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Accurate analytical solutions for nonlinear oscillators with discontinuous

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Abstract. In this study, three approximate analytical methods have been proposed to prepare an accurate analytical solution for nonlinear oscillators with fractional potential. The basic idea of the approaches and their applications to nonlinear discontinuous equations have been completely presented and discussed. Some patterns are also presented to show the accuracy of the methods. Comparisons between Energy Balance Method (EBM) , Variational Iteration Method (VIM) and Hamiltonian Approach (HA) shows that the proposed approaches are very close together and could be easily extend to conservative nonlinear vibrations.

Keywords: natural frequency; nonlinear oscillators; discontinuities; perturbation method

1. Introduction

The general nonlinear oscillators with discontinuities is

$$u'' + f\,(u, u', u'') = 0 \tag{1}$$

with initial conditions $u(0)=A$ and $u'(0)=0$. Here f is a known discontinuous function. If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Many researchers have been works on new approximate analytical solutions for nonlinear equations without possible small parameters. Many new approaches have been proposed to overcome the traditional perturbation methods such as :homotopy perturbation method (Shaban *et al.* 2010, Bayat 2013a), Hamiltonian approach (Bayat *et al.* 2011a, 2012a, 2013a, b, 2014a, b), energy balance method (He 2002, Bayat *et al.* 2011b, Pakar *et al.* 2011a, b, Mehdipour 2010), variational iteration method (Dehghan 2010, Pakar *et al*. 2012), amplitude frequency formulation (Bayat 2011c, 2012b, Pakar *et al.* 2013a, He 2008), max-min approach (Shen *et al.* 2009, Zeng *et al.* 2009), variational approach (He 2007, Bayat *et al.* 2012c, 2013c, 2014c, Pakar *et al.* 2012b), and the other analytical and numerical (Xu 2009, Alicia *et al.* 2010, Bor-Lih *et al.* 2009, Wu 2011, Odibat *et al*. 2008, He 1999, Ganji *et al.* 2009, He 2004, Pakar *et al.* 2014).

The main objective of this paper is to approximately solve nonlinear oscillators with fractional

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potential by applying the Energy Balance Method (EBM), Variational Iteration Method (VIM) and Hamiltonian Approach (HA). Some comparisons of the results are presented to have better comparison between the methods. It has been demonstrated that these three methods could be strong mathematical tools for solving nonlinear vibration equations.

2. Basic idea of energy balance method

In order to clarify this method, consider the following general oscillator (He 2002)

$$u'' + f\left(u(t)\right) = 0 \tag{2}$$

In which *u* and *t* are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u)\right)dt \tag{3}$$

Where $T=2\pi/\omega$ is period of the nonlinear oscillator**,** $F(u)=\int f(u)du$**.**
Its Hamiltonian, therefore, can be written in the form

$$\Delta H = \frac{1}{2}u'^2 + F(u) = F(A) \tag{4}$$

Or

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{5}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, *A*. So let us consider such initial conditions

$$u(0) = A, \qquad u'(0) = 0 \tag{6}$$

He use the following trial function to determine the angular frequency ω

$$u = A\cos(\omega t) \tag{7}$$

Substituting Eq. (7) into Eq. (5), He obtain the following residual equation

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + F\left(A\cos(\omega t)\right) - F(A) = 0 \tag{8}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make *R* zero for all values of *t* by appropriate choice of ω. Since Eq. (6) is only an approximation to the exact solution, *R*, can not be made zero everywhere. Collocation at $\omega t=\pi/4$ gives

$$\omega = \sqrt{\frac{2F(A) - F\left(A\cos(\omega t)\right)}{A^2\sin(\omega t)}} \qquad rad/\sec \tag{9}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\dfrac{2F(A)-F\left(A\cos(\omega t)\right)}{A^2\sin(\omega t)}}} \tag{10}$$

3. Basic idea of variational iteration method

We re-write Eq. (1) in the following form (He 1999)

$$u'' + \omega^2 u = F(u), \qquad F(u) = \omega^2 u - f(u) \tag{11}$$

We consider that the angular frequency of the oscillator is ω, and we choose the trial function u_0=$A\cos(\omega t)$ The angular frequency ω is identified with the physical understanding that no secular terms should appear in $u_1(t)$,which leads to

$$\int_0^T \cos(\omega t)\left[\omega^2 u_0 - f(u_0)\right]dt = 0, \qquad T = \frac{2\pi}{\omega} \tag{12}$$

From this equation, ω can easily be found. It should be specially pointed out that the more accurate the identification of the multiplier, the more faster the approximations converge to its exact solution, and for this reason, we identify the multiplier from Eq. (11) rather than Eq. (1).

