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Nonlinear in-plane free oscillations of suspended cable investigated by homotopy analysis method

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Abstract. An analytical solution for the nonlinear in-plane free oscillations of the suspended cable which contains the quadratic and cubic nonlinearities is investigated via the homotopy analysis method (HAM). Different from the existing analytical technique, the HAM is indeed independent of the small parameter assumption in the nonlinear vibration equation. The nonlinear equation is established by using the extended Hamilton's principle, which takes into account the effects of the geometric nonlinearity and quasi-static stretching. A non-zero equilibrium position term is introduced due to the quadratic nonlinearity in order to guarantee the rule of the solution expression. Therefore, the mth-order analytic solutions of the corresponding equation are explicitly obtained via the HAM. Numerical results show that the approximate solutions obtained by using the HAM are in good agreement with the numerical integrations (i.e., Runge-Kutta method). Moreover, the HAM provides a simple way to adjust and control the convergent regions of the series solutions by means of an auxiliary parameter. Finally, the effects of initial conditions on the linear and nonlinear frequency ratio are investigated.

Keywords: suspended cable; nonlinear free oscillation; homotopy analysis method; Runge-Kutta method

1. Introduction

Cables have been widely applied in many mechanical systems and civil structures (Irvine 1981), such as the cable-supported bridges, suspended roofs, post-tensioned concretes, guyed towers and elevators. Generally speaking, the nonlinear dynamics of the cable is very complicated and remains an important area of research in the last few years.

Natural oscillation of an elastic suspended cable in the linearized theory was studied by Irvine and Caughey (1974). Then, the rich nonlinear phenomena of cable dynamics were investigated in early works, such as Hagedorn and Schafer (1980), Rega *et al*. (1984), Luongo *et al*. (1984). The quadratic and cubic nonlinearity terms in the equation of motion of the suspended cable-the former associated with the initial curvature and the latter with stretching of the cable axis - strongly influence the dynamics of the element (Benedettini and Rega 1989, Nayfeh *et al*. 1992).

Among all these researches, the perturbation technology (Nayfeh 1981) is one of the most well-

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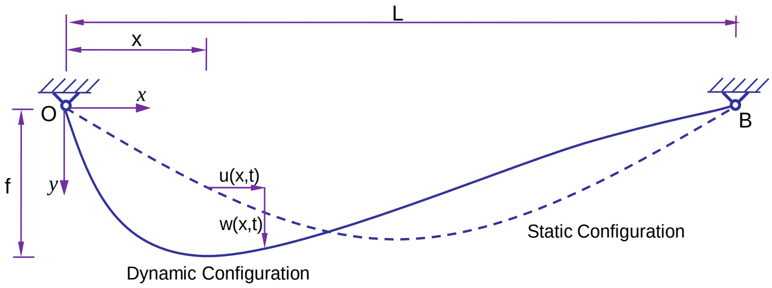


Fig. 1 Two different configurations of the suspended cable

known and important methods in nonlinear dynamics. The abundant nonlinear phenomena of the suspended cable were studied by the multiple scales method (Rega 2004a, Zhao and Wang 2006, Wang and Zhao 2009) and the Lindstedt-Poincare method (Hagedorn and Schafer 1980, Luongo *et al.* 1984). However, these methods should be based on the assumption that the nonlinear systems need to include the small parameters, the so-called *perturbation quantity*, so we often scale the damping and forcing terms to balance the influence of the nonlinearities (Arafat and Nayfeh 2003).

However, tremendous nonlinear cases which do not contain any small parameters exist in both science and engineering, for instance, the free vibrations of the suspended cable. Given the fact that there are some restrictions in the classical perturbation method, so some other techniques have been proposed recently in order to overcome the above mentioned limitation, such as the δ expansion method (Bender *at al.* 1989), the Lyapunov's artificial parameter method (Lyapunov 1992) and the Adomian decomposition method (Adomian *et al.* 1994). Nevertheless, all these non-perturbation methods cannot provide a simple way to control the convergence of the approximation series and adjust regions of convergence.

