Structural Engineering and Mechanics, *Vol. 51, No. 3 (2014) 519-530*
DOI: http://dx.doi.org/10.12989/sem.2014.51.3.519

An accurate novel method for solving nonlinear mechanical systems

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(Received September 5 2013, Revised April 18, 2014, Accepted June 20, 2014)

Abstract. This paper attempts to investigate the nonlinear dynamic analysis of strong nonlinear problems by proposing a new analytical method called Hamiltonian Approach (HA). Two different cases are studied to show the accuracy and efficiency of the method. This approach prepares us to obtain the nonlinear frequency of the nonlinear systems with the first order of the solution with a high accuracy. Finally, to verify the results we present some comparisons between the results of Hamiltonian approach and numerical solutions using Runge-Kutta's [RK] algorithm. This approach has a powerful concept and the high accuracy, so it can be apply to any conservative nonlinear problems without any limitations.

Keywords: approximate frequency; Hamiltonian approach; analytical investigations

1. Introduction

Many phenomena and physical and engineering problems are modeled such as nonlinear differential equations. Therefore, solving these nonlinear differential equations is very important to obtain more information from the behavior of the system or the problem. We have two approaches to solve differential equations, if it is a linear differential equation, it is possible to prepare an analytic solution for it and if it is a nonlinear one, we should use numerical solutions because it is very difficult to have an analytical. Recently, a particular attention has been done to the new developed methods to prepare an approximate analytic solution of nonlinear differential equations such as : variational iteration method (Wazwaz 2007), Homotopy perturbation method (Shou 2009) energy balance method (Ganji *et al.* 2009),max-min method (Zeng 2009), amplitude frequency- formulation (Ren *et al.* 2011), parameter expansion method (Kaya *et al.* 2009) and other methods (Bayat *et al.* 2011a, b, c, 2012a, b, 2013a, b, c, 2014a, b, c, Pakar *et al.* 2011a, 2012a, b, 2013a, b).

Through the continuous investigations of these methods, many scientific works have been conducted as follows. Bayat *et al.* (2012a) considered the recent new approaches and have a complete comparison in their valuable review paper, obtaining nonlinear frequency of the problems by numerical and recent new approximate approaches for different kinds of nonlinear

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

systems is one of the advantages of their study. Fu *et al*. (2011) prepares an analytical solutions by means of energy balance method for nonlinear vibration of micro electromechanical system. They governed the nonlinear equation based on Euler-Bernoulli hypothesis. They reduced a PDE problem to an ODE by using Galerkine method and then applying the energy balance method. The results are in good agreement with numerical ones. Pirbodaghi *et al*. (2010) tried to analysis symmetrically conservative Multi-Degree Of Freedom (MDOF) System with cubic nonlinearity. The second-order coupled differential equations were solved analytically with homotopy analysis method. Beléndez et al. (2008) investigated a modified version of homotopy perturbation method for to achieve high order solution of nonlinear oscillator with discontinuities. Bayat *et al*. (2012b) obtained the nonlinear frequency of tapered beams by means of hamiltonian approach. They considered the effect of significant parameters on the nonlinear frequency of the system.

In this study, we have considered a new application of Hamiltonian approach for two high nonlinear mechanical cases. Some comparisons are presented to show the accuracy of the proposed method. It has been indicated that the Hamiltonian approach could be easily extended to any conservative nonlinear oscillators.

2. Basic idea of Hamiltonian approach

Previously, He (2002) had introduced the energy balance method based on collocation and the Hamiltonian. This approach is very simple but strongly depends upon the chosen location point. Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{u} + f\ (u, \dot{u}, \ddot{u}) = 0 \tag{1}$$

With initial conditions

$$u(0) = A, \qquad \dot{u}(0) = 0. \tag{2}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation *A*. It is easy to establish a variational principle for Eq. (1), which reads

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}\dot{u}^2 + F(u) \right\} dt \tag{3}$$

Where *T* is period of the nonlinear oscillator, $\partial F/\partial u = f$.