According to the VIM, we can construct a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{u_n''(\tau) + \omega^2 u_n(\tau) - \tilde{F}_n\right\}d\tau \tag{13}$$

where λ is a general Lagrange multiplier (He 1999), which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{F}_n$ is considered as a restricted variation, i.e., $\delta\tilde{F}_n = 0$. Under this condition, its stationary conditions of the above correction functional can be written as

Follows

$$\begin{aligned} &\lambda''(\tau) + \omega^2\lambda(\tau) = 0, \\ &\lambda(\tau)\big|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)\big|_{\tau=t} = 0. \end{aligned} \tag{14}$$

The Lagrange multiplier, therefore, can be readily identified by

$$\lambda = \frac{1}{\omega}\sin\,\omega(\tau - t) \tag{15}$$

Which leads to following iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\omega}\sin\,\omega(\tau - t)\left\{u_n''(\tau) + f_n\right\}d\tau. \tag{16}$$

As we will see in the forthcoming illustrative examples, we always stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

4. Basic idea of Hamiltonian approach

Recently, He (2010) has proposed the Hamiltonian approach. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider Eq. (2) as general oscillator with initial conditions $u(0)=A$, $u'(0)=0$.

It is easy to establish a variational principle for Eq. (2), which reads

$$J(u)=\int_0^{T/4}\left\{-\frac{1}{2}u'^2+F(u)\right\}dt \tag{17}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial u=f$.

In the Eq (17), $\frac{1}{2}u'^2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (17) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(u)=\frac{1}{2}u'^2+F(u)=\text{constant}=H_0 \tag{18}$$

From Eq. (18), we have

$$\frac{\partial H}{\partial A}=0 \tag{19}$$

Introducing a new function, $\bar{H}(u)$, defined as

$$\bar{H}(u)=\int_0^{T/4}\left\{\frac{1}{2}u'^2+F(u,u',u'')\right\}dt=\frac{1}{4}TH \tag{20}$$

Eq. (19) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial T}\right)=0 \tag{21}$$

Or

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial (1/\omega)}\right)=0 \tag{22}$$

From Eq. (22) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

5. Application

5.1 Application of energy balance method

In the present paper, we consider the following nonlinear oscillators with discontinuities (Ganji *et al.* 2009)

$$u'' + u + \varepsilon u\left|u\right| = 0 \tag{23}$$

With the initial condition:

$$u(0) = A, \quad u'(0) = 0 \tag{24}$$

Here the discontinuous function is $f(u) = u + \varepsilon u\left|u\right|$.
Its variational principle can be easily obtained

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \begin{Bmatrix} -\frac{1}{3}\varepsilon u^3 & u < 0 \\ \frac{1}{3}\varepsilon u^3 & u \ge 0 \end{Bmatrix} \right) d\tau \tag{25}$$

Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \begin{Bmatrix} -\frac{1}{3}\varepsilon u^3 & u < 0 \\ \frac{1}{3}\varepsilon u^3 & u \ge 0 \end{Bmatrix} = \frac{1}{2}A^2 + \begin{Bmatrix} -\frac{1}{3}\varepsilon A^3 & A < 0 \\ \frac{1}{3}\varepsilon A^3 & A \ge 0 \end{Bmatrix} = 0 \tag{26}$$

or

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \begin{Bmatrix} -\frac{1}{3}\varepsilon u^3 & u < 0 \\ \frac{1}{3}\varepsilon u^3 & u \ge 0 \end{Bmatrix} - \frac{1}{2}A^2 - \begin{Bmatrix} -\frac{1}{3}\varepsilon A^3 & A < 0 \\ \frac{1}{3}\varepsilon A^3 & A \ge 0 \end{Bmatrix} = 0 \tag{27}$$

In Eq. (26) the kinetic energy (E) and potential energy (T) can be respectively expressed as

$$E = \frac{1}{2}u'^2 \tag{28}$$

and

$$T = \frac{1}{2}u^2 + \begin{Bmatrix} -\frac{1}{3}\varepsilon u^3 & u < 0 \\ \frac{1}{3}\varepsilon u^3 & u \ge 0 \end{Bmatrix} \tag{29}$$