A basic idea of homotopy from the topology is introduced by Liao (1992), then a general analytic method for solving nonlinear problems was proposed in 1992, namely the homotopy analysis method (HAM). Different from the other perturbation and non-perturbation techniques, the analytical method is indeed independent of the small parameter assumption (Liao 1995) and also provides a simple way to adjust and control the convergent regions of the approximate series solutions by means of an auxiliary parameter. Besides, it was proved that this approach logically includes other non-perturbation methods (Liao 2003). Therefore, the method has been successfully applied in many types of nonlinear problems in engineering (Hoseini *et al.* 2008, Pirbodaghi *et al.* 2009, Qian *et al.* 2012) and theoretical studies (Wen and Cao 2007, Feng and Chen 2009, Chen and Liu 2009) in the last few years. Nevertheless, to the best of our knowledge, there is little literature on the oscillate analysis of suspended cables based on the HAM up to now, so such an attempt is made in this study.

The structure of the paper is organized as follows: firstly, we obtain the nonlinear equation of an elastic cable based on the Hamilton's principle, which takes into account the effects of the geometric nonlinearity and the *quasi-static* stretching assumption. In addition, the partial differential equation of planar motion is reduced to one ordinary equation via the Galerkin procedure by assuming a modal deflection shape in section 2. We extend the application of the HAM to construct the approximate solutions for the governing equation and pursue the numerical series solutions through the HAM in section 3 and 4, respectively. In the end of the paper, the conclusions and future directions of the research are made and given (section 5).

2. Governing equation

2.1 Model of the suspended cable

Fig. 1 displays the homogeneous elastic cable with uniform cross-sectional hanging between two fixed supports at the same level. Two different configurations are distinguished: the initial deformed configuration of static equilibrium under its own weight and the dynamic configuration occupied during the vibration. The displacements are described by $u(x,t)$ and $w(x,t)$ along the longitudinal x and vertical y directions, respectively.

In the present paper, the Cartesian coordinate system O-xy is chosen and the origin dot O is placed at the left fixed point. By applying the Hamilton's principle and the quasi-static stretching assumption, we could express the in-plane nonlinear partial differential equation of motion of the suspended cable without considering the bending, torsional, shear rigidities, the damping and forcing terms as (Rega 2004b)

$$m\frac{\partial^2 w}{\partial t^2}-H\frac{\partial^2 w}{\partial x^2}-\frac{EA}{L}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\int_0^L\left[\frac{\partial w}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x=0 \tag{1}$$

where, m, the mass per unit length of the cable; A, the uniform cross-sectional area of the cable; E, the modulus of elasticity of the cable; H, the horizontal component of the cable tension ($H=mgL^2/8f$, $H/EA<<1$); L, the span of the cable; f, the sag of the cable at the mid-span; g, the acceleration due to gravity.

In our study, because the sag-to-span ratio is sufficiently small ($f/L\leq 1/8$), the static equilibrium configuration of the cable is described well through a parabola

$$y(x)=4f\left[\frac{x}{L}-\left(\frac{x}{L}\right)^2\right] \tag{2}$$

2.2 Development of the equations of motion

In order to make the subsequent section be more general, the following non-dimensional quantities are adopted (Zhao *et al.* 2005)

$$w^*=\frac{w}{L}\qquad x^*=\frac{x}{L}\qquad y^*=\frac{y}{L}\qquad f^*=\frac{f}{L}\qquad t^*=\sqrt{\frac{g}{8f}}t\qquad \alpha=\frac{EA}{H} \tag{3}$$

As a result, Eq. (1) can be written as

$$\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial x^2}-\alpha\left(\frac{\partial^2 w}{\partial x^2}+\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)\int_0^1\left[\frac{\partial w}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x=0 \tag{4}$$

where the asterisks in Eq. (4) are omitted for simplicity and $y(x)=4fx(1-x)$ is the non-dimensional initial parabolic shape of the suspended cable.

