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A novel two-dimensional approach to modelling functionally graded beams resting on a soil medium

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Abstract. The functionally graded beam (FGB) is investigated in this study on both dynamic and static loading in case of resting on a soil medium rather than on the usual Winkler-Pasternak elastic foundation. The powerful ABAQUS software was used to model the problem applying finite element method. In the present study, two different soil models are taken into account. In the first model, the soil is assumed to be an elastic plane stress medium. In the second soil model, the Drucker-Prager yield criterion, which is one of the most well-known elastic-perfectly plastic constitutive models, is used for modelling the soil medium. The results are shown to evaluate the effects of the different soil models, stiffness values of the elastic soil medium on the normal and shear stress and free vibration properties. A comparison was made to those from the existing literature. Numerical results show that considering real soil as a continuum space affects the results of the bending and the modal properties significantly.

Keywords: functionally graded materials; beams; ABAQUS; Winkler-Pasternak; foundation; FEM; composite

1. Introduction

Traditionally, layered composites have been widely used in many engineering structures in order to satisfy high performance demands. However, stress concentrations may exist within those structures, and these can result in serious material failure due to the severe interface effects between two different materials (Şimşek and Reddy 2013a, b). It is known that a Japanese scientific group used functionally graded materials (FGM) in 1984 to overcome such defects. FGMs are an especial type of composite material in which the properties of the material are subjected to change continuously through the interfaces. Such variation is due to continuous changes in the volume fraction. Since the variation of properties is continuous, the severe stress effect of variation of materials characteristics at the interface is mainly avoided. Due to their excellent properties, FGMs are used in a range of different areas, instruments and structures. These

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include aircraft, space vehicles, engineering structures and the biomedical sector. In this context, it has critical significance to consider the mechanical behaviour of functionally graded (FG) structures such as beams, plates and shells. Several researchers have reported bending and buckling, along with vibrations, when conducting analyses of functionally graded structures. An analytical solution is introduced by Sankar (2001) for FGB. Here Sankar assumes an exponential change in Young's modulus of materials via depth of beam. In the similar studies an element is introduced (Chakraborty *et al*. 2003) and also vibration of FGM is investigated (Aydogdu and Taskin 2007). Moreover, forced and free vibration have been investigated by Şimşek and Kocatürk (2009). The Euler-Bernoulli FG beam is studied under harmonic moving load. Mahi *et al*. (2010) proposed an approach in investigating free vibration, including the thermal effects. Following these studies, there were a number of related investigations published. These studies dealt with the static, dynamic, and buckling behaviour of functionally graded (FG) structure. (Khalili *et al*. 2010, Alibeigloo 2010, Şimşek 2010, Yan *et al*. 2011, Alshorbgy *et al*. 2011, Fallah and Aghdam 2011, Şimşek *et al*. 2012, Şimşek *et al*. 2013, Şimşek and Reddy 2013a, b, Komijani *et al*. 2013).

Some of the applications of structures resting on elastic foundations include pavement and railroads, pipelines and some aero-space structural applications (Civalek and Öztürk 2010). So far, it has been common to consider the foundation as an array of uniformly distributed springs positioned beneath the beam. However, this assumption, widely held in the civil engineering field, is not realistic in terms of conception, and erroneous results are often produced.

This paper aims to model the FGB resting on a soil space continuum and considers two types of behaviour in the soil foundation, firstly, where the soil is assumed to be an elastic material and secondly, where it is assumed to be an elastic-plastic material. The innovation in this study is in the improvement of the modelling of the FGB resting in/on an accurately conceived soil medium continuum. The author is of the opinion that this method may produce more accurate results as compared to results obtained when conceptualizing the soil as springs.

2. Theoretical background

2.1 Governing equations

In the numerical simulation the orthotropic element is used to define change in materials properties with the depth of the beam, and this does not affect the general isotropic assumption for the governing equation. When the plane stress condition is assumed, the constitutive equation of an elastic solid element can be expressed as (Ying *et al*. 2008)

$$\sigma_x = C_{11}\frac{\partial u}{\partial x} + C_{13}\frac{\partial w}{\partial z} \tag{1}$$

$$\sigma_z = C_{13}\frac{\partial u}{\partial x} + C_{11}\frac{\partial w}{\partial z} \tag{2}$$

$$\tau_{xz} = C_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{3}$$

where u and w are the displacement components, σ_x and σ_z are normal stresses, τ_{xz} is shear stress. The elastic constants C_{ij} are given below (Ying *et al*. 2008)

$$C_{11} = C_{33} = \frac{E}{1-\nu^2} \tag{4}$$

$$C_{13} = \nu C_{11} \tag{5}$$

$$C_{55} = \frac{E}{2(1+\nu)} \tag{6}$$

where E is the elasticity modulus and ν is the Poisson's ratio, and they are functions of the z coordinate. By excluding body forces, equations of motion can be expressed as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{7}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} = \rho \frac{\partial^2 w}{\partial t^2} \tag{8}$$

where ρ is the mass density of the beam, and t denotes time. (Ying *et al*. 2008)

2.2 Material properties

The elastic modulus of a beam is considered to vary exponentially in its depth according to the following equation

$$E = E_0 e^{kz} \tag{9}$$

$$k = h^{-1} ln\left(\frac{E_h}{E_0}\right) \tag{10}$$

Here E_0 and E_h are the elastic moduli relevant to bottom and top surfaces of the FG beam, and h is the depth of the FG beam.