Throughout the oscillation, it holds that $H=E+T$ constant. We use the following trial function to determine the angular frequency ω

$$u = A\cos(\omega t) \tag{30}$$

Substituting Eq. (30) into Eq. (27) ,we obtain the following residual equation

$$\begin{aligned} R(t) = {} & \frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}A^2\cos^2(\omega t) + \begin{Bmatrix} -\frac{1}{3}\varepsilon A^3\cos^3(\omega t) & A\cos(\omega t) < 0 \\ \frac{1}{3}\varepsilon A^3\cos(\omega t) & A\cos(\omega t) \ge 0 \end{Bmatrix} \\ & - \frac{1}{2}A^2 - \begin{Bmatrix} -\frac{1}{3}\varepsilon A^3 & A < 0 \\ \frac{1}{3}\varepsilon A^3 & A \ge 0 \end{Bmatrix} = 0 \end{aligned} \tag{31}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (30) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t=\pi/4$ gives

$$R\left(t\right)=\frac{1}{4}A^2\omega^2-\frac{1}{4}A^2+\begin{cases}-\frac{\sqrt{2}}{12}\varepsilon A^3 & A<0\\ \frac{\sqrt{2}}{12}\varepsilon A^3 & A\geq 0\end{cases}-\begin{cases}-\frac{1}{3}\varepsilon A^3 & A<0\\ \frac{1}{3}\varepsilon A^3 & A\geq 0\end{cases}=0 \tag{32}$$

In Eq. (32), ω is passive and the other parameters are active, including A and ε.
By solving Eq. (32) there will be

$$\omega_{EBM}=\begin{cases}\frac{1}{3}\sqrt{9+3\varepsilon A\sqrt{2}-12\varepsilon A} & A<0\\ \frac{1}{3}\sqrt{9-3\varepsilon A\sqrt{2}+12\varepsilon A} & A\geq 0\end{cases} \tag{33}$$

5.2 Application of variational iteration method

In Eq. (23), the discontinuous function is $f(u)=u+\varepsilon u|u|$ we can determine the angular frequency

$$\int_0^T \cos(\omega t)\left[\omega^2 A\cos(\omega t)-\left(A\cos(\omega t)+\varepsilon A\cos(\omega t)\left|A\cos(\omega t)\right|\right)\right]dt,\qquad T=\frac{2\pi}{\omega} \tag{34}$$

Nothing that $|\cos(\omega t)|=\cos(\omega t)$ when $-\frac{\pi}{2}<\omega t<\frac{\pi}{2}$ and $|\cos(\omega t)|=-\cos(\omega t)$ when $\frac{\pi}{2}<\omega t<\frac{3\pi}{2}$, so we write Eq. (34) in the following form

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\left(\omega^2-1\right)A\cos^2(\omega t)-\varepsilon A^2\cos^3(\omega t)\right]dt+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left[\left(\omega^2-1\right)A\cos^2(\omega t)-\varepsilon A^2\cos^3(\omega t)\right]dt=0 \tag{35}$$

From the above equation, one can easily conclude that:

$$\omega_{VIM}=\sqrt{1+\frac{8}{3\pi}\varepsilon A} \tag{36}$$

We re-write Eq.(16) in the following form:

$$u_{n+1}(t)=u_n(t)+\int_0^t\frac{1}{\omega}\sin\left(\tau-t\right)\left\{u_n''(\tau)+u_n(\tau)+\varepsilon u_n(\tau)\left|u_n(\tau)\right|\right\}d\tau \tag{37}$$

By the above iteration formula, we can calculate the first-order approximation:

$$u_1(t)=\begin{cases}A\cos(\omega t)+\int_0^t\frac{1}{\omega}\sin\left(\tau-t\right)\left\{\left(1-\omega^2\right)A\cos(\omega t)+\varepsilon A^2\cos^2(\omega t)\right\}d\tau, & -\frac{\pi}{2}<\omega t<\frac{\pi}{2}\\ A\cos(\omega t)+\int_0^t\frac{1}{\omega}\sin\left(\tau-t\right)\left\{\left(1-\omega^2\right)A\cos(\omega t)-\varepsilon A^2\cos^2(\omega t)\right\}d\tau, & \frac{\pi}{2}<\omega t<\frac{3\pi}{2}\end{cases} \tag{38}$$

