cos\theta}{l_{\mathrm{A}}}\delta x\delta y \tag{12}$$

where θ and l_A are angle and length of line OA. It is indispensable to consider the second differentiation terms in analyzing the compression softening caused by external prestress force. There are n contact points between the beam and external tendon, the first and the last contact points locate at two ends of the beam, the others are at the deviators. In the jth segment of external tendon as shown in Fig. 6, the coordinates of the jth and (j+1)th contact points are (x_j,e_j) and (x_{j+1},e_{j+1}) before vibrating. Taking into account the horizontal displacement of the point at beam axis caused by the bending of the beam axis, the coordinates of the contact points $(\bar{x}_j,\bar{y}_j)$ and $(\bar{x}_{j+1},\bar{y}_{j+1})$ when $\dot{y}_i$=0 are:

$\bar{x}_j = x_j - \int_0^{x_j} \frac{1}{2}(y_i')^2 dx - e_j y_i'(x_j)$, $\bar{x}_{j+1} = x_{j+1} - \int_0^{x_{j+1}} \frac{1}{2}(y_i')^2 dx - e_{j+1} y_i'(x_{j+1})$

$\bar{y}_j = e_j + y_i(x_j)$, $\bar{y}_{j+1} = e_{j+1} + y_i(x_{j+1})$

The deformation of the jth segment in x and y axes δx_j and δy_j are:

$\delta x_j = (\bar{x}_{j+1} - \bar{x}_j) - (x_{j+1} - x_j) = b_j A_i^2 + c_j A_i$, $\delta y_j = (\bar{y}_{j+1} - \bar{y}_j) - (e_{j+1} - e_j) = d_j A_i$,

$b_j = -\frac{i\pi}{4l}\left[\frac{1}{2}\sin\frac{2i\pi x_{j+1}}{l} - \frac{1}{2}\sin\frac{2i\pi x_j}{l} + \frac{i\pi}{l}(x_{j+1} - x_j)\right]$, $c_j = \frac{i\pi}{l}\left(e_j \cos\frac{i\pi x_j}{l} - e_{j+1}\cos\frac{i\pi x_{j+1}}{l}\right)$,

$d_j = \sin\frac{i\pi x_{j+1}}{l} - \sin\frac{i\pi x_j}{l}$

Substituting δx_j and δy_j into Eq. (12), and neglecting the terms involving high order infinitesimal A_i^3 and A_i^4 (A_i can be infinitesimal), give the deformation of the jth segment of external tendon

$$\Delta l_j = \alpha_j A_i + \beta_j A_i^2 \tag{13}$$

where $\alpha_j = d_j \sin\theta_j + c_j \cos\theta_j$, $\beta_j = b_j \cos\theta_j + c_j^2 \frac{(\sin\theta_j)^2}{2l_j} + d_j^2 \frac{(\cos\theta_j)^2}{2l_j} - c_j d_j \frac{\sin\theta_j \cos\theta_j}{l_j}$, l_j and θ_j are the length and angle of the jth segment of external tendon respectively. The whole deformation of the external tendon is

$$\Delta l_{\mathrm{t}} = A_i \psi_i + A_i^2 \zeta_i \tag{14}$$

where $\psi_i = \sum_{j=1}^{n-1} \alpha_j$, $\zeta_i = \sum_{j=1}^{n-1} \beta_j$. $A_i\psi_i$ is the first order deformation caused by the ith vibration mode, i.e., $A_i \psi_i = -\int_0^l m_{\mathrm{t}} y_i'' dx$; $A_i^2 \zeta_i$ is the second order deformation, which is indispensable to analyze the compression softening of the prestress force. The axial force of external tendon is

$$P_{\mathrm{t}1} = P_{\mathrm{t}}^0 + E_{\mathrm{t}} A_{\mathrm{t}} (A_i \psi_i + A_i^2 \zeta_i)/l_{\mathrm{t}} \tag{15}$$