2.3 Soil governing equation

2.3.1 Elastic model

Here in the first case, the soil is assumed to be an elastic plane stress medium. In this condition the stress perpendicular to the calculated plane is taken as zero ($\boldsymbol{\sigma_{zz}}$=0) and the following equation can be applied (Yu 2006)

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right) \tag{11}$$

$$\sigma_{zz} = \frac{E}{1-\nu^2}\left(\varepsilon_{zz} + \nu\varepsilon_{xx}\right) \tag{12}$$

$$\sigma_{zx} = \frac{E}{1-\nu^2}\left(1-\nu\right)\varepsilon_{zx} \tag{13}$$

2.3.2 Elastic-plastic model

The Drucker-Prager yield criterion (Drucker and Prager 1952) is one of the best known elastic-perfectly plastic constitutive models used for modelling the soil medium. This model can be said to be an addition of the von Mises criterion in which the frictional behaviour of materials is taken into account. While in principle stress space the von Mises criterion is seen as an infinite cylinder, the Drucker-Prager model is defined as a cone with the apex placed at the origin (Yu 2006). The linear Drucker-Prager criterion abides by the following equation

$$f = \sqrt{J_2} - aI_1 - k = 0 \tag{14}$$

Where I_1 and J_2 are the first and second stress invariants, a is a representative parameter of soil friction, and k is a function of soil friction and cohesion. As outlined previously, in most studies the soil is generally designed to be a set of springs underneath of foundation. In the model of Ying *et al*. (2008), the FGM beam rests on the usual Winkler elastic medium with modulus K_W and the shear layer with modulus K_P. However, in the present study, the elastic medium is modelled as a soil space continuum. Both models and the differences between them are presented in Fig. 1.

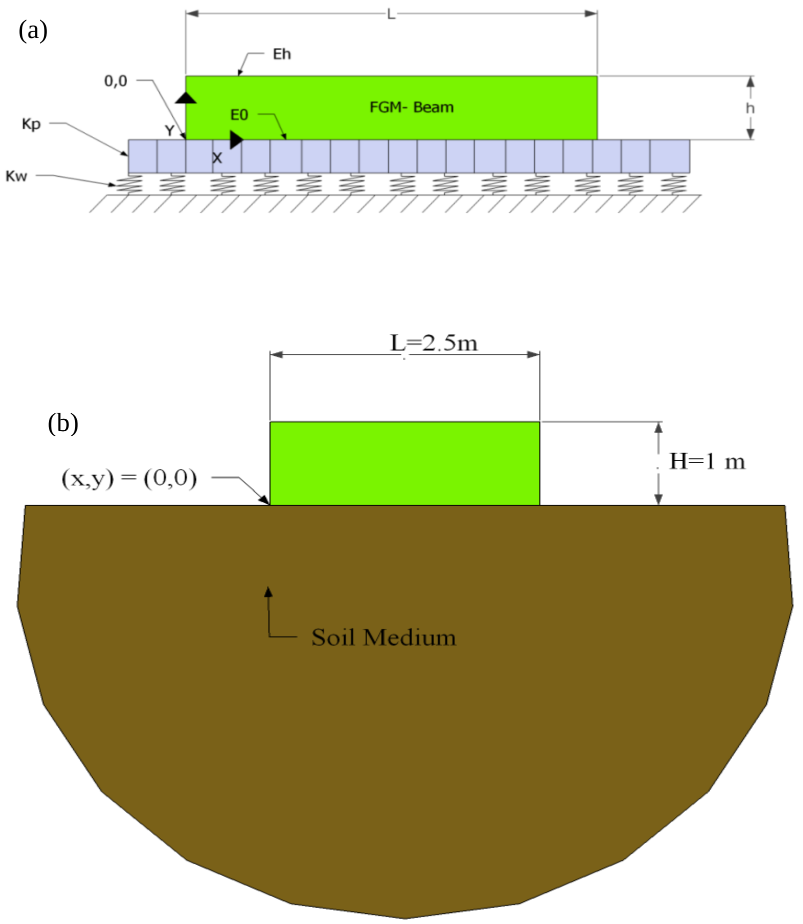

Fig. 1 Schematic illustration of the FGM beam (a) Ying *et al*. (2008) model (b) current model

Table 1 Soil properties

Soil Type	Density (kg/m^3)	*E* (pa)	*ν*	Φ (degrees)	Yield stress (Pa)
Elastic	1800	60e6	0.3	30	-
Drucker-Prager Elastoplastic	1800	60e6	0.3	30	2000

3. Developed model in ABAQUS

As mentioned above, a functionally graded beam (FGB) resting on continuum soil was modelled using the ABAQUS package (see Fig. 1). In this problem, the Poisson ratio of the FG beam was considered to be 0.3 and the ratio of elastic modulus (E_h/E_0) was assumed to be E_h/E_0=10which aligns with 'Case 2' as mentioned in the study by Ying *et al*. (2008). In this model the mass density of material is assumed to be constant.

For the purposes of this analysis, a sinusoidal varying distributed load is assumed to act on the top surface of the FG beam. The following equation is used for load distribution

$$q(x) = q_0 sin\left(\frac{n\pi x}{l}\right) \tag{15}$$

Here n is the half-wave number and q_0=10000 kN/m. In order to realistically model the frictional behaviour of the soil, the internal friction angle Φ is taken into account. All mechanical properties of the soil models considered in the present study are given in Table 1.