Which yields

$$u(t)=\begin{cases} A\cos(\omega t)+\dfrac{1}{2\omega}A\left(\omega^2-1\right)t\ \sin(\omega t)+\dfrac{\varepsilon A^2}{6\omega^2}\left(\cos(2\omega t)+2\cos(\omega t)\right)-\dfrac{\varepsilon A^2}{2\omega^2}, & -\dfrac{\pi}{2}<\omega t<\dfrac{\pi}{2} \\ A\cos(\omega t)+\dfrac{1}{2\omega}A\left(\omega^2-1\right)t\ \sin(\omega t)-\dfrac{\varepsilon A^2}{6\omega^2}\left(\cos(2\omega t)+2\cos(\omega t)\right)+\dfrac{\varepsilon A^2}{2\omega^2}, & \dfrac{\pi}{2}<\omega t<\dfrac{3\pi}{2} \end{cases} \tag{39}$$

Where the angular frequency ω is defined as Eq. (36).

The above results are in good agreement with the results obtained by the homotopy perturbation reported in He (2004)

In order to compare with traditional perturbation solution, we write Nayfeh's result (1973)

$$u(t)=A\cos\left(1+\frac{4}{3\pi}\varepsilon A\right)t+.... \tag{40}$$

5.3 Application of Hamiltonian approach

The Hamiltonian of Eq. (23) is constructed as

$$H=-\frac{1}{2}u'^2+\frac{1}{2}u^2+\frac{1}{3}\varepsilon u^2\left|u\right| \tag{41}$$

Integrating Eq. (41) with respect to t from 0 to $T/4$, we have

$$\bar{H}(u)=\int_0^{T/4}\left(\frac{1}{2}u'^2+\frac{1}{2}u^2+\frac{1}{3}\varepsilon u^3\right)dt+\int_{T/4}^{T/2}\left(\frac{1}{2}u'^2+\frac{1}{2}u^2-\frac{1}{3}\varepsilon u^3\right)dt. \tag{42}$$

We use the following trial function

$$u(t)=A\cos(\omega t) \tag{43}$$

If we Substitute Eq. (43) into Eq. (42), its results are

$$\begin{aligned}\bar{H}(u)&=\int_0^{T/2}\left(\frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{1}{2}A^2\cos^2(\omega t)+\frac{1}{3}\varepsilon A^3\cos^3(\omega t)\right)dt\\&+\int_{T/4}^{T/2}\left(\frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{1}{2}A^2\cos^2(\omega t)-\frac{1}{3}\varepsilon A^3\cos^3(\omega t)\right)dt\\&=\int_0^{\pi/4}\left(\frac{1}{2}A^2\omega\sin^2 t+\frac{1}{2\omega}A^2\cos^2 t+\frac{1}{3\omega}\varepsilon A^3\cos^3 t\right)dt\\&+\int_{\pi/2}^{\pi}\left(\frac{1}{2}A^2\omega\sin^2 t+\frac{1}{2\omega}A^2\cos^2 t-\frac{1}{3\omega}\varepsilon A^3\cos^3 t\right)dt\\&=\frac{\pi}{4}\omega A^2+\frac{\pi}{4\omega}A^2+\frac{4}{9\omega}\varepsilon A^3\end{aligned} \tag{44}$$

Setting

$$\frac{\partial}{\partial\Delta}\left(\frac{\partial\bar{H}}{\partial\left(1/\omega\right)}\right)=-\frac{\pi}{2}A\omega^2+\frac{\pi}{2}A+\frac{4}{3}\varepsilon A^2 \tag{45}$$

Table 1 Comparison of time history response of EBM, VIM and HA.