The strain energy of external tendon is $U_{\mathrm{t}1} = \frac{P_{\mathrm{t}1}^2 l_{\mathrm{t}}}{2E_{\mathrm{t}} A_{\mathrm{t}}}$, neglecting the terms involving high order infinitesimal A_i^3 and A_i^4 gives

$$U_{\mathrm{t}1} = U_{\mathrm{t}0} + P_{\mathrm{t}}^0 \left(A_i \psi_i + A_i^2 \zeta_i\right) + \frac{E_{\mathrm{t}} A_{\mathrm{t}}}{2l_{\mathrm{t}}} (A_i \psi_i)^2 \tag{16}$$

Substituting each term into the equation of energy method: $K_{b0}+U_{t0}+U_{b0}=U_{t1}+U_{b1}$, and deducting the term $\int_0^l y_i'' P_{\mathrm{t}}^0 m_{\mathrm{t}} dx$ in U_{b1} and the term $P_{\mathrm{t}}^0 A_i \psi_i = -P_{\mathrm{t}}^0 \int_0^l m_{\mathrm{t}} y_i'' dx$ in U_{t1} mutually, give

$$\frac{1}{4} A_i^2 \omega_i^2 l m = \frac{EI}{4} A_i^2 \left(\frac{i\pi}{l}\right)^4 l + P_{\mathrm{t}}^0 A_i^2 \zeta_i + \frac{E_{\mathrm{t}} A_{\mathrm{t}}}{2l_{\mathrm{t}}} A_i^2 \psi_i^2 \tag{17}$$

Table 1 Relationship between influence coefficient $C_{\rm pi}$ and the number of contact points n

n	Straight tendon				Parabolic tendon			
	$C_{\rm p1}$	$C_{\rm p2}$	$C_{\rm p3}$	$C_{\rm p4}$	$C_{\rm p1}$	$C_{\rm p2}$	$C_{\rm p3}$	$C_{\rm p4}$
2	1	1	1	1	1	1	1	1
3	0.189	1	0.91	1	0.203	0.995	0.904	0.995
4	0.088	0.316	1	0.829	0.09	0.338	0.991	0.812
5	0.05	0.189	0.385	1	0.049	0.191	0.411	0.992
7	0.023	0.088	0.189	0.316	0.02	0.084	0.189	0.322
9	0.013	0.05	0.11	0.189	0.009	0.045	0.106	0.187
11	0.008	0.032	0.072	0.125	0.004	0.026	0.066	0.12

The natural frequency is obtained

$$\omega_i = \frac{i\pi}{l}\sqrt{\frac{1}{m}\left[EI\left(\frac{i\pi}{l}\right)^2 - C_{\rm p\it i}P_{\rm t}^0 + \frac{2\psi_i^2 E_{\rm t}A_{\rm t}l}{(i\pi)^2 l_{\rm t}}\right]}, i = 1,2,3,... \tag{18}$$

where the influence coefficient of the external prestress force C_{pi}=−4$l\zeta i/(i\pi)^2$. In comparison with the axial force N in Eq. (1), the external prestress force does affect the natural frequencies of beam, but its effect depends on the coefficient $C_{\rm p\it i}$ which is related to the layout of external tendon. Relationship between influence coefficient $C_{\rm p\it i}$ and the number of contact points n is listed in Table 1, the deviators are distributed uniformly along the span, the layout of tendon is symmetric, the beam length l=16 m, the eccentricities at two ends are e_1=e_n=0.2 m. For the parabolic tendon, all contact points locate at a parabola (see Fig. 3), the mid-span maximum eccentricity of the parabola is 1m. Table 1 indicates that for the beam with only 2 contact points, i.e., without deviator, the influence coefficient is 1, the effect of external prestress force is the same effect of axial compression, because the loss of eccentricity is maximum. As the number of contact points increases, the loss of eccentricity moderates, the influence coefficient decreases. In general, as the number of vibration mode increases, the influence coefficient increases because the number of contact points in every half-wave sinusoid decreases, just like the number of contact points of the whole beam decreases in the first vibration mode. The influence coefficient of straight tendon is close to the coefficient of parabolic tendon, which indicates the coefficient mainly depends on the number of contact points. For the beam with large number of contact points, there is almost no loss of eccentricity, the external tendon is closely a unbonded tendon, hence the influence coefficient decreases from 1 to almost 0, which is consistent with the conclusion pointed out by Hamed (2006), i.e., no effect of prestress force of bonded or unbonded tendon on the natural frequencies.