The soil medium is assumed to have a width three times that of beam width, and a depth five times that of beam depth. The whole model consists of 2400 4-node bilinear plane stress elements. Fig. 2 illustrates the generated mesh.

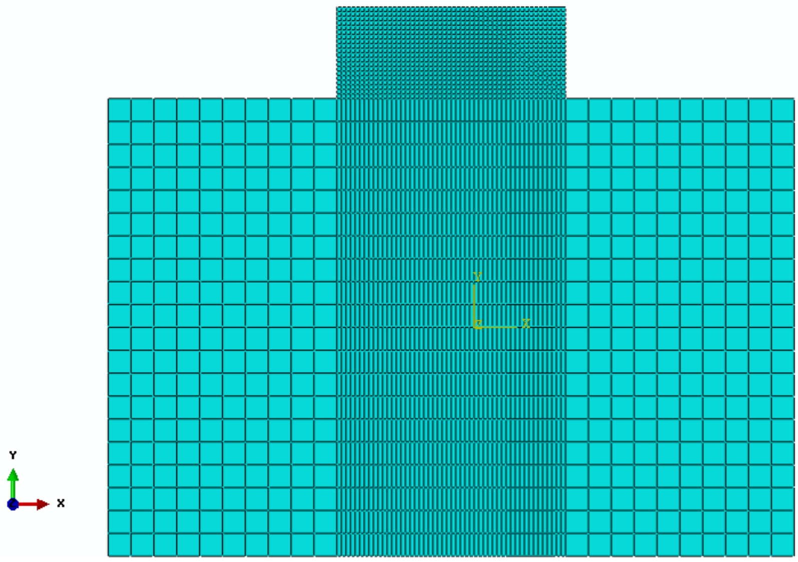

Fig. 2 Generated FEM mesh

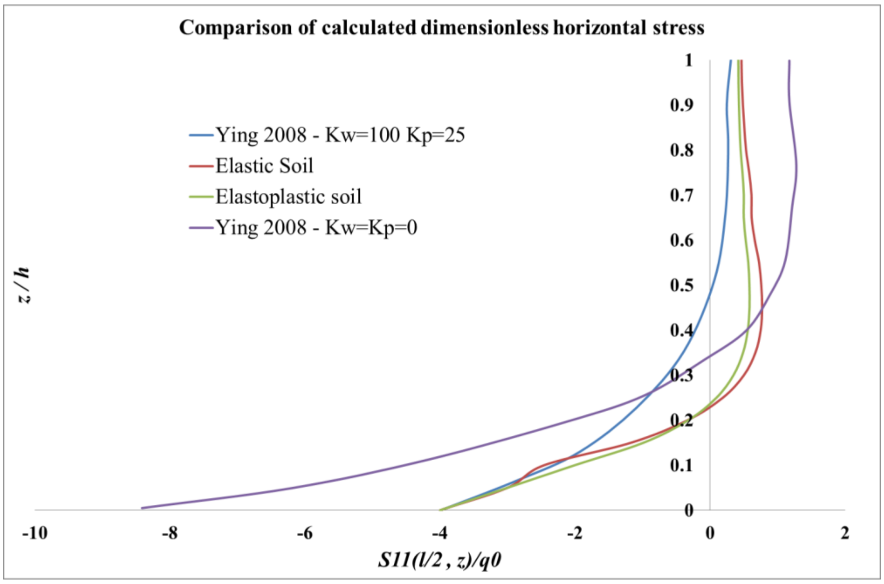

Fig. 3 Induced horizontal stress along the beam depth in the middle section

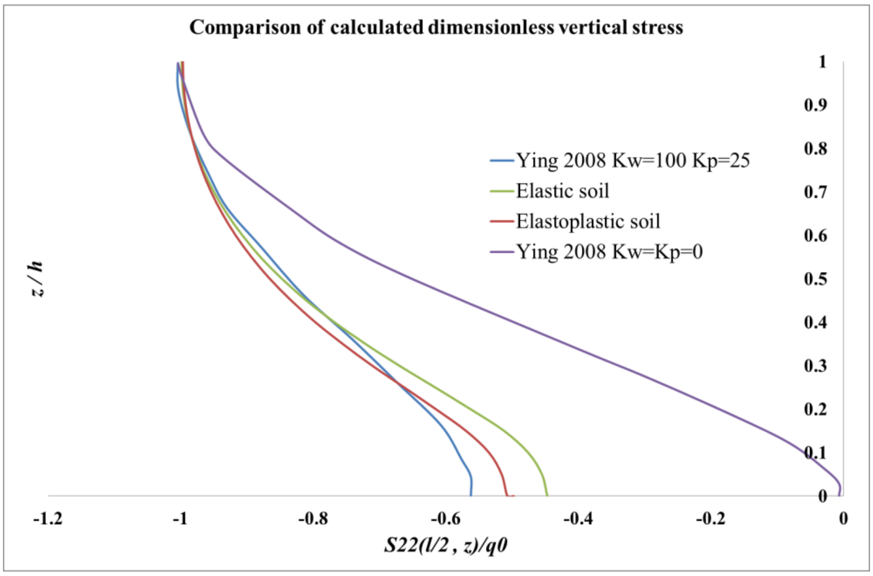

Fig. 4 Induced vertical stress along the beam depth in the middle section

4. Numerical results

4.1 Verification and numerical results of present model

The results of the model used here were first verified by comparing them to the results found in Ying *et al*. (2008). The soil medium in this research was considered to behave in an elastic manner and and the results were again compared with those of Ying *et al*. (2008). The results showed a