In the Eq. (3), $\frac{1}{2}\dot{u}^2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (3) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(u) = \frac{1}{2}\dot{u}^2 + F(u) = \text{constant} \tag{4}$$

From Eq. (4), we have

$$\frac{\partial H}{\partial A} = 0 \tag{5}$$

Introducing a new function, $\bar{H}(u)$, defined as

$$\bar{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2}\dot{u}^2 + F(u) \right\} dt = \frac{1}{4} TH \tag{6}$$

Eq. (5) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T} \right) = 0 \tag{7}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)} \right) = 0 \tag{8}$$

From Eq. (8) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

3. Applications

In order to assess the advantages and the accuracy of the Hamiltonian approach, we will consider the following examples:

3.1 Example 1

We first consider the following nonlinear oscillator as it is shown in Fig. 1 (Nayfe 1973)

$$\left(1 + Ru^2\right)\ddot{u} + Ru\dot{u}^2 + \omega_0^2 u + \frac{1}{2}\frac{R\,g\,u^3}{l} = 0 \qquad u(0) = A, \quad \dot{u}(0) = 0 \tag{9}$$

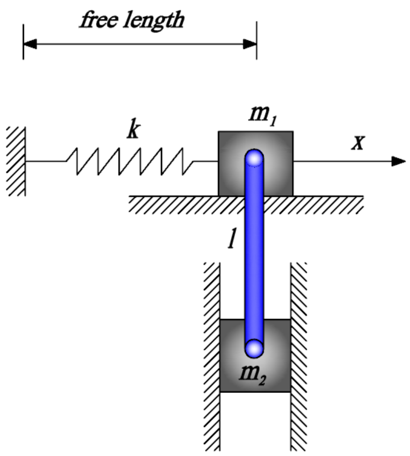


Fig. 1 The geometry of the problem

where

$$\omega_0^{\,2} = \frac{k}{m_1} + \frac{Rg}{l} \;\;,\quad R = \frac{m_2}{m_1} \;\;,\quad \left| u = \frac{x}{l} \right| << 1 \tag{10}$$

The Hamiltonian of Eq. (9) is constructed as

$$H = -\frac{1}{2}\dot{u}^2 - \frac{1}{2}R\,u^2\dot{u}^2 + \frac{1}{2}\omega_0^2 u^2 + \frac{1}{8}\frac{Rg\,u^4}{l} \tag{11}$$

Integrating Eq. (12) with respect to *t* from 0 to *T/*4, we have

$$\bar{H}(u) = \int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2 - \frac{1}{2}R\,u^2\dot{u}^2 + \frac{1}{2}\omega_0^2 u^2 + \frac{1}{8}\frac{Rg\,u^4}{l}\right)dt \tag{12}$$

Assume that the solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain

$$\begin{aligned}
\bar{H} &= \int_0^{T/4}\left(-\frac{1}{2}\omega^2 A^2\sin(\omega t)^2 - \frac{1}{2}R\omega^2 A^4\sin^2(\omega t)\cos^2(\omega t) + \frac{1}{2}\omega_0^{\,2}A^2\cos^2(\omega t) + \frac{1}{8}\frac{Rg}{l}A^4\cos^4(\omega t)\right)dt \\
&= \int_0^{\pi/2}\left(-\frac{1}{2}\omega A^2\sin t^2 - \frac{1}{2}R\,\omega A^4\sin^2 t\cos^2 t + \frac{1}{2\omega}\omega_0^{\,2}A^2\cos^2 t + \frac{1}{8\omega}\frac{Rg}{l}A^4\cos^4 t\right)dt \\
&= -\frac{1}{8}\omega\pi A^2 - \frac{1}{32}\omega R\,\pi A^4 + \frac{1}{8}\frac{\pi}{\omega}\omega_0^{\,2}A^2 + \frac{3}{128}\frac{Rg\pi}{\omega l}A^4
\end{aligned} \tag{14}$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right) = -\frac{1}{4}\omega^2\pi A - \frac{1}{8}R\,\pi\omega^2 A^3 + \frac{1}{4}\pi\,\omega_0^2\,A + \frac{3}{32}\frac{Rg\,\pi}{l}A^3 = 0 \tag{16}$$