3. Numerical examples

For the externally prestressed simply supported beam shown in Fig. 3, l=16 m, width 0.4 m and height 0.8 m of rectangular cross section, E=32.5 GPa, the mass per unit length of the beam m=6 t/m; E_t=200 GPa, the deviators are distributed uniformly along the span, the initial prestress σ_t^0=1000 MPa, the layout of tendon is symmetric, e_1=0.2 m. Table 2 lists the 1st, 2nd and 3rd natural frequencies of externally prestressed beams. In 8 calculation beams, there are two

Table 2 The 1st, 2nd and 3rd natural frequencies of externally prestressed beam

Beam	e_2 (m)	e_3 (m)	A_t (mm^2)	ω_1(rad/s)			ω_2(rad/s)			ω_3(rad/s)		
				M①	M②	M③	M①	M②	M③	M①	M②	M③
1	0.8		1668	12.98	12.45	12.94	46.74	47.63	46.74	105.1	106.2	105.1
2	0.8	1.0	1668	13.45		13.39	46.8		46.8	105.5		105.5
3	1.1		1668	13.96	13.27	13.89	46.73	48.51	46.73	105.1	107.1	105.1
4	1.1	1.4	1668	14.8		14.7	46.8		46.8	105.4		105.4
5	0.8		2502	13.57	12.71	13.48	46.66	47.91	46.66	105	106.5	105
6	0.8	1.0	2502	14.23		14.11	46.76		46.76	105.4		105.4
7	1.1		2502	14.96	13.78	14.81	46.65	49.08	46.65	105	107.7	105
8	1.1	1.4	2502	16.11		15.92	46.76		46.76	105.4		105.4

Note: M① stands for the method of this paper, M② for the method of Miyamoto (2000); M③ for FEM

Table 3 The 1st, 2nd and 3rd natural frequencies of M① without initial prestress force

Beam	0	1	2	3	4	5	6	7	8
ω_1(rad/s)	11.72	13.02	13.46	14	14.81	13.63	14.26	15	16.14
ω_2(rad/s)	46.89	46.89	46.89	46.89	46.89	46.89	46.89	46.89	46.89
ω_3(rad/s)	105.5	105.6	105.6	105.6	105.7	105.7	105.7	105.6	105.7

Note: Beam 0 stands for the beam without external tendon

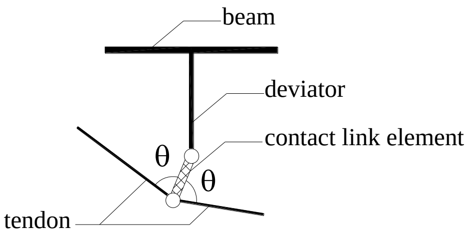

Fig. 7 Contact link element

eccentricities e_2=0.8, 1.1 m, two areas of the tendon A_t=1668,2502 mm^2 and two layouts of external tendon, trapezoid without mid-span deviator (beam 1,3,5,7, see Fig. 1) and approximate parabola (beam 2,4,6,8, $e2<e3$, see Fig. 3) respectively. M① stands for the method in this paper, M② for the method of Miyamoto (2000); M③ for FEM. Miyamoto (2000) only presented the analytical solution of trapezoidal external tendon, hence Table 2 only lists the results of trapezoidal external tendon. Generally speaking, the validity of theoretical results would be tested by experimental results. However, in the field and laboratory experiments, Saiidi (1994) found that the natural frequencies increases as the prestress force increases, which is not consistent with the theoretical conclusion. For this disparity, they opined that the prestress force causes closure of micro-cracks in the concrete, which increases the flexural stiffness and natural frequencies. In author’s view, to test the validity of theoretical results by FEM results, is also acceptable. Therefore FEM analysis was carried out to test the validity of analytical formula.