Time	displacement			velocity			acceleration		
	EBM	VIM	HA	EBM	VIM	HA	EBM	VIM	HA
0	2	2	2	0	0	0	-4.7582	-4.7514	-4.7162
0.5	1.4341	1.4408	1.4389	-2.1502	-2.1481	-2.1332	-3.4119	-3.4127	-3.3930
1	0.0567	0.0637	0.0703	-3.0836	-3.0857	-3.0693	-0.1350	-0.1509	-0.1659
1.5	-1.3528	-1.3493	-1.3377	-2.2721	-2.2845	-2.2832	3.2184	3.1959	3.1544
2	-1.9968	-2.0019	-1.9951	-0.1749	-0.1960	-0.2159	4.7505	4.7417	4.7046
2.5	-1.5109	-1.5265	-1.5330	2.0213	2.0029	1.9726	3.5945	3.6155	3.6149
3	-0.1700	-0.1909	-0.2107	3.0737	3.0731	3.0541	0.4044	0.4520	0.4968
3.5	1.2671	1.2522	1.2298	2.3868	2.4116	2.4220	-3.0145	-2.9661	-2.9001
4	1.9871	1.9896	1.9802	0.3493	0.3912	0.4307	-4.7276	-4.7127	-4.6696
4.5	1.5827	1.6059	1.6195	-1.8859	-1.8496	-1.8022	-3.7655	-3.8036	-3.8189
5	0.2827	0.3173	0.3500	-3.0539	-3.0480	-3.0238	-0.6726	-0.7512	-0.8253
5.5	-1.1773	-1.1500	-1.1159	-2.4938	-2.5288	-2.5487	2.8009	2.7243	2.6314
6	-1.9711	-1.9692	-1.9556	-0.5225	-0.5847	-0.6434	4.6894	4.6644	4.6116
6.5	-1.6495	-1.6788	-1.6980	1.7444	1.6887	1.6229	3.9244	3.9761	4.0040
7	-0.3945	-0.4424	-0.4875	3.0242	3.0104	2.9786	0.9387	1.0473	1.1497
7.5	1.0837	1.0432	0.9965	2.5927	2.6357	2.6629	-2.5782	-2.4714	-2.3498
8	1.9487	1.9408	1.9213	0.6941	0.7758	0.8530	-4.6362	-4.5972	-4.5307

If we solve Eq. (45) the approximate frequency of the system is

$$\omega_{HA}=\sqrt{1+\frac{8}{3\pi}\varepsilon A} \tag{46}$$

Hence, the approximate solution can be readily obtained:

$$u(t)=A\cos\left(\sqrt{1+\frac{8}{3\pi}\varepsilon A}\;\;t\right) \tag{47}$$

6. Result and discussion

In this section, we compare the results of energy balance method, variational iteration method and Hamiltonian approach in table and figures to show the accuracy of these three methods.

Table 1 shows values of displacements and velocity and acceleration for different time points. As it is obvious from the table, the results are very close together for different time values.

To better understanding the motion of the problem, we have compared the results of the energy balance solution with the variational iteration solution and Hamiltonian approach in some figures for two case to show the displacement time history response, velocity time history response, acceleration time history response and phase curve. Fig. 1 is for A=1, ε=0.5. Fig. 2 is for A=10, ε=0.2. The motion of the problem is periodic and it is function of the initial condition. In the phase cure, it has not seen any problem in the extreme points. Fig. 3 is the effect of epsilon on frequency base on amplitude. The increases of epsilon is increases the nonlinear frequency of the problem.

The three proposed methods have been correctly applied to nonlinear problem with discontinuity. The methods are very strong for solving nonlinear conservative problems.