Solving the above equation, an approximate frequency as a function of amplitude equals

$$\omega_{HA} = \frac{1}{2}\frac{\sqrt{3RgA^2 + 8\omega_0^2 l}}{\sqrt{A^2 l^2 R + 2l^2}}, \tag{17}$$

According to Eqs. (13) and (17), we can obtain the following approximate solution

$$u(t) = A\cos\left(\frac{1}{2}\frac{\sqrt{3RgA^2 + 8\omega_0^2 l}}{\sqrt{A^2 l^2 R + 2l^2}}t\right) \tag{18}$$

3.2 Example 2

Consider the motion of a mass *m* moving without friction along a circle of radius *R* that is rotating with a constant angular velocity Ω about its vertical diameter as shown in figure 2. The forces acting on the mass are gravitational force *mg*, the centrifugal of the circle O and the reaction force .The following governing equation has been obtained (Nayfe 1973)

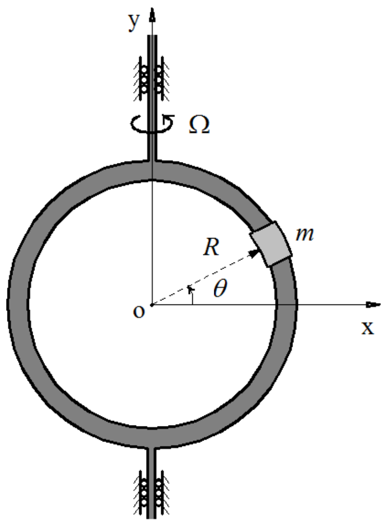


Fig. 2 Particle moving without friction on a rotating circular

$$m R^2\ddot{\theta} - m R^2\Omega^2 \sin(\theta)\cos(\theta) + mgR\sin(\theta) = 0, \qquad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{19}$$

By using the Taylor's series expansion for cos($\theta(t)$), sin($\theta(t)$) and by some manipulation in Eq. (19) we can re-write Eq. (19) in the following form

$$\alpha\ddot{\theta} - \beta\left(\theta - \frac{1}{6}\theta^3\right)\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right) + \lambda\left(\theta - \frac{1}{6}\theta^3\right) = 0, \qquad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{20}$$

Where

$$\alpha = mR^2, \ \beta = mR^2\Omega^2, \ \ \lambda = mgR \tag{21}$$

The Hamiltonian of Eq. (20) is constructed as

$$H = -\frac{1}{2}\alpha\dot{\theta}^2 - \frac{1}{2}\beta\theta^2 + \frac{1}{6}\beta\theta^4 - \frac{1}{48}\beta\theta^6 + \frac{1}{1152}\beta\theta^8 + \frac{1}{2}\lambda\theta^2 - \frac{1}{24}\lambda\theta^4 \tag{22}$$

Integrating Eq. (22) with respect to t from 0 to $T/4$, we have;

$$\bar{H}(\theta) = \int_0^{T/4}\left(-\frac{1}{2}\alpha\dot{\theta}^2 - \frac{1}{2}\beta\theta^2 + \frac{1}{6}\beta\theta^4 - \frac{1}{48}\beta\theta^6 + \frac{1}{1152}\beta\theta^8 + \frac{1}{2}\lambda\theta^2 - \frac{1}{24}\lambda\theta^4\right)dt \tag{23}$$

Assume that the solution can be expressed as

$$\theta(t) = A\cos(\omega t) \tag{24}$$

Substituting Eq. (24) into Eq. (23), we obtain

$$\bar{H}=\int_0^{T/4}\begin{pmatrix}-\frac{1}{2}\alpha A^2\omega^2\sin^2(\omega t)-\frac{1}{2}\beta A^2\cos^2(\omega t)+\frac{1}{6}\beta A^4\cos^4(\omega t)-\frac{1}{48}\beta A^6\cos^6(\omega t)\\+\frac{1}{1152}\beta A^8\cos^8(\omega t)+\frac{1}{2}\lambda A^2\cos^2(\omega t)-\frac{1}{24}\lambda A^4\cos^4(\omega t)\end{pmatrix}dt$$