The boundary conditions associated to Eq. (4) are given by

$$w(x,t)=0\quad \text{at}\quad x=0\quad \text{and}\quad x=1 \tag{5}$$

Neglecting the nonlinear terms in Eq. (4), the linearized equation of motion could be obtained

$$\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial x^2}=\alpha\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\int_0^1\frac{\partial w}{\partial x}\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x \tag{6}$$

The mode shapes and natural frequencies of the suspended cable can be ascertained by solving Eqs. (5) and (6), so the in-plane nth (n=*odd*) symmetric mode shapes are derived by

$$\varphi_n(x)=\xi_n\left[1-\cos(\omega_n x)-\tan\left(\frac{1}{2}\omega_n\right)\sin(\omega_n x)\right]\qquad(n=1,3,5...) \tag{7}$$

where the mode shapes $\varphi_n(x)$ are normalized, so

$$\int_0^1\varphi_n^2(x)=1 \tag{8}$$

One obtains

$$\xi_n=\sqrt{\frac{2\omega_n\cos^2(\omega_n/2)}{[2+\cos(\omega_n)]\omega_n-3\sin(\omega_n)}} \tag{9}$$

where the natural frequency ω_n in Eq. (9) is obtained by solving the following transcendental equation

$$\tan\left(\frac{1}{2}\omega_n\right)=\frac{1}{2}\omega_n-\frac{1}{2\lambda^2}\omega_n^3\qquad(n=1,3,5...) \tag{10}$$

where $\lambda^2=EA/mgL(8f/L)^3$ is the Irvine parameter. The eigenvalue problems specified by Eq. (10) are strongly nonlinear with respect to this parameter.

The nth (n=*even*) in-plane anti-symmetric mode shapes and corresponding natural frequencies are

$$\varphi_n(x)=\sqrt{2}\sin(n\pi x)\qquad\omega_n=n\pi\qquad(n=2,4,6...) \tag{11}$$

2.3 Discrete modal

The suspended cable could be assumed to be a multi-degree-of-freedom (*MDOF*) dynamic system, which is composed of symmetric modes and anti-symmetric modes with respect to the mid-span of the cable. The Galerkin method can be employed to simplify the nonlinear oscillation equation of motion. Considering the boundary conditions, the solutions of Eq. (1) could be expanded into the following expression

$$w(x,t)=\sum_{n=1}^{N}v_n(t)\varphi_n(x) \tag{12}$$

where N is the number of modes used in the approximation, $v_n(t)$ is an unknown function of time which is a generalized coordinate of system response and $\varphi_n(x)$ is a space coordinate function which satisfies the associated linear problem.

A set of nonlinear ordinary differential equations are yielded by substituting Eqs. (12) into (4). For the sake of simplicity, the present study is only restricted to N=1. By multiplying the first symmetric mode shape function $\varphi_1(x)$ and integrating the outcomes over $x \in [0,1]$, the equation of motion is finally reduced to

$$\ddot{v}_1(t) + b_1 v_1(t) + b_2 v_1^2(t) + b_3 v_1^3(t) = 0 \tag{13}$$

where the dots indicate differentiation with respect to t and the expressions of the coefficients $b_i(i$=1,2,3,) in the Eq. (13) are presented in Appendix. In Eq. (13), the initial conditions are assumed to be

$$v_1(0) = b_0 + \delta \qquad \dot{v}_1(0) = 0 \tag{14}$$

where b_0 is the initial condition which denotes the non-dimensional maximum amplitude of oscillations and δ is the non-zero equilibrium position term.