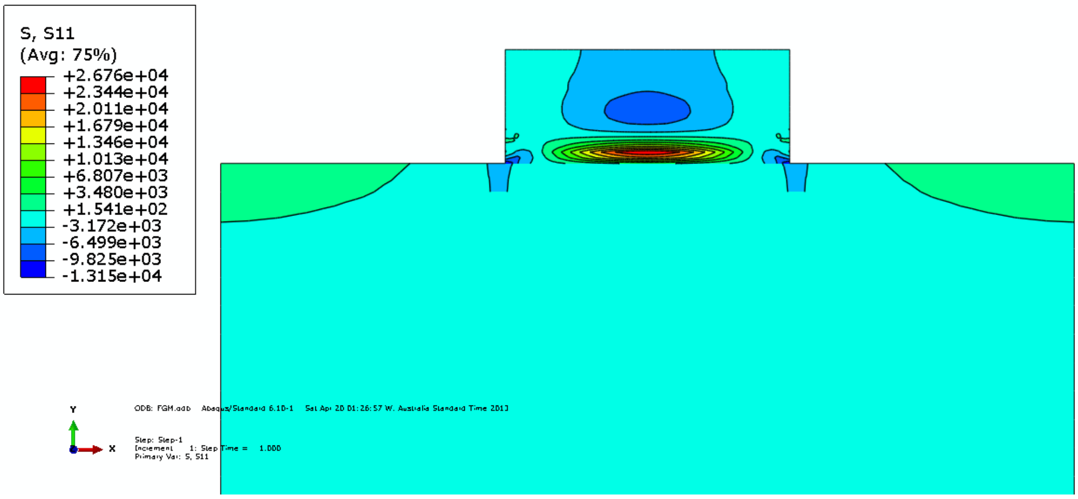

Fig. 5 Horizontal stress generated in whole-of-model assuming elastic soil

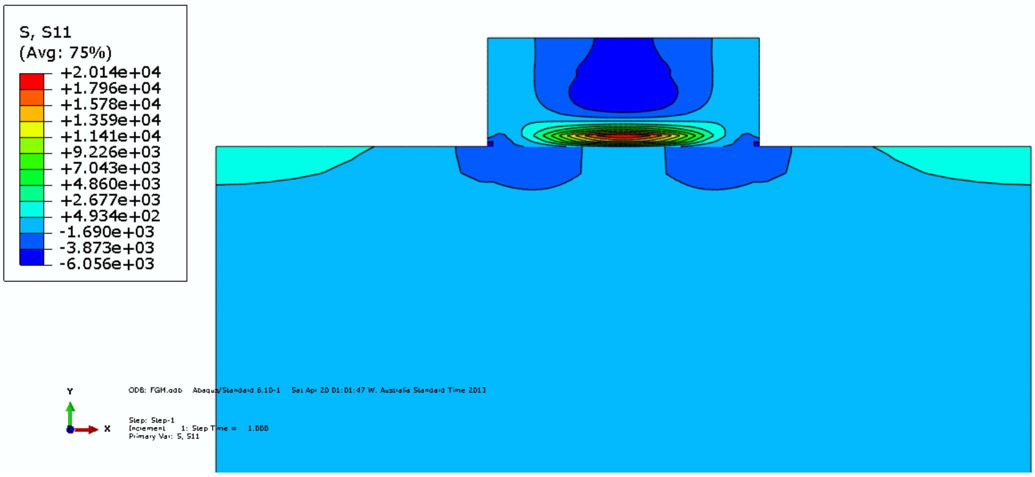

Fig. 6 Horizontal stress generated in whole-of-model assuming elastoplastic soil

positive agreement in trend but a change in values due to the soil being considered to be a continuum space. In the second stage, the soil was also considered to be an elastic-plastic medium and the results are presented in Figs. 3 and 4. In subsequent figures, horizontal stress is indicated by S11 while vertical stress is indicated by S22.

Figs. 3 and 4 illustrate the distribution of horizontal and vertical stresses on the depth of the beam at the middle section. Both elastic soil and elastoplastic soil were modelled and the results are presented together for the purposes of comparison. It can be seen that by assuming a soil medium continuum, the results produce different distributions of stress in the beam. The stress distribution lies between the stresses, as calculated by Ying *et al*. (2008), for two extreme cases of Kw=Kp=0 and Kw=100 and Kp=25 (Ying *et al*. 2008). In Fig. 4 the results of vertical stress are completely bracketed by the results of two extreme conditions of Winkler spring. However, this is not completely true for Fig. 3, where the results of horizontal stress are investigated. In this case,