$$=\int_0^{\pi/2}\begin{pmatrix}-\frac{1}{2}\alpha A^2\omega\sin^2 t-\frac{1}{2\omega}\beta A^2\cos^2 t+\frac{1}{6\omega}\beta A^4\cos^4 t-\frac{1}{48\omega}\beta A^6\cos^6 t\\+\frac{1}{1152\omega}\beta A^8\cos^8 t+\frac{1}{2\omega}\lambda A^2\cos^2 t-\frac{1}{24\omega}\lambda A^4\cos^4 t\end{pmatrix}dt \tag{25}$$

$$=-\frac{1}{8}A^2\pi\omega\alpha-\frac{1}{8}\frac{\beta\pi}{\omega}A^2+\frac{1}{32}\frac{\beta\pi}{\omega}A^4-\frac{5}{1536}\frac{\beta\pi}{\omega}A^6+\frac{35}{294912}\frac{\beta\pi}{\omega}A^8+\frac{1}{8}\frac{\lambda\pi}{\omega}A^2-\frac{1}{128}\frac{\lambda\pi}{\omega}A^4$$

Setting

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=-\frac{1}{4}\omega^2\alpha\pi A-\frac{1}{4}\beta\pi A+\frac{1}{8}\beta\pi A^3-\frac{5}{256}\beta\pi A^5+\frac{35}{36864}\beta\pi A^7+\frac{1}{4}\lambda\pi A-\frac{1}{32}\lambda\pi A^3 \tag{26}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega=\sqrt{\frac{\beta}{\alpha}-\frac{1}{2}\frac{\beta}{\alpha}A^2+\frac{5}{64}\frac{\beta}{\alpha}A^4-\frac{35}{9216}\frac{\beta}{\alpha}A^6-\frac{\lambda}{\alpha}+\frac{1}{8}\frac{\lambda}{\alpha}A^2} \tag{27}$$

By substituting Eq.(21) in to Eq.(27) we have:

$$\omega_{HA}=\sqrt{\Omega^2-\frac{g}{R}+\frac{1}{8}\frac{g}{R}A^2-\frac{1}{2}\Omega^2A^2+\frac{5}{64}\Omega^2A^4-\frac{35}{9216}\Omega^2A^6} \tag{28}$$

Hence, the approximate solution can be readily obtained:

$$\theta(t)=A\cos\left(\sqrt{\Omega^2-\frac{g}{R}+\frac{1}{8}\frac{g}{R}A^2-\frac{1}{2}\Omega^2A^2+\frac{5}{64}\Omega^2A^4-\frac{35}{9216}\Omega^2A^6}\;t\right) \tag{29}$$

4. Results and discussions

In this section to verify the results some comparison are presented to show the accuracy of the proposed approach with numerical method.

In example 1: Table 1 is the comparison of the Hamiltonain approach and the numerical solution using Runge-Kutta algorithm (Appendix A) for two different cases.

Case 1: $g=10,\quad m_1=20,\quad m_2=10,\quad l=2,\ k=200,\quad A=0.6$

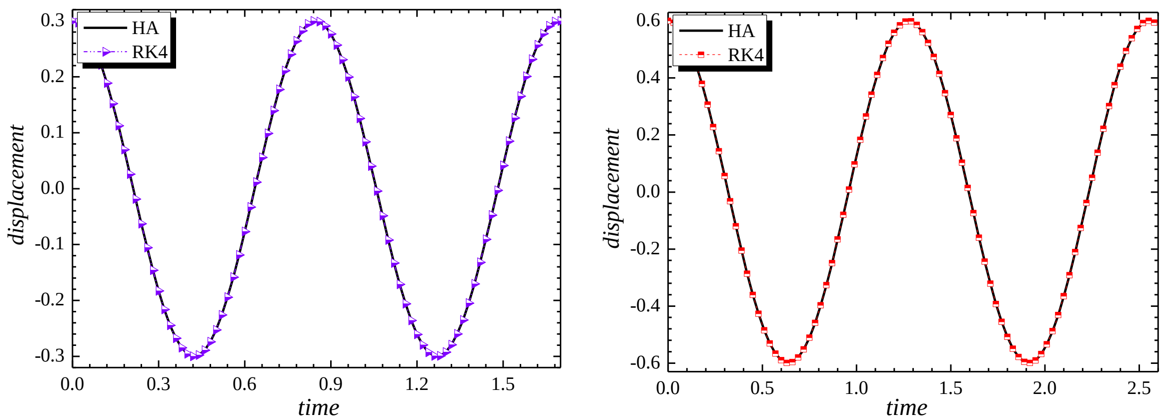
Case 2: $g=10,\quad m_1=15,\quad m_2=3,\quad l=3,\ k=250,\quad A=0.9$

