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Buckling of an elastic plate due to surface-attached thin films with intrinsic stresses

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Abstract. We analyze the buckling of a thin elastic plate due to intrinsic stresses in thin films attached to the surfaces of the plate. In the case of cylindrical buckling, it is shown that for a plate with clamped edges compressive intrinsic film stresses can cause flexural buckling of the plate, while tensile intrinsic film stresses cannot. For a plate with free edges, film intrinsic stresses, compressive or tensile, cannot cause buckling.

Keywords: cylindrical buckling; elastic plate; thin film; intrinsic stress

1. Introduction

It is well known that the material near the surface of a solid may behave differently from the bulk material by having its own effective material properties (Miller and Shenoy 2000, Shenoy 2005, Tian and Rajapakse 2007, Guo and Zhao 2007, Mi *et al*. 2008, Dong and Pan 2011). Manufacturing processes may also affect the surface of a body, resulting in surface stresses, etc. When the body is large, its surface effects can usually be neglected. However, for small structures, the surface effects become more pronounced and have been extensively discussed (Streitz *et al*. 1994, Liang *et al*. 2002, Cuenot *et al*. 2004, Yang 2004, Villain *et al*. 2004). A simple and direct way of studying surface effects is to treat the material near the surface of a body as a separate phase with its own physical properties. A general nonlinear theory for a material surface on a body was formulated by Gurtin and Murdoch (1975) which includes the effects of surface stresses.

Specifically, during the deposition process of a thin film on a substrate, an intrinsic stress is generated in the film material, this film intrinsic stress will induces a state of stress in the plate called the residual stress in order that the entire structure (film plus plate) remain in equilibrium. To investigate the influence of combined intrinsic plus residual stress state on the resonant frequency of an electroded quartz resonator, Tiersten *et al*. (1981) derived the equations for small dynamic fields superposed on the static bias from the rotationally invariant equations of nonlinear elasticity. Actually, the stress equations of motion for the curved surface are equivalent to those obtained from the surface elasticity of Gurtin and Murdoch (1975). However, the latter theory does

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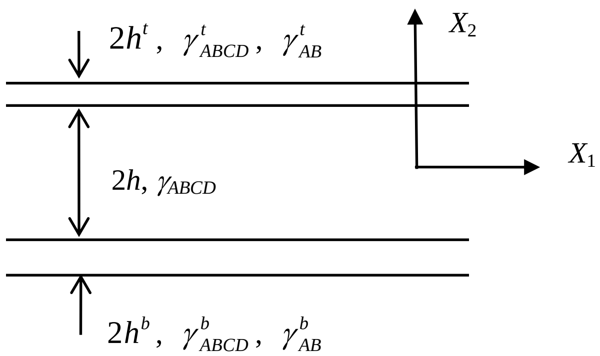


Fig. 1 An elastic plate with two thin films and their thickness, stiffness, and intrinsic stress

not consider the residual stress in the substrate induced by intrinsic stress, which is of the utmost importance for frequency calculation of quartz plate with electrode included. By means of a Taylor expansion, the two-dimensional equations for an elastic plate carrying thin films with intrinsic stresses were derived by Tiersten *et al*. (1981) by joining the extensional equations for the thin films to the extensional and flexural equations of elastic plates.

Recently, due to the development of nano-scale structures, there have been growing interest and new derivations (Kornev and Srolovitz 2004, Lu *et al*. 2006, Wang and Zhao 2009) of two-dimensional equations for elastic plates carrying surface films with intrinsic stresses. It was predicted by Kornev and Srolovitz (2004) and Wang and Zhao (2009) that the intrinsic stresses in the surface films on an (simply-supported or free-standing) elastic plate can cause buckling of the plate.