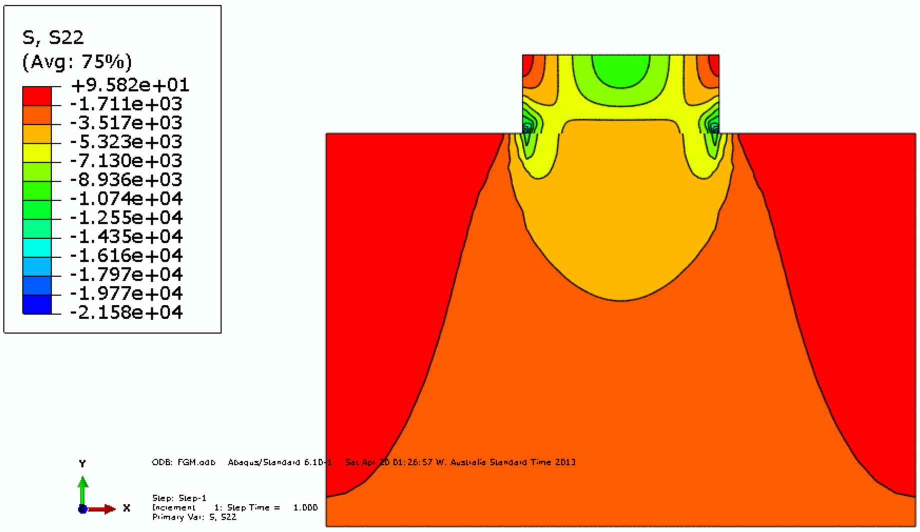

Fig. 7 Vertical stress generated in whole-of-model, assuming elastic soil

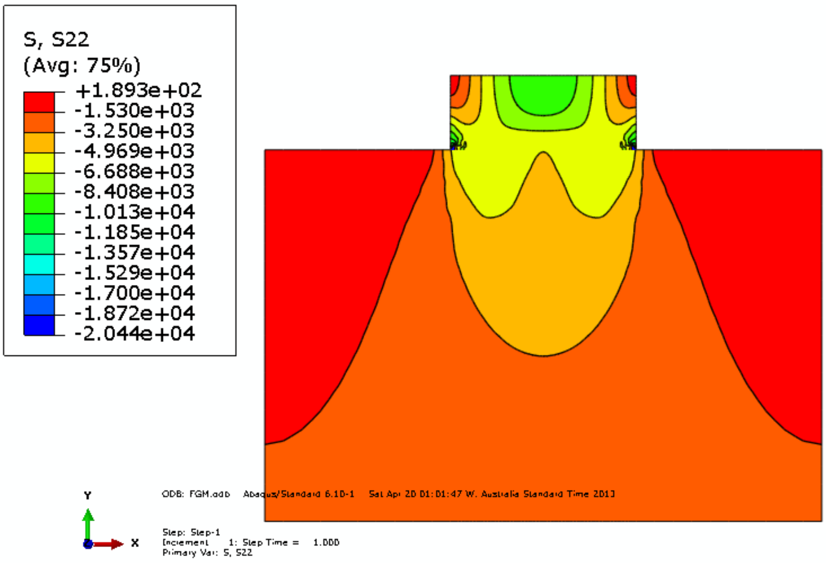

Fig. 8 Vertical stress generated in whole-of-model, assuming elastoplastic soil

the stress calculated from assuming soil medium as foundation is generally below the upper extreme Winkler spring case (*Kw*=100 and *Kp*=25); however, the results of lower extreme Winkler spring case (*Kw*=*Kp*=0) exceed the stress results for the soil medium foundation. This is due to the natural line of the beam. As can be observed, the stress sign changes from the top of the beam to the bottom. This means that the lower extreme case will have a higher negative magnitude at the bottom and a higher positive magnitude at the top. From this perspective, the results of assuming the soil medium as foundation are still bracketed by the two extreme cases.