The results show the high accuracy of the solution for whole domain.

Fig. 3 is displacement comparison for different parameters; it is obvious from the figure that the behavior of the oscillation is periodic as a function of amplitude. Fig. 4 is shown the effect of spring stiffness on the frequency of the system with different amplitudes. By increasing the stiffness of the spring the nonlinear frequency is increased too, it is decreased by increasing of the

Table 1 Comparison of time history response of Hamiltonian approach with Runge-Kutta for example 1

	Case 1			Case 2		
	Displacement			Displacement		
Time	HA	RK4	Error %	HA	RK4	Error %
0	0.6	0.6	0	0.9	0.9	0
0.05	0.4954	0.5016	1.2291	0.6048	0.6170	1.9686
0.1	0.2182	0.2259	3.4305	-0.0871	-0.0910	4.3076
0.15	-0.1351	-0.1410	4.2204	-0.7219	-0.7317	1.3432
0.2	-0.4413	-0.4499	1.9015	-0.8832	-0.8841	0.1092
0.25	-0.5937	-0.5943	0.0967	-0.4651	-0.4770	2.4792
0.3	-0.5392	-0.5428	0.6784	0.2580	0.2687	3.9967
0.35	-0.2967	-0.3052	2.7721	0.8119	0.8180	0.7523
0.4	0.0492	0.0525	6.3687	0.8332	0.8367	0.4172
0.45	0.3779	0.3883	2.6733	0.3080	0.3163	2.6093
0.5	0.5749	0.5771	0.3829	-0.4192	-0.4341	3.4282
0.55	0.5716	0.5731	0.2750	-0.8715	-0.8741	0.2960
0.6	0.3690	0.3770	2.1309	-0.7521	-0.7587	0.8644
0.65	0.0378	0.0374	1.0297	-0.1394	-0.1413	1.3056
0.7	-0.3065	-0.3178	3.5411	0.5647	0.5804	2.6942
0.75	-0.5441	-0.5487	0.8480	0.8985	0.8988	0.0351
0.8	-0.5919	-0.5922	0.0407	0.6428	0.6516	1.3454
0.85	-0.4335	-0.4402	1.5130	-0.0344	-0.0404	0.1174
0.9	-0.1240	-0.1264	1.8839	-0.6891	-0.7025	1.9035
0.95	0.2287	0.2397	4.5685	-0.8918	-0.8917	0.0073
1	0.5018	0.5093	1.4759	-0.5095	-0.5182	1.6898



(a) (b)

Fig. 3 Comparison of analytical solution of displacement with the RK4 solution for cases (a) g=10, m_1=20, m_2=10, l=2, k=100, A=0.3 (b) g=10, m_1=20, m_2=5, l=2, k=50, A=0.6

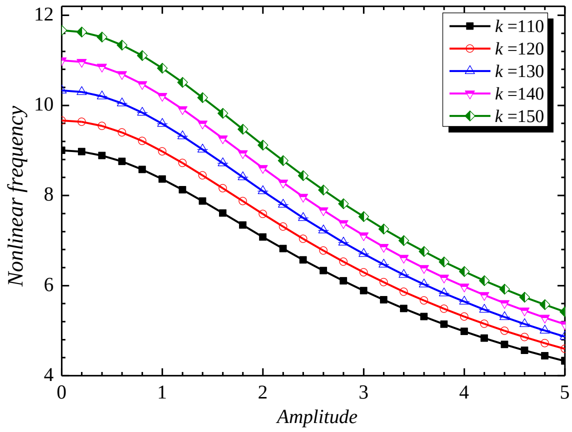


Fig. 4 Effect of stiffness on nonlinear frequency for various parameter of amplitude g=10, m_1=15, m_2=5, l=2