3. Solution via homotopy analysis method

In this section, the nonlinear response of the suepended cable is explored by the HAM which transforms a nonlinear problem into an infinite number of linear problems with an embedding parameter q that typically varies from 0 to 1. As have been discussed in section 2.3, the first order discrete equation of motion via the Galerkin procedure could be obtained. Introducing a new time scale $\tau = \omega t$ (ω is the nonlinear vibration frequency) and taking into account the quadratic nonlinear term, we suppose

$$v_1(t) = u(\tau) + \delta \tag{15}$$

Under the new time scale transformation, the new form of Eq. (13) is expressed as

$$\omega^2 \ddot{u}(\tau) + b_1\left[u(\tau) + \delta\right] + b_2\left[u(\tau) + \delta\right]^2 + b_3\left[u(\tau) + \delta\right]^3 = 0 \tag{16}$$

where the notation $\ddot{u}(\tau) = \mathrm{d}^2 u(\tau)/\mathrm{d}\tau^2$ is used to alleviate the text.

The corresponding initial conditions are

$$u(0) = b_0 \qquad \dot{u}(0) = 0 \tag{17}$$

It should be pointed out that our HAM approximation highly depends on the initial condition b_0. Given the fact that the free oscillations of a conservative system without damping effect could be expressed by a series of periodic functions which satisfy the initial conditions, the displacement solution in Eq. (16) can be denoted by the following base functions

$$\left\{\cos(k\tau) \middle| k = 1, 2, 3...\right\} \tag{18}$$

Obviously, the solutions of Eq. (16) could be expressed as

$$u(\tau) = \sum_{k=1}^{+\infty} C_k \cos(k\tau) \tag{19}$$

Considering the rule of solution expression and the initial conditions in Eq. (17), it is straightforward that the initial guess of $u(\tau)$ could be chosen as

$$u_0(\tau) = b_0 \cos\tau \tag{20}$$

To construct the homotopy function, one could define the linear auxiliary operator as

$$\mathcal{L}\left[\Phi(\tau;q)\right] = \omega_0^2\left[\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2} + \Phi(\tau;q)\right] \tag{21}$$

which has the property

$$\mathcal{L}\left[C_1\sin(\tau) + C_2\cos(\tau)\right] = 0 \tag{22}$$

where C_1 and C_2 are integral constants to be determined by the initial conditions. Furthermore, it is noticed that the rule of solution expression plays an important role in defining the linear operator.

According to Eq. (16), we could define the nonlinear operator as

$$\begin{aligned}&\mathcal{N}\left[\Phi(\tau;q),\Delta(q),\Omega(q)\right] = \\ &\Omega^2(q)\left[\frac{\partial^2\Phi(\tau;q)}{\partial\tau^2}\right] + b_1\left[\Phi(\tau;q)+\Delta(q)\right] + b_2\left[\Phi(\tau;q)+\Delta(q)\right]^2 + b_3\left[\Phi(\tau;q)+\Delta(q)\right]^3\end{aligned} \tag{23}$$

where the unknown function $\Phi(\tau;q)$ is a mapping of $u(\tau)$, the unknown functions $\Omega(q)$ and $\Delta(q)$ are some kinds of mapping of the unknown frequency ω and the equilibrium position δ. In accordance with the HAM, we construct the so-called zeroth-order deformation equation as

$$(1-q)\mathcal{L}\left[\Phi(\tau;q) - u_0(\tau)\right] = q\hbar H(\tau)\mathcal{N}\left[\Phi(\tau;q),\Delta(q),\Omega(q)\right] \tag{24}$$

which is subjected to the initial conditions

$$\Phi(0;q) = b_0 \qquad \frac{\partial\Phi(\tau;q)}{\partial\tau}\Big|_{\tau=0} = 0 \tag{25}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is an auxiliary convergence control parameter and $H(\tau)$ is an auxiliary function, $\mathcal{L}/\mathcal{N}$ is an auxiliary linear/nonlinear operator and $u_0(\tau)$ is an initial guess of $u(\tau)$.