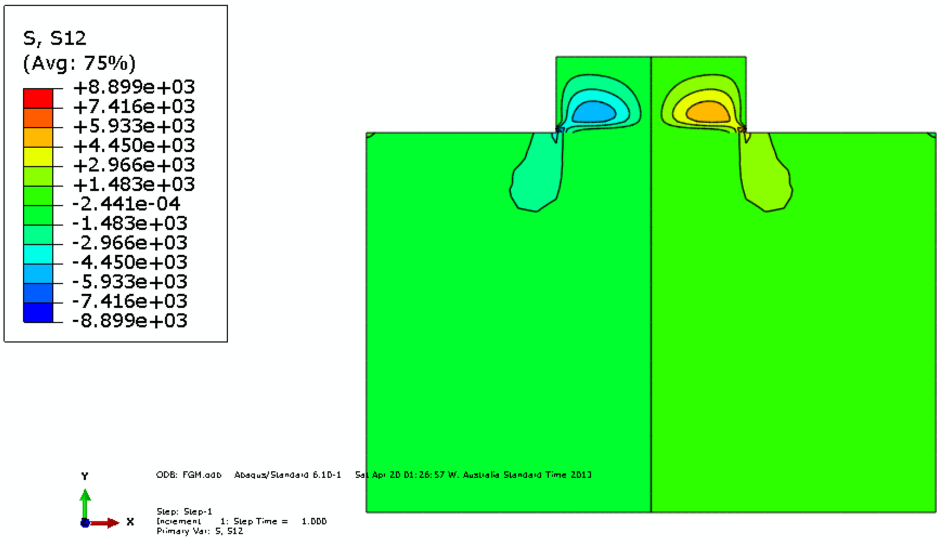

Fig. 9 Shear stress generated in whole-of-model, assuming elastic soil

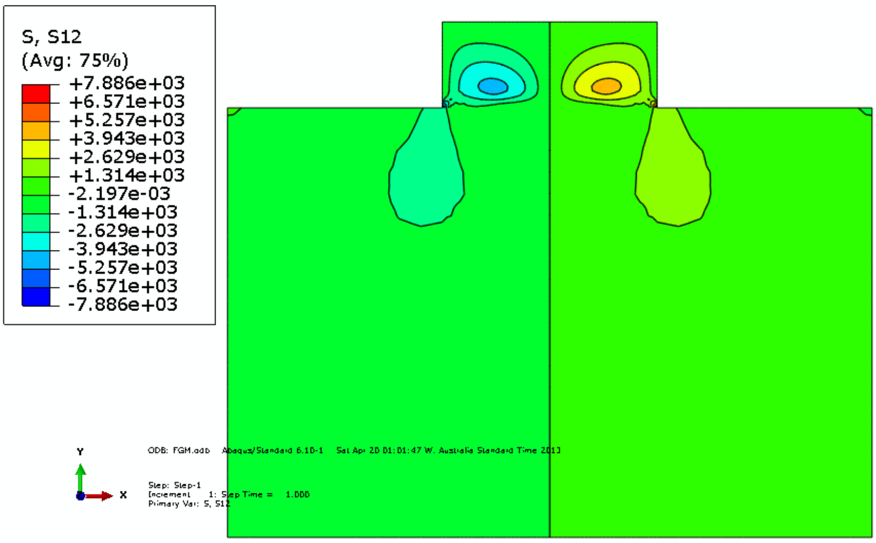

Fig. 10 Shear stress generated in whole-of-model, assuming elastoplastic soil

Figs. 5 to 10 represent the contours of the horizontal, vertical and shear stresses in both beams and the soil medium for the two cases of elastic and elastoplastic soils. In these figures, instead of using the concept of *Kw* and *Kp*, the soil has been modelled as a continuum. It can be seen that the 'softer' behaviour of the soil medium has a notable effect on the distribution of stresses. For example, while the maximum horizontal stress contour is calculated to be 2.676e4 Pa for the elastic soil, it decreases to 2.01e4 Pa when assuming elastoplastic behaviour in the soil.

Figs. 7 and 8 show the vertical stress distribution. The distributed vertical stress contours are greatly affected by assumptions regarding soil behaviour. The elastoplastic soil stress is more

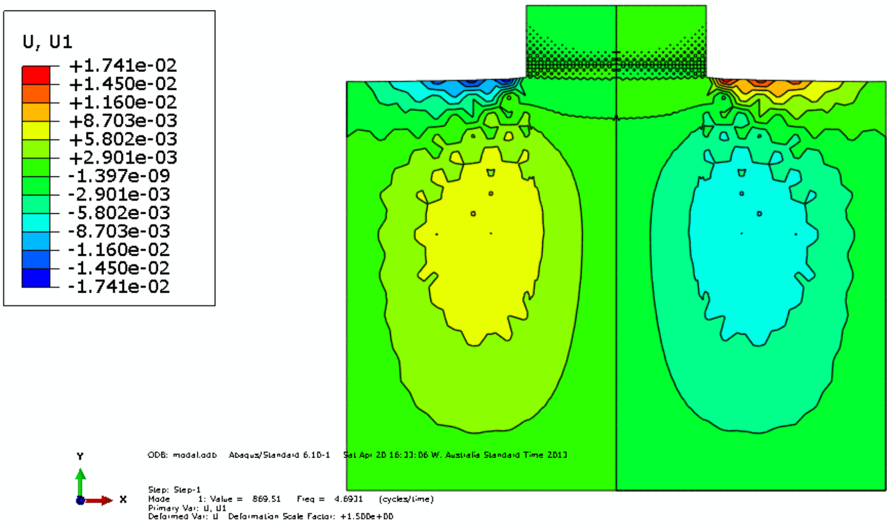

Fig. 11 Mode shape of horizontal displacement (Mode 1)

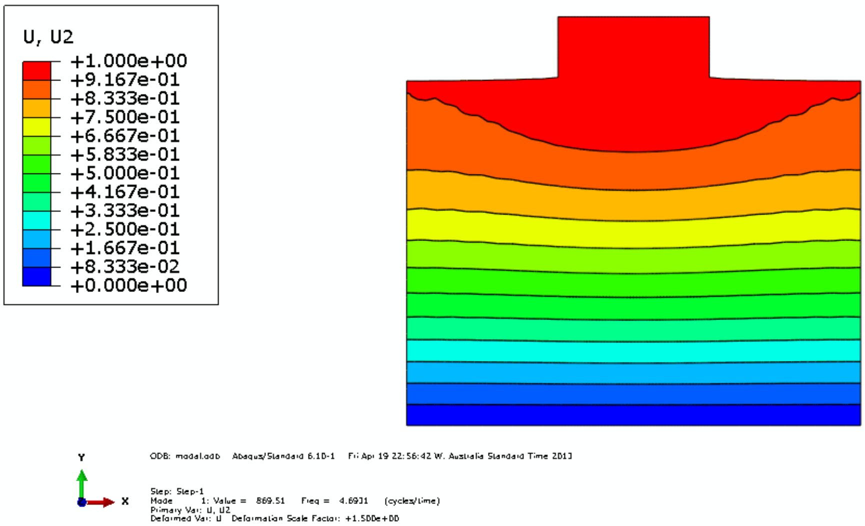

Fig. 12 Mode shape of vertical displacement (Mode 1)

uniformly spread along the bottom of the beam while in the elastic soil, the bottom corners of the beam carry more vertical stress than the part of the beam that is located in the middle of the bottom.