There exists the frictionless slide between the external tendon and deviator, thus the increment of prestress force in external tendon depends on the deformation of whole beam and is assumed uniform at all section. The concise method to overcome this additional difficulty is to add a contact link element at angle bisector between external tendon and deviator as shown in Fig. 7. The link element is subjected to only axial force, thus is chosen to simulate the external tendon. The contact link element can simulate the frictionless slide between the external tendon and deviator, and the position assures the uniform increment of prestress force of external tendon, the length of contact link element is 1cm, the axial stiffness is E_tA_t, the same as the external tendon axial stiffness. The deviator is simulated by the rigid beam element. The mass of the beam element is simplified by assuming that the distributed mass of the element can be lumped as point masses at two ends.

The equation to calculate the natural frequencies is

$$\{\varphi_i\} = \omega_i^2[\delta][M]\{\varphi_i\} \tag{19}$$

where $\{\varphi_i\}$ is the ith natural mode shape, $[M]$ is the diagonal lumped mass matrix, $[\delta]$ is the vertical flexibility matrix. The coefficient δ_{ij} of $[\delta]$ is the vertical displacement in ith lumped mass due to the unit vertical force applied in jth lumped mass, which is computed by standard procedures of structural analysis to obtain the flexibility matrix. The influence coefficient of the external prestress force $C_{\mathrm{p}i}$ in Eq. (18) is taken into consideration, hence the stiffness matrix of beam element to calculate ω_i is

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} - \frac{C_{\mathrm{p}i}P_{\mathrm{t}}^0}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{20}$$

Various parameters are involved in coefficient δ_{ij} of $[\delta]$, such as EI and l of beam, E_tA_t, l_t and the layout of external tendon, the frictionless slide of contact link element between the external tendon and deviator. Obtaining $[\delta]$ by ordinary structural analysis, the natural frequencies ω_i and the natural mode shape $\{\varphi_i\}$ are determined by Matlab software. The beam is discretized into 16 beam elements with the same length for beam 2, 4, 6 8 in Table 2, and is discretized into 15 beam elements with the same length for the beam 1, 3, 5, 7. Before proceeding further, it is appropriate to ascertain the correctness of the preceding procedure for finding ω_1~ω_3. For this purpose, ω_1~ω_3 of the simply supported beam without external tendon subjected to the external axial compression 0 or 2502kN, are obtained by structural analysis and Matlab software. The results are compared with the Eq. (1) analytical solutions. When the axial compression is 2502kN, the error of ω_3 is maximum, that is, FEM value ω_3=104.8 rad/s, analytical value ω_3=104.808 rad/s. This ascertains the validity of above FEM procedure. The lumped mass method has high accuracy in calculating low order natural frequencies, there are 15 lumped masses for the beam 2, 4, 6 8 or 14 lumped masses for the beam 1, 3, 5, 7, ω_1~ω_{15} or ω_1~ω_{14} can be found, so the high accuracy of ω_1~ω_3 is expected. Hence, the results of M③ in Table 2 can be used to validate the accuracy of the results of M① and M②.