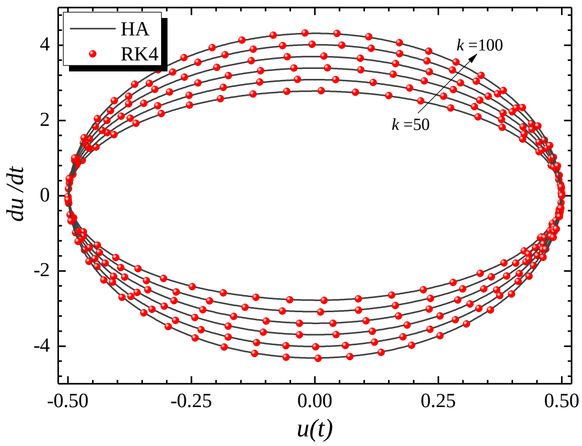


Fig. 5 Effect of sprig stiffness on phase plane for g=10, m_1=15, m_2=20, l=5, A=0.5

amplitude. Fig. 5 shows the phase plan of the problem by considering the effects of Effect of sprig stiffness.

For the second example: In this example as same as the previous one, we obtain the same results and compare them with the numerical solutions.

The results for different time steps are shown in Table 2. The results have a very good agreement with Runge-kutta's algorithm for two different cases for example 2.

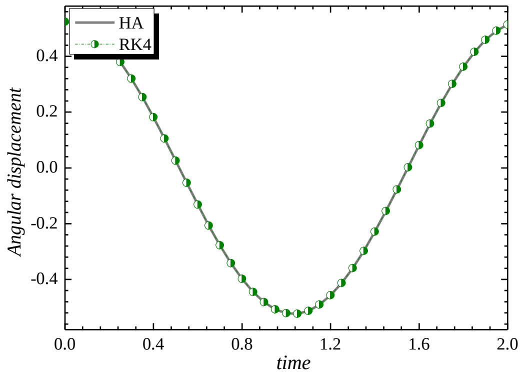
Case 1: $g=10,\quad m=2,\quad R=1.5,\quad \Omega=2,\quad A=\pi/3$

Case 2: $g=10,\quad m=2,\quad R=0.4,\quad \Omega=2.2,\quad A=\pi/2$

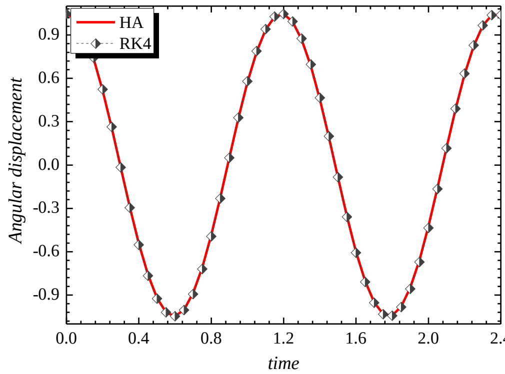
Figure 6 indicates the periodical behavior of the system in two different cases in which the angles and the radius are different. Figure 7 is shown the effect of velocity of the system respect to the amplitude and nonlinear frequency of the system. By decreasing of the velocity the nonlinear frequency is increased. Increasing of the amplitude reached that to the peak point and after that it has a rapid decreasing by increasing the amplitude of the system. Figure 8 is the phase plan curve by considering the effect of rotating circle radius on phase plan.