The shear stress distribution in beam and soil is represented in Figs. 9 and 10. It can be observed that the assumption of plastic behaviour in the soil results in less maximum shear in the bottom corners of the beam (8.8999e3 in elastic soil and 7.886e3 in elastoplastic soil). Moreover, the distributed contours of shear stress are less in the case of plastic soil behaviour when compared to elastic soil behaviour.

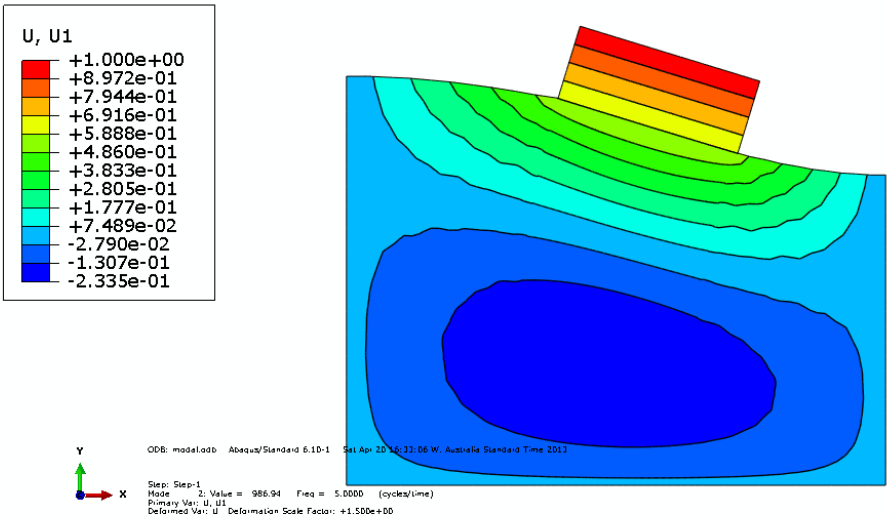

Fig. 13 Mode shape of horizontal displacement (Mode 2)

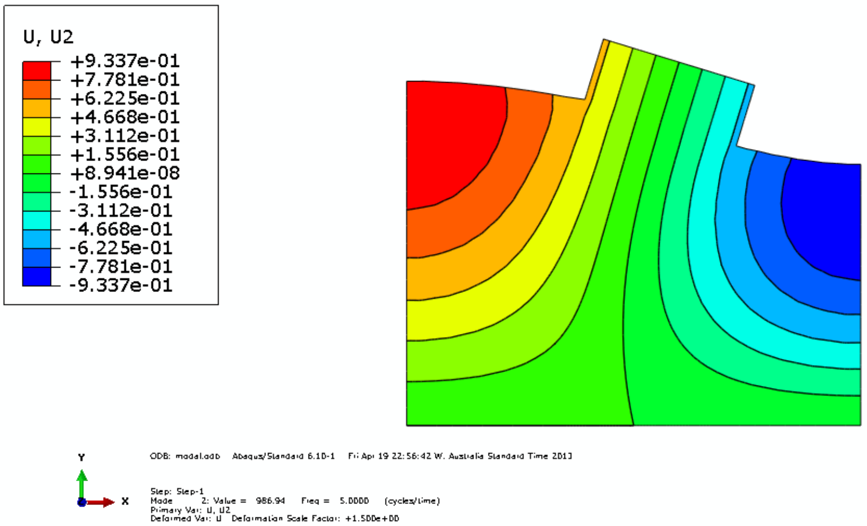

Fig. 14 Mode shape of vertical displacement (Mode 2)

Figs. 11 to 16 illustrate the first three mode shapes for the whole model. The contours for horizontal and vertical displacement are presented, separately. It can be seen that the whole model, including the interaction of the beam and secondary soil, and the influence of the assumed soil medium, cannot be neglected.

Figs. 11 and 12 show the first mode shape. It can be seen that the most dominant behaviour for both horizontal and vertical displacement is transverse vibration.

Figs. 13 and 14 show the second mode shape. Here the longitudinal vibration can be observed as the dominant component for both vertical and horizontal displacement.

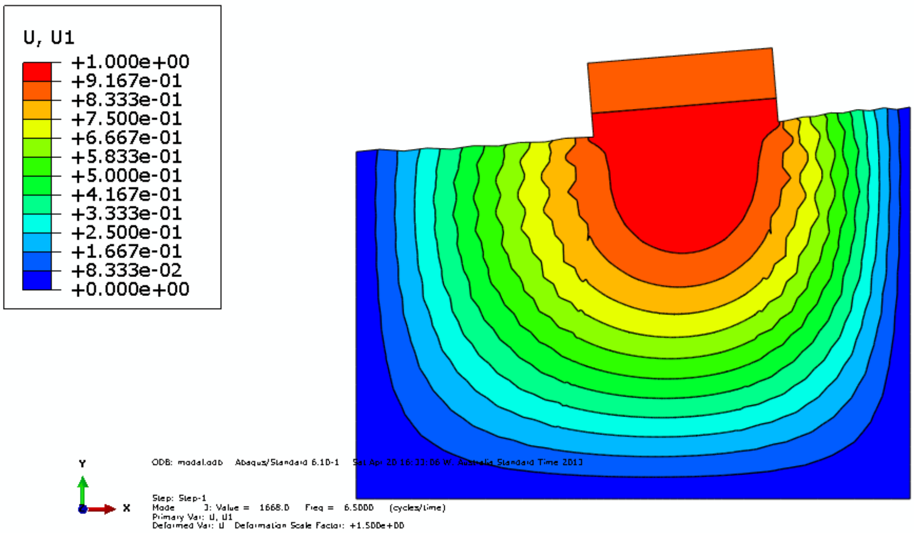

Fig. 15 Mode shape of horizontal displacement (Mode 3)