In Table 2, ω_1 of M① is greater than ω_1 of M②, which indicates that M② underestimates the effect of external tendon on ω_1. In M②, since $y(0.5l)$ is replaced by $y(x)$, ΔP_t which originally is proportional to the maximum vibration displacement $y(0.5l)$, now is wrongly reduced to be proportional to $y(x)$, $y(x) \leq y(0.5l)$, therefore, the effect of external tendon on ω_1 is underestimated. In M①, ψ_i representing the effect of external tendon is naught for even number vibration modes, and is quite small for the 3rd, 5th,….odd number vibration modes except the 1st vibration mode, because the even number vibration modes are antisymmetric, which do not cause deformation of external tendon with symmetric layout, hence ψ_i=0, i=2,4,…; the odd number vibration modes are symmetric, the deformations of external tendon caused by positive and negative vibration displacements deduct mutually, the whole deformation is quite small, hence $\psi_i \approx 0$, i=3,5,…. In M①, the effect of external tendon mainly influences ω_1, slightly influences ω_3 and does not influence ω_2. However in Eq. (7) of M②, the value of $\eta(e\cos\theta+a\sin\theta)$ representing the effect of external tendon is the same for each natural frequency, thus the effect of external tendon on ω_2 and ω_3 is overestimated, hence ω_2 and ω_3 of M② is greater than ω_2 and ω_3 of M①. ω_1~ω_3 of M① are almost equal to ω_1~ω_3 of M③, which indicates the high accuracy and rational assumptions of M①. In practical prestressed beam, the value of P_t^0 is greatly less than the term $EI(i\pi/l)^2$ in Eq. (18), the influence of P_t^0 is quite limited, especially for high order vibration mode, the influence coefficient C_{p1} is also greatly less than 1 for the beam with 2 or more deviators. Table 3 lists the 1st, 2nd and 3rd natural frequencies of M① without initial prestress force, i.e., P_t^0=0, the beam 1~8 frequencies neglecting the effect of the prestress force is very close to the frequencies in Table 2 considering the effect of the prestress force, which indicates the effect of the external prestress force can be neglected for beam 1~8, i.e., for the beam with 2 or more deviators due to slight loss of external tendon eccentricity. A_t=0 for beam 0 in Table 3, i.e., neglecting the effect of external tendon, ω_1 of beam 0 is significantly less than ω_1 of beam 1~8 in Table 2, which indicates the external tendon noticeably influences ω_1. As the eccentricity and area of tendon increases, ω_1 in Table 2 increases. However ω_2, ω_3 of beam 0 are very close to the frequencies of beam 1~8 in Table 2, which indicates that except ω_1, the effect of external tendon on other frequencies can be neglected.

6. Conclusions

In this study, the effect of external prestress force and tendon on the frequencies of a simply supported beam was investigated. At first, the conclusion mathematically rigorously proven by Hamed (2006) was introduced, i.e., no effect of both bonded and unbounded prestress force on the natural frequencies of beams as opposed to some research works. Then, Miyamoto (2000) analytical solution of externally prestressed simply supported beam was introduced and discussed, 3 problems were found. The energy method considering the second order effects was adopted to solve the problems. The frequency results by Miyamoto (2000) analytical solution, the energy method in this study and FEM were compared. The results obtained from this study are summarized below:

1) There exist 3 problems in Miyamoto (2000) analytical solution, 2 problems in the proportional assumption that the increment of tendon tension is proportional to the vibration displacement at midspan of beam, 1 problem in the effect of the external prestress force which is treated as the effect of axial compression.

2) The influence coefficient of the external prestress force, i.e., the compression softening effect caused by external prestress force depends on the layout of external tendon, mainly depends on the number of contact points. As the number of contact points increases from minimum value 2 to a large number, the influence coefficient decreases from 1 to almost 0, which is consistent with the conclusion pointed out by .Hamed (2006). The loss of external tendon eccentricity results in the compression softening effect of external prestress force. The compression softening effect of external prestress force can be neglected for the beam with 2 or more deviators due to slight loss of

external tendon eccentricity. Therefore, Miyamoto (2000) treatment of the effect of external prestress force as the effect of axial compression without considering the number of contact points overestimates the compression softening effect for the beam with deviators.

3) The comparison among the results obtained by the energy method in this study, FEM and Miyamoto (2000) solution validates the accuracy of the energy method. The comparison indicates that the Miyamoto (2000) proportional assumption underestimates the effect of external tendon on the 1st natural frequency, and overestimates the effect on other natural frequencies.

4) The external tendon noticeably influences the first natural frequency. As the eccentricity and area of tendon increase, the first natural frequency increases. The external tendon almost does not influence the other frequencies except the first natural frequency. Therefore, the effect of external tendon must be taken into consideration only in calculating the first natural frequency, and can be neglected in calculating other frequencies.

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