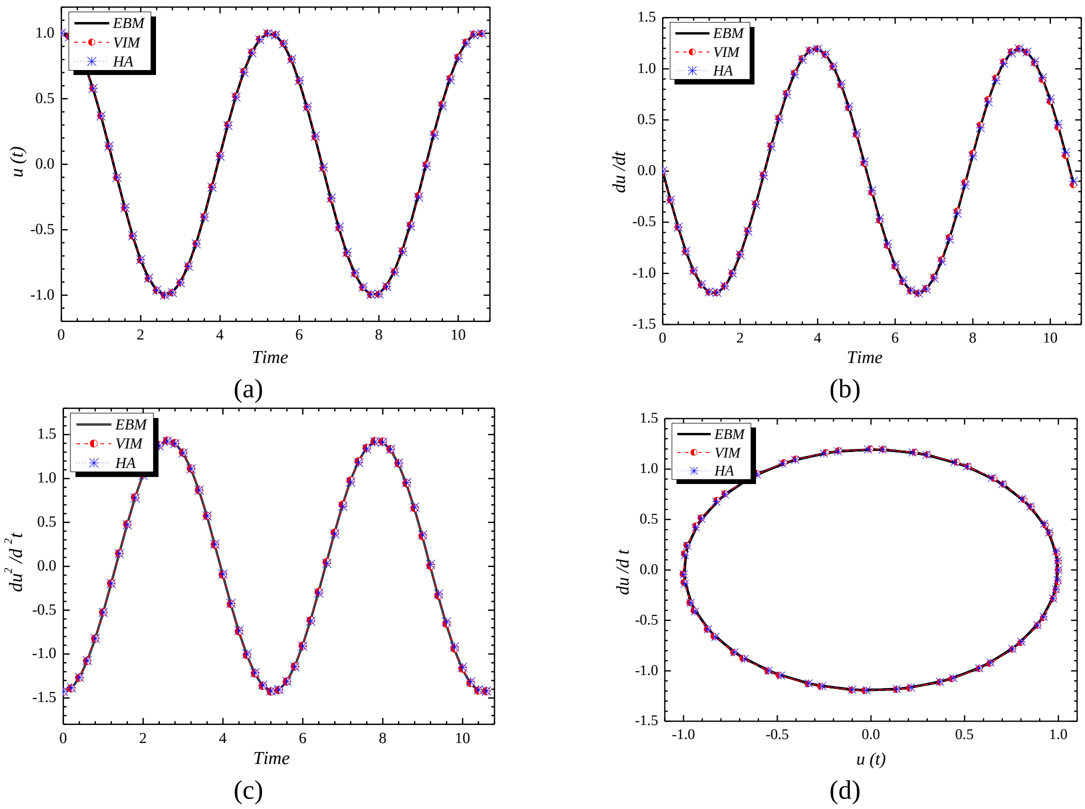


Fig. 1 Comparison of the energy balance solution with the variational iteration solution and Hamiltonian approach. (a) displacement time history response (b) velocity time history response (c) acceleration time history response (d) phase curve. for A=1, ε=0.5

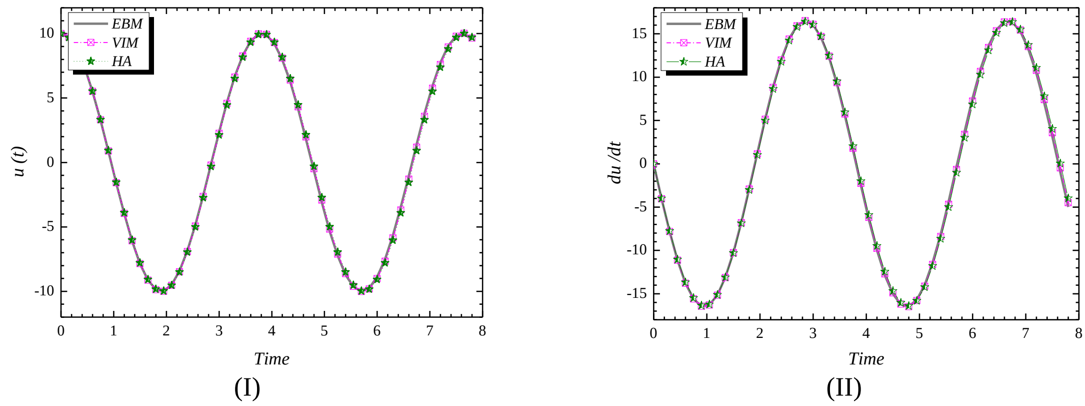
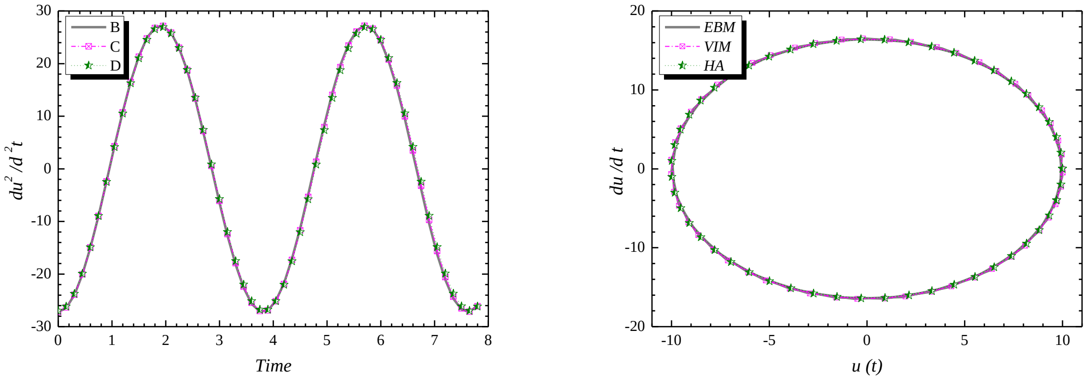