Motivated by existing works (Kornev and Srolovitz 2004, Wang and Zhao 2009), in this paper we examine the buckling of an elastic plate due to intrinsic stresses in surface films using the existing equations from Tiersten *et al*. (1981) (Hereafter, referred to Tiersten's equations for brevity). Tiersten's equations were derived under the assumption that the intrinsic stresses in the surface films are finite, but all other stresses induced by the film intrinsic stresses are infinitesimal. Therefore, in the equation for flexure in Tiersten *et al*. (1981), while the contribution to flexure from the film intrinsic stresses is considered, the contribution to flexure from the induced or residual stresses in the elastic plate is ignored. As a consequence, the equation for flexure in Tiersten *et al*. (1981) is not accurate for studying buckling from intrinsic film stresses in general, but it is valid when a plate has clamped edges with the film stresses balanced by the reactions at the edges without inducing any residual stresses in the elastic plate.

2. Tiersten's equations

The notation in Tiersten *et al*. (1981) is complicated for more general applications than buckling. For our purpose we simplify the notation by removing certain superscripts and renaming the deflection function, bending and twisting moments, and shear forces, etc., using more common notations. Consider an elastic plate with two surface films as shown in Fig. 1. The two films were treated as different in Tiersten *et al*. (1981) but will soon be assumed identical in this paper. The plate normal is along the X_2 direction which will be called the vertical direction for convenience. X_1 and X_3 are the in-plane coordinates. The plate thickness is $2h$. Its stiffness tensor after relaxing the plate thickness stresses is γ_{ABCD} (the plane-stress stiffness tensor), where A, B, C and D are the in-plane tensor indices that assume values of 1 and 3 only but skip 2. The thickness and stiffness of the top surface film are $2h^t$ and γ^t_{ABCD}. The intrinsic stress in this film is γ^t_{AB} which may be anisotropic. The bottom surface film is similar, with a superscript b for its parameters. There are no other loads on the structure except the film intrinsic stresses.

Let the deflection function of the middle plane of the plate be $w(X_1, X_3)$. From Eq. (4.42) of of Tiersten *et al*. (1981), we have the flexural equation of equilibrium

$$M_{AB,AB} + q = 0 \tag{1}$$

where M_{AB} are the bending and twisting moments given by the following plate constitutive relation or the moment-curvature relation (Eq. (4.49) of Tiersten *et al*. 1981)

$$\begin{aligned} M_{AB} = &-\frac{2}{3}h^3\left[\gamma_{ABCD} + 3\left(\frac{h^t}{h}\gamma^t_{ABCD} + \frac{h^b}{h}\gamma^b_{ABCD}\right)\right]w_{,CD} \\ &+ 2h\left(h^t\gamma^t_{AB} - h^b\gamma^b_{AB}\right) - 2h^2\left(h^t\gamma^t_{AC} + h^b\gamma^b_{AC}\right)w_{,BC}. \end{aligned} \tag{2}$$

In Eq. (1), q is the effective flexural or vertical load due to the finite film intrinsic stresses when the plate is deflected infinitesimally, which is given by Eq. (4.50) of Tiersten *et al*. (1981) plus an in-plane divergence operation as implied by Eqs. (4.42), (4.46) and (4.47) of Tiersten *et al*. (1981) as

$$q = 2\left(h^t\gamma^t_{AC} + h^b\gamma^b_{AC}\right)w_{,CA} \tag{3}$$

(Note that according to the notation of Tiersten *et al*. (1981) we have $q = k^{t1}_{A2,A} + k^{b1}_{A2,A}$). The total shear resultants or more precisely the total vertical forces over the plate cross sections including the contributions from the film intrinsic stresses when the plate is deflected are given by Eqs. (4.46), (4.47) and (4.50) of Tiersten *et al*. (1981) as

$$Q_{A2} = M_{BA,B} + 2\left(h^t\gamma^t_{AC} + h^b\gamma^b_{AC}\right)w_{,C} \tag{4}$$

with which Eq. (1) can be written as

$$Q_{A2,A} = 0 \tag{5}$$

3. Cylindrical deformation of a plate with identical films

Consider the case of cylindrical deformation independent of X_3. We also assume that the top and bottom films are identical. Both the plate and the films are of isotropic materials. Let the Young's modulus and the Poisson's ratio of the plate be E and v, and those of the films be E^t and v^t. Then the equations in the previous section reduce to