For the sake of simplicity in our research, we choose

$$H(\tau) = 1 \tag{26}$$

Therefore, with the increase in the embedding parameter q from 0 to 1, $\Phi(\tau;q)$ varies continuously from the initial guess $u_0(\tau)$ to the exact solution $u(\tau)$. So does $\Omega(q)$ from its initial frequency ω_0 to the nonlinear physical frequency ω. Similarly, $\Delta(q)$ varies from the initial approximation δ_0 to the equilibrium position δ of the system. Herein, the ω_0 and δ_0 are unknown zeroth-order parameters which would be determined later.

Using the Taylor series expansion and considering the so-called deformation derivatives, we yield

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)q^m \qquad \Delta(q) = \delta_0 + \sum_{m=1}^{+\infty}\delta_m q^m \qquad \Omega(q) = \omega_0 + \sum_{m=1}^{+\infty}\omega_m q^m \tag{27}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m\Phi(\tau;q)}{\partial q^m}\Big|_{q=0} \qquad \delta_m = \frac{1}{m!}\frac{\partial^m\Delta(q)}{\partial q^m}\Big|_{q=0} \qquad \omega_m = \frac{1}{m!}\frac{\partial^m\Omega(q)}{\partial q^m}\Big|_{q=0} \tag{28}$$

The $\hbar$ is an important auxiliary parameter that determines the convergent regions of the system. Furthermore, assuming that the auxiliary parameter $\hbar$ is properly chosen, all the series solutions are converged at q=1, thus the series solutions could be written as

$$u(\tau)=\Phi(\tau;1)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau) \qquad \delta=\Delta(1)=\delta_0+\sum_{m=1}^{+\infty}\delta_m \qquad \omega=\Omega(1)=\omega_0+\sum_{m=1}^{+\infty}\omega_m \tag{29}$$

For the sake of brevity and simplicity, we define the following vectors

$$\mathsf{U}_m=\{u_0(\tau),u_1(\tau),...,u_m(\tau)\} \qquad \Delta_m=\{\delta_0,\delta_1,...,\delta_m\} \qquad \Omega_m=\{\omega_0,\omega_1,...,\omega_m\} \tag{30}$$

Differentiating the zeroth-order deformation equation m times with respect to the embedding parameters q, then dividing the resulted equation by m! and finally setting q=0, we could obtain the mth-order deformation equation as

$$\mathcal{L}\left[u_m(\tau)-\chi_m u_{m-1}(\tau)\right]=\hbar R_m(\mathsf{U}_{m-1},\Delta_{m-1},\Omega_{m-1}) \tag{31}$$

which is subjected to the initial conditions

$$u_m(0)=0 \qquad \dot{u}_m(0)=0 \qquad (m\geq 1) \tag{32}$$

where

$$\chi_m=\begin{cases}0, & m\leq 1\\ 1, & m>1\end{cases} \tag{33}$$

and

$$\begin{aligned} R_m &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}\left[\Phi(\tau;q),\Delta(q),\Omega(q)\right]}{\partial q^{m-1}}\Big|_{q=0} \\ &= \sum_{k=0}^{m-1}\sum_{p=0}^{k}\omega_p\omega_{k-p}\ddot{u}_{m-1-k}(\tau)+b_1\left[u_{m-1}(\tau)+\delta_{m-1}\right] \\ &\quad + b_2\left\{\sum_{k=0}^{m-1}\left[u_k(\tau)u_{m-1-k}(\tau)+\delta_k\delta_{m-1-k}+2u_k(\tau)\delta_{m-1-k}\right]\right\} \\ &\quad + b_3\left\{\sum_{k=0}^{m-1}\sum_{p=0}^{k}u_p(\tau)u_{k-p}(\tau)u_{m-1-k}(\tau)+\sum_{k=0}^{m-1}\sum_{p=0}^{k}\delta_k\delta_{k-p}\delta_{m-1-k}\right\} \\ &\quad + 3b_3\left\{\sum_{k=0}^{m-1}\sum_{p=0}^{k}u_p(\tau)u_{k-p}(\tau)\delta_{m-1-k}+\sum_{k=0}^{m-1}\sum_{p=0}^{k}\delta_p\delta_{k-p}u_{m-1-k}(\tau)\right\} \end{aligned} \tag{34}$$