Fig. 2 Comparison of the energy balance solution with the variational iteration solution and Hamiltonian approach. (I) displacement time history response (II) velocity time history response (III) acceleration time history response (IV) phase curve. for A=10, ε=0.2



(III) (IV)

Fig. 2 Continued

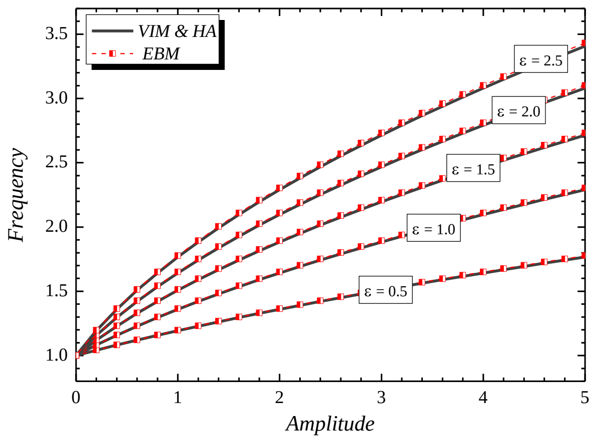


Fig. 3 Influence of epsilon on frequency base on amplitude

7. Conclusions

In this study, energy balance method, variational iteration method and Hamiltonian approach were proposed and discussed. The basic ideas of the methods were illustrated and also their applications to discontinue nonlinear problem were considered. The methods are very useful to obtain the nonlinear frequency of the conservative problems. Some patterns were also presented to show the accuracy of these new methods. The methods do not require any linearization or small perturbation, and adequately accurate to both linear and nonlinear problems. The methods can easily extend to engineering nonlinear problems.

Reference

Bayat, M. and Pakar, I. (2011a), "Nonlinear free vibration analysis of tapered beams by Hamiltonian approach", *J. Vib.*, **13**(4), 654-661.

Bayat, M. and Pakar, I. (2011b), "Application of He's Energy Balance Method for Nonlinear vibration of thin circular sector cylinder", *Int. J. Phy. Sci.*, **6**(23),5564-5570.

Bayat, M., Pakar, I. and Shahidi, M. (2011c), "Analysis of nonlinear vibration of coupled systems with cubic nonlinearity", *Mechanika*, **17**(6), 620-629.

Bayat, M. and Pakar, I. (2012a), "Accurate analytical solution for nonlinear free vibration of beams", *Struct. Eng. Mech.*, **43**(3), 337-347.

Bayat, M., Pakar, I. and Domaiirry, G. (2012b), "Recent developments of Some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review", *Lat. Am. J. Solid. Struct.*, **9**(2),145-234 .

Bayat, M., Pakar, I. and Bayat, M. (2013a), "Analytical solution for nonlinear vibration of an eccentrically reinforced cylindrical shell", *Steel Compos. Struct.*, **14**(5), 511-521.

Bayat, M. and Pakar, I. (2013b), "Nonlinear dynamics of two degree of freedom systems with linear and nonlinear stiffnesses", *Earthq. Eng. Eng. Vib.*, **12**(3), 411-420.

Bayat, M. and Pakar, I. (2013c), "On the approximate analytical solution to non-linear oscillation systems", *Shock Vib.*, **20**(1), 43-52.

Bayat, M., Pakar, I. and Cveticanin, L. (2014a), "Nonlinear free vibration of systems with inertia and static type cubic nonlinearities : an analytical approach", *Mech. Mach. Theor.*, **77**, 50-58.

Bayat, M., Pakar, I. and Cveticanin, L. (2014b), "Nonlinear vibration of stringer shell by means of extended Hamiltonian approach", *Arch. Appl. Mech.*, **84**(1), 43-50.

Bayat, M., Bayat, M. and Pakar, I. (2014c), "Nonlinear vibration of an electrostatically actuated microbeam", *Lat. Am. J. Solid. Struct.*, **11**(3), 534-544.

Kuo, B.L. and Lo, C.Y. (2009), "Application of the differential transformation method to the solution of a damped system with high nonlinearity", *Nonlin. Anal.*, **70**(4), 1732-1737.