$$\gamma_{1111} = \frac{E}{1-\nu^2} = \bar{E}, h'' = h', \quad \gamma_{11}^t = \gamma_{11}^b, \quad \gamma_{1111}^t = \gamma_{1111}^b = \frac{E^t}{1-(\nu^t)^2} = \bar{E}^t, \tag{6}$$

$$M_{11} = -\frac{2}{3}h^3\left[\bar{E} + 6\frac{h^t}{h}\bar{E}^t\right]w_{,11} - 4h^2h^t\gamma^t w_{,11} \tag{7}$$

$$q = 4h^t\gamma^t w_{,11} \tag{8}$$

$$Q_{12} = -\frac{2}{3}h^3\left[\bar{E} + 6\frac{h^t}{h}\bar{E}^t\right]w_{,111} - 4h^2h^t\gamma^t w_{,111} + 4h^t\gamma^t w_{,1} \tag{9}$$

Substitution of (9) into (5) gives the equation for $w(X_1)$

$$\begin{aligned} Q_{12,1} &= M_{11,11} + q \\ &= -\frac{2}{3}h^3\left[\bar{E} + 6\frac{h^t}{h}\bar{E}^t\right]w_{,1111} - 4h^2h^t\gamma^t w_{,1111} + 4h^t\gamma^t w_{,11} \\ &= -(D + 4h^2h^t\gamma^t)w_{,1111} + 4h^t\gamma^t w_{,11} = 0, \end{aligned} \tag{10}$$

where

$$D = \frac{2\bar{E}}{3}h^3 + 4\bar{E}^t h^t h^2 \tag{11}$$

is the bending stiffness of the elastic plate with the films. We note that in Eq. (10) D is modified by the intrinsic stress γ^t in the films.

4. Cylindrical buckling of a plate with clamped edges

As an example consider a clamped plate within $-L<X_1<L$. The general solution to Eq. (10) can be written as

$$w = A_1 \sin\beta X_1 + A_2 \cos\beta X_1 + A_3 X_1 + A_4 \tag{12}$$

where A_i (i=1, 2, 3, 4) are undetermined constants and β must satisfy

$$(D + 4h^2h^t\gamma^t)\beta^2 + 4h^t\gamma^t = 0 \tag{13}$$

Consider symmetric solutions first

$$w = A_2 \cos\beta X_1 + A_4 \tag{14}$$

Eq. (14) needs to satisfy $w(\pm L)=0$ and $w'(\pm L)=0$ which imply that

$$\sin\beta L = 0\,, \qquad \beta L = n\pi\,, \qquad n=1,\,2,\,3,\,\ldots \tag{15}$$

and

$$A_4 / A_2 = -\cos\beta L \tag{16}$$

Similarly, for antisymmetric solutions, we consider

$$w = A_1 \sin\beta X_1 + A_3 X_1 \tag{17}$$

The boundary conditions imply that

$$\tan\beta L = \beta L \tag{18}$$

and

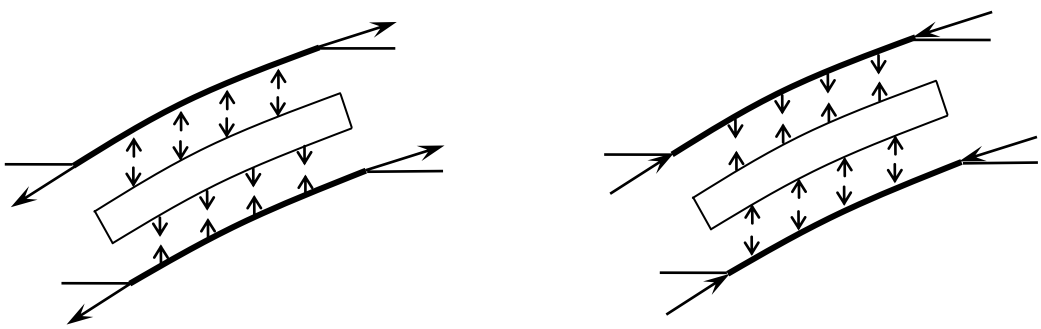
$$A_3 / A_1 = -\beta\cos\beta L \tag{19}$$