It could be observed that $u_m(\tau)$, δ_{m-1} and ω_{m-1} are three unknown functions. Considering that only one equation could be utilized to solve $u_m(\tau)$, so two additional algebraic equations are required to determine the δ_{m-1} and ω_{m-1}. Moreover, it is found that the right hand side of the mth-order deformation equation could be expressed as

$$R_m(\mathsf{U}_{m-1},\Delta_{m-1},\Omega_{m-1})=c_{m,0}(\Delta_{m-1},\Omega_{m-1})+\sum_{k=1}^{\mu_m}c_{m,k}(\Delta_{m-1},\Omega_{m-1})\cos(k\tau) \tag{35}$$

where $c_{m,0}$ is the coefficient of the constant term, $c_{m,k}$ is the coefficient of cos($k\tau$) and μ_m is the positive integral dependent on order m. According to the property of the auxiliary linear operator $\mathcal{L}$, it is worth noticing that if $c_{m,1}\neq 0$, the solution of the *m*th-order deformation equation would contain the so-called secular term ($\tau\cos\tau$) which breaches the rule of solution expression. Moreover, when $c_{m,0}\neq 0$, the solution contains a drift constant term $c_{m,0}/\omega^2$, which breaks the rule of solution expression as well. Hence, in order to avoid the presence of such terms, their coefficients must be set to zero

$$c_{m,0}\left(\Delta_{m-1},\Omega_{m-1}\right)=0 \qquad c_{m,1}\left(\Delta_{m-1},\Omega_{m-1}\right)=0 \qquad \left(m=1,2,3,...\right) \tag{36}$$

which provides us with two additional algebraic equations for solving ω_{m-1} and δ_{m-1}. Consequently, as long as the values of b_1, b_2, b_3 and b_0 are given, the periodic solutions can be determined by the analytical approach.

For instance, when m=1

$$\begin{aligned} R_1\left[u_0(\tau),\delta_0,\omega_0\right] &= \omega_0^2\ddot{u}_0(\tau)+b_1\left[u_0(\tau)+\delta_0\right]+b_2\left[u_0(\tau)+\delta_0\right]^2+b_3\left[u_0(\tau)+\delta_0\right]^3 \\ &= \left[b_0b_1+\frac{3}{4}b_0^3b_3+2b_0b_2\delta_0+3b_0b_3\delta_0^2-b_0\omega_0^2\right]\cos(\tau) \\ &\quad +\left[\frac{1}{2}b_0^2b_2+\frac{2}{3}b_0^2b_3\delta_0\right]\cos(2\tau)+\left(\frac{1}{4}b_0^3b_3\right)\cos(3\tau) \\ &\quad +\left(\frac{1}{2}b_0^2b_2+b_1\delta_0+\frac{3}{2}b_0^2b_3\delta_0+b_2\delta_0^2+b_3\delta_0^3\right) \end{aligned} \tag{37}$$

Satisfying the rule of solution expression, the coefficients $c_{1,0}(\delta_0,\ \omega_0)$ and $c_{1,1}(\delta_0,\ \omega_0)$ must vanish, so we could get two additional algebraic equations about ω_0 and δ_0

$$\begin{cases} \frac{1}{2}b_0^2b_2+b_1\delta_0+\frac{3}{2}b_0^2b_3\delta_0+b_2\delta_0^2+b_3\delta_0^3=0 \\ b_1+\frac{3}{4}b_0^2b_3+2b_2\delta_0+3b_3\delta_0^2-\omega_0^2=0 \end{cases} \tag{38}$$