Cordero, A., Hueso, J.L., Martínez, E. and Torregros, J.R. (2010), "Iterative methods for use with nonlinear discrete algebraic models", *Math. Comput.Model.*, **52**(7-8), 1251-1257.

Dehghan, M. and Tatari, M. (2008), "Identifying an unknown function in a parabolic equation with over specified data via He's variational iteration method", *Chaos, Soliton. Fract.*, **36**(1), 157-166.

Ganji, D.D., Rafei, M., Sadighi, A. and Ganji, Z.Z. (2009), "A comparative comparison of He's method with perturbation and numerical methods for nonlinear vibrations equations", *Int. J. Nonlin. Dyn. Eng. Sci.*, **1**(1), 1-20.

He, J.H. (2007), "Variational approach for nonlinear oscillators", *Chaos, Soliton. Fract.*, **34**(5), 1430-1439.

He, J.H. (2008), "An improved amplitude-frequency formulation for nonlinear oscillators", *Int. J. Nonlin. Sci. Numer. Simul.*, **9**(2), 211-212.

He, J.H. (2002), "Preliminary report on the energy balance for nonlinear oscillators", *Mech. Res. Commun.*, **29**(2), 107-111.

He, J.H. (1999), "Variational iteration method: a kind of nonlinear analytical technique: some examples", *Int. J. Nonlin. Mech.*, **34**(4), 699-708.

He, J.H. (2004), "The homotopy perturbation method for nonlinear oscillators with discontinuities", *Appl. Math. Comput.*, **151**(1), 287-292.

Mehdipour, I., Ganji, D.D. and Mozaffari, M. (2010), "Application of the energy balance method to nonlinear vibrating equations", *Curr. Appl. Phys.*, **10**(1), 104-112.

Nayfeh, A.H. (1973), *Perturbation Methods*, Wiley Online Library.

Odibat, Z., Momani, S. and Erturk, V.S. (2008), "Generalized differential transform method: application to differential equations of fractional order", *Appl. Math. Comput*., **197**(2) , 467-477.

Pakar, I., Bayat, M. and Bayat, M. (2014), "Accurate periodic solution for nonlinear vibration of thick circular sector slab", *Steel Compos. Struct.*, **16**(5), 521-531.

Pakar, I. and Bayat, M. (2011a), "Analytical solution for strongly nonlinear oscillation systems using energy balance method", *Int. J. Phy. Sci.*, **6**(22), 5166-5170.

Pakar, I., Bayat, M. and Bayat, M. (2012a), "On the approximate analytical solution for parametrically excited nonlinear oscillators", *J. Vibroeng.*, **14**(1), 423-429.

Pakar, I. and Bayat, M. (2012b), "Analytical study on the non-linear vibration of Euler-Bernoulli beams", *J.*

Vibroeng., **14**(1), 216-224.

Pakar, I. and Bayat, M. (2013a), “An analytical study of nonlinear vibrations of buckled Euler_Bernoulli beams”, *Acta Physica Polonica A*, **123**(1), 48-52.

Pakar, I. and Bayat, M. (2013b), “Vibration analysis of high nonlinear oscillators using accurate approximate methods”, *Struct. Eng. Mech.*, **46**(1), 137-151.

Shaban, M., Ganji, D.D. and Alipour, A.A. (2010), “Nonlinear fluctuation, frequency and stability analyses in free vibration of circular sector oscillation systems”, *Curr. Appl. Phys.*, **10**(5), 1267-1285.

Shen, Y.Y. and Mo, L.F. (2009), “The max-min approach to a relativistic equation”, *Comput. Math. Appl*., **58**(11), 2131-2133.

Wu, G. (2011), “Adomian decomposition method for non-smooth initial value problems”, *Math. Comput. Model.*, **54**(9-10), 2104-2108.

Xu, N. and Zhang, A. (2009), “Variational approachnext term to analyzing catalytic reactions in short monoliths”, *Comput. Math. Appl.*, **58**(11-12), 2460-2463.

Xu, L. (2008), “Variational approach to solution of nonlinear dispersive $K(m, n)$ equation”, *Chaos, Soliton. Fract.*, **37**(1),137-143.

Zeng, D.Q. and Lee, Y.Y. (2009), “Analysis of strongly nonlinear oscillator using the max–min approach”, *Int. J. Nonlin. Sci. Numer. Simul*., **10**(10), 1361-1368.