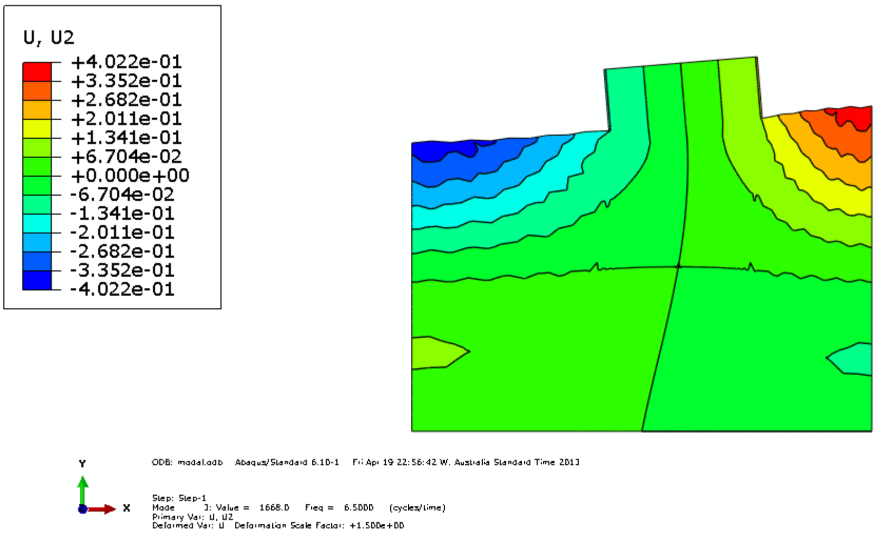

Fig. 16 Mode shape of vertical displacement (Mode 3)

Figs. 15 and 16 illustrate the third mode shape for horizontal and vertical displacement. In these two cases, the behavior is complicated and the dominant displacement component cannot easily be detected.

With regard to the first and the second mode shapes, Ying *et al*. (2008) have reported the same trend as calculated here, for the bottom line of the studied beam resting on springs.

Table 2 lists the first ten normalized frequencies for the whole model, including the soil and beam together.

Table 2 First ten frequencies for the model

Mode No.	Normalized Frequency
1	4.69308
2	4.99995
3	6.50001
4	7.20574
5	7.22848
6	7.60174
7	9.18774
8	9.21017
9	10.0308
10	10.5471

The fundamental frequencies calculated by Ying *et al*. (2008) range from 2.7026 for $Kw=Kp=0$ to 5.1749 for $Kw=10^6$ and $Kp=25$. In the present study the first frequency is 4.69308, which is between the two values calculated by Ying *et al*. (2008).

Generally it can be indicated that assuming a soil foundation at the bottom of the beam instead of simplifying the condition to Winkler springs can cause a noticeable change in results. The response mechanism of the beam especially can be modified according to the assumption of the foundation being a continuous soil medium.

Furthermore, assumptions about the behaviour of the soil medium itself can have an effect on the results. Where the linear elastic continuum medium provides responses similar to those arising from assuming a Winkler springs model, the difference (especially in terms of stress distribution) is more noticeable when more complex behaviour (such as the elastoplastic mechanism) is considered for the soil medium.

5. Conclusions

This study focused on a new approach to modelling the behaviour of FGM beams on soil. So far, the work in this area has been based on considering soil as a Winkler spring model; this study adds to the body of knowledge in this area by modelling the soil as a continuum space.

In this study, a functionally graded beam was modelled using ABAQUS software. The innovation in this study lay in the consideration of the soil medium as a continuum rather than as a classical elastic medium represented by springs. The study proved that this new consideration has a significant impact on the results compared to the classical method. A modal analysis was performed to observe the behaviour of the FGM beam resting on a ‘proper’ soil medium. As expected, the results were significantly different to those of Ying *et al*. (2008). The main reason is that Ying *et al*. (2008) considered the soil as a set of springs, while in reality the soil medium itself absorbed some of the energy of the system. The Ying *et al*. (2008) model was constructed upon the elasticity of the foundation and the two-dimensional theory of elasticity. The first assumption in which the foundation is considered to be an elastic material can be improved upon by considering the foundation as elastoplastic or plastic. The results were for two soil phases; firstly, the elastic continuum space phase and secondly, the elastic-plastic phase. They followed a similar trend to the results of Ying *et al*. (2008), but showed a difference of 25% in the value of dimensionless horizontal stress. The results of dimensionless vertical stress also occasionally showed a 10% difference, depending on the different values of z/h. The first normalized frequency was confirmed to be within the same range as that found by Ying *et al*. (2008). As can be seen, the Ying *et al*. (2008) model is very dependent on the *Kw* and *Kp* values, and for a specific range of *Kw* and *Kp* (i.e., *Kw*=100, *Kp*=25), it corresponds very well with the results from the current study and for the other values (i.e., *Kw*=*Kp*=0) show difference. In conclusion, this study has opened up a new approach to gaining a better understanding of the behaviour of FGM resting on soils.

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