Eq. (18) has a series of real and positive roots. The smallest one is $\beta L \cong 4.493$ which is larger than the smallest root in Eq. (15). Therefore the lowest buckling mode is symmetric, associated with $\beta L=\pi$ from Eq. (15).

With β determined from (15) or (18), from (13) we can determine the following dimensionless intrinsic stress for buckling

$$\frac{4h^t\gamma^t}{D/h^2} = -\frac{\beta^2h^2}{1+\beta^2h^2} \tag{20}$$

The γ^t in Eq. (20) is negative. This shows that films with compressive intrinsic stresses cause buckling. This can be explained as follows. According to Tiersten *et al*. (1981), the films produce an effective distributed vertical load $k^{t1}_{A2,A} + k^{b1}_{A2,A}$ on the plate through the interactions between the films and the plate. This vertical load is stabilizing when γ^t>0 (see Fig. 2 (a)) and is



(a) (b)

Fig. 2 A differential element showing film intrinsic stresses and interactions betweens the films and the plate (a) Tensile film stresses and stabilizing effects (b) Compressive film stresses and destabilizing effects when γ^t<0 (see Fig. 2(b)). In fact it is more direct to look at the vertical components of the film intrinsic stresses which are stabilizing when γ^t>0 (see Fig. 2 (a)) and are destabilizing when γ^t<0 (see Fig. 2(b)). Basically films with compressive stresses will try to expand and are thus destablizing. Since β is inversely proportional to L, the critical buckling stress determined by Eq. (20) goes to zero for large values of L. Therefore compressive surface film stresses can cause buckling for sufficiently long plates. As a numerical example, consider an aluminum plate with copper films. L=100 mm, h=1 mm, h^t=0.1 mm. For aluminum $E=7.17\times10^{10}$ Pa, and ν=0.33. For copper $E=11.9\times10^{10}$ Pa, and ν=0.326. The lowest compressive intrinsic stress causing buckling is found to be $|\gamma^t|=2.64\times10^8$ Pa.

5. A plate with free edges

In the case of a free plate, there exist tensile or compresses residual stresses in the elastic plate induced by the intrinsic film stresses, as dictated by the free-edge conditions of vanishing total extensional resultant in the plate and the films together, and the equilibrium of the plate or any part of it. These residual stresses were treated as infinitesimal in Tiersten *et al*. (1981) and their contribution to the flexural equation when the plate is deflected infinitesimally was neglected. As a consequence Eq. (3) for q only has the contribution from the film intrinsic stresses, without the contribution from the residual stresses in the plate. If we still use this incomplete q for a free plate, it will lead to the conclusion that a free plate can buckle under compressive film intrinsic stresses. However, a more accurate analysis using a complete q also including the contribution from the residual stresses leads to a different conclusion as explained below.

To analyze the case of a free plate or buckling of a plate carrying thin films with intrinsic stresses in general, Tiersten's flexural equation in Tiersten *et al*. (1981) needs to be generalized to include contributions from not only the film intrinsic stresses but also all the induced stresses or the residual stresses in the elastic plate. Then the effective flexural load q in Eq. (3) takes the following form

$$q = N_{AC} w_{,CA} \tag{21}$$

where N_{AC} are the total in-plane extensional resultants of the plane-stress type including contributions from both the films and the plate. With the new q in Eq. (21), for the one-dimensional case we are considering, $q=N_{11}w_{,11}$ and Eq. (10) becomes

$$-(D+4h^2h^t\gamma^t)w_{,1111} + N_{11}w_{,11} = 0 \tag{22}$$

Since for a free plate the total extensional resultant N_{11} over a cross section including the film intrinsic stresses and the plate residual stresses is zero as required by equilibrium, its contribution to flexure when the plate is deflected infinitesimally is also zero. Therefore, in a free plate, the total effective vertical load q=0. There is no destablizing mechanism and buckling cannot happen.