Table 2 Comparison of time history response of Hamiltonian approach with Runge-Kutta for example 2

	Case 1			Case2		
	Angular displacement			Angular displacement		
Time	HA	RK4	Error %	HA	RK4	Error %
0	1.0472	1.0472	0	1.5708	1.5708	0
0.2	0.9729	0.9688	0.4251	1.0833	1.1130	2.6698
0.4	0.7605	0.7489	1.5548	-0.0766	-0.0753	1.8397
0.6	0.4403	0.4283	2.7942	-1.1890	-1.2097	1.7116
0.8	0.0575	0.0567	1.3852	-1.5633	-1.5649	0.1011
1	-0.3334	-0.3211	3.8325	-0.9673	-1.0059	3.8414
1.2	-0.6770	-0.6619	2.2781	0.2292	0.2250	1.8688
1.4	-0.9245	-0.9168	0.8470	1.2834	1.2955	0.9353
1.6	-1.0409	-1.0402	0.0653	1.5410	1.5471	0.3974
1.8	-1.0095	-1.0081	0.1410	0.8420	0.8892	5.3024
2	-0.8349	-0.8269	0.9664	-0.3796	-0.3724	1.9214
2.2	-0.5418	-0.5311	2.0255	-1.3656	-1.3701	0.3321
2.4	-0.1719	-0.1697	1.2946	-1.5039	-1.5176	0.9019
2.6	0.2225	0.2107	5.6157	-0.7088	-0.7638	7.2050
2.8	0.5853	0.5674	3.1611	0.5263	0.5160	2.0003



(I)



(II)

Fig. 6 Comparison of analytical solution of angular displacement with the RK4 solution for cases (I) g=10, m=2, R=0.8, Ω=1.8, $A=\pi/6$ (II) g=10, m=2, R=0.3, Ω=1.2, $A=\pi/3$

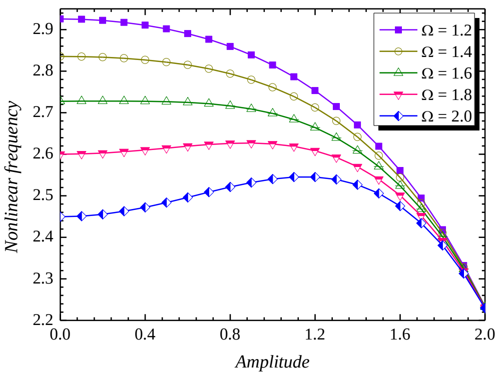


Fig. 7 Effect of amplitude on nonlinear frequency for various parameter of velocity, g=10, R=2

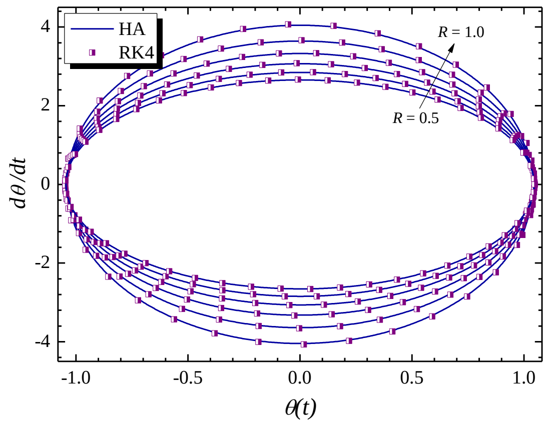


Fig. 8 Effect of rotating circle radius on phase plan for g=10, m=2, Ω=2, A= $\pi/3$

5. Conclusions

In this study, we tried to propose a new analytical approach for nonlinear vibration equations. The successful application of the Hamiltonian approach for two strong nonlinear cases shows that, this new approach could give a reasonable and accurate solution form the problem and also could give engineering sense in nonlinear system. In fact, we can suggest Hamiltonian approach for nonlinear conservative problems without any limitations to obtain nonlinear response of the problem and also obtain the effect of significant parameters on the nonlinear behavior of the system.

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Appendix A. Basic idea of Runge-Kutta

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows

$$\dot{u} = f\,(t, u), \quad u(t_0) = u_0 \tag{A.1}$$

u is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} u_{n+1} &= u_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A.2}$$

for n=0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f\,(t_n, u_n), \\ k_2 &= f\left(t_n + \frac{1}{2}h,\ u_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \frac{1}{2}h,\ u_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\,(t_n + h,\ u_n + hk_3). \end{aligned} \tag{A.3}$$

Where u_{n+1} is the RK4 approximation of $u(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.