6. Conclusions

Tiersten's two-dimensional equations for elastic plates carrying thin surface films with finite intrinsic stresses and infinitesimal induced stresses are insufficient for buckling analysis in general. For a clamped plate compressive intrinsic film stresses can cause buckling in a long plate while tensile intrinsic film stresses cannot. For a free plate film intrinsic stresses cannot cause buckling whether they are compressive or tensile.

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References

Cuenot, S., Fretigny, C., Champagne, S.D. and Nysten, B. (2004), "Surface tension effect on the mechanical properties of nanomaterials measured by atomic force microscopy", *Phys. Rev. B*, **69**, 165410.

Dong, C.Y. and Pan, E. (2011), "Boundary element analysis of nanoinhomogeneities of arbitrary shapes with surface and interface effects", *Eng. Anal. Bound. Elem.*, **35**, 996-1002.

Guo, J.G. and Zhao, Y.P. (2007), "The size-dependent bending elastic properties of nanobeams with surface effects", *Nanotechnology*, **18**, 295701.

Gurtin, M.E. and Murdoch, A.I. (1975), "A continuum theory of elastic material surfaces", *Arch. Ration. Mech. An.*, **57**(4), 291-323.

Kornev, K.G. and Srolovitz, D.J. (2004), "Surface stress-driven instability of a free film", *Appl. Phys. Lett.*, **85**(13), 2487-2489.

Mi, C., Jun, S., Kouris, D.A. and Kim, S.Y. (2008), "Atomistic calculations of interface elastic properties in noncoherent metallic bilayers", *Phys. Rev. B*, **77**, 075425

Miller, R.E. and Shenoy, V.B. (2000), "Size-dependent elastic properties of nanosized structural elements", *Nanotechnology*, **11**, 139-147.

Liang, L.H., Li, J.C. and Jiang, Q. (2002), "Size-dependent elastic modulus of Cu and Au thin films", *Solid State Commun.*, **121**, 453-455.

Lu, P., He, L.H., Lee, H.P. and Lu, C. (2006), "Thin plate theory including surface effects", *Int. J. Solids Struct.*, **43**(16), 4631-4647.

Shenoy, V.B. (2005), "Atomistic calculations of elastic properties of metallic fcc crystal surfaces", *Phys. Rev. B*, **71**, 094101.

Streitz, F.H., Cammarata, R.C. and Sieradzki, K. (1994), "Surface-stress effects on elasticproperties. I. Thin metal films", *Phys. Rev. B*, **49**, 10699

Tian, L. and Rajapakse, R.K.N.D. (2007), "Elastic field of an isotropic matrix with a nanoscale elliptical inhomogeneity", *In. J. Solids Struct.*, **44**(22), 7988-8005.

Tiersten, H.F., Sinha, B.K. and Meeker, T.R. (1981), "Intrinsic stress in thin films deposited on anisotropic substrates and its influence on the natural frequencies of piezoelectric resonators", *J. Appl. Phys.*, **52**(9), 5614-5624.

Villain, P., Beauchamp, P. K., Badwi, F., Goudeau, P. and Renault, P.O. (2004), "Atomistic calculation of size effects on elastic coefficients in nanometre-sized tungsten layers and wires", *Scr. Mater.*, **50**(9), 1247-1251.

Wang, Z.Q. and Zhao, Y.P. (2009), "Self-instability and bending behaviors of nano plates", *Acta Mech. Solida Sin.*, **22**(6), 630-643.

Yang, F.Q. (2004), "Size-dependent effective modulus of elastic composite materials: spherical nanocavities at dilute concentrations", *J. Appl. Phys.*, **95**(7), 3516-3520.