The solutions of Eq. (38) are

$$\begin{cases} \delta_0=-\frac{b_2}{3b_3}-\frac{\Gamma_1}{3b_3}\sqrt[3]{\frac{2}{\Gamma_2+\sqrt{\Gamma_2^2+32\Gamma_1^3}}}+\frac{1}{6b_3}\sqrt[3]{\frac{\Gamma_2+\sqrt{\Gamma_2^2+32\Gamma_1^3}}{2}} \\ \omega_0=\frac{1}{2}\sqrt{4b_1+3b_0^2b_3+8b_2\delta_0+12b_3\delta_0^2} \end{cases} \tag{39}$$

where

$$\Gamma_1=-2b_2^2+6b_1b_3+9b_0^2b_3^2 \qquad \Gamma_2=-16b_2^3+72b_1b_2b_3 \tag{40}$$

Eliminating the secular term and considering the expression of linear operator, the first-order deformation equation becomes

$$\omega_0^2\left[\ddot{u}_1(\tau)+u_1(\tau)\right]=\hbar\left[\left(\frac{1}{2}b_0^2b_2+\frac{2}{3}b_0^2b_3\delta_0\right)\cos(2\tau)+\left(\frac{1}{4}b_0^3b_3\right)\cos(3\tau)\right] \tag{41}$$

It is easy to solve the linear ordinary differential equation with the initial conditions ($u_1(0)=0$, $\dot{u}_1(0)=0$), so the first-order approximation is

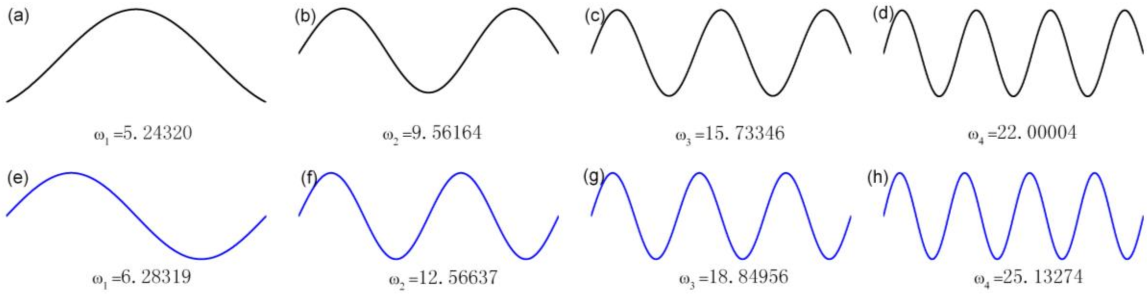


Fig. 2 First eight in-plane mode shapes and natural frequencies of the suspended cable: (a)-(d) the first four symmetric mode shapes and natural frequencies; (e)-(h) the first four anti-symmetric mode shapes and natural frequencies

$$u_1(\tau)=\frac{\hbar}{96\omega_0^2}\Big[\left(16b_0^2b_2+3b_0^3b_3+48b_0^2b_3\delta_0\right)\cos\tau-\left(16b_0^2b_2+48b_0^2b_3\delta_0\right)\cos 2\tau+\left(3b_0^3b_3\right)\cos 3\tau\Big] \tag{42}$$

Following the same procedure, the mth-order ($m\geq 2$) approximation of ω_{m-1}, δ_{m-1} and $u_m(\tau)$ can be obtained and their expressions become more and more complicated. Therefore, the general periodic solution $u_m(\tau)$ of Eq. (16) is obtained from

$$u_m(\tau)=\chi_m u_{m-1}(\tau)+C_1\sin\tau+C_2\cos\tau+\frac{\hbar}{\omega_0^2}\sum_{k=2}^{\mu_m}\frac{c_{m,k}\left(\Delta_{m-1},\Omega_{m-1}\right)}{1-k^2}\cos k\tau \tag{43}$$

where C_1 must be forced to zero in order to obey the rule of solution expression and C_2 is a constant that could be determined by the initial conditions given by Eq. (32). Accordingly, the mth-order analytic approximate solutions of the δ, ω and u(τ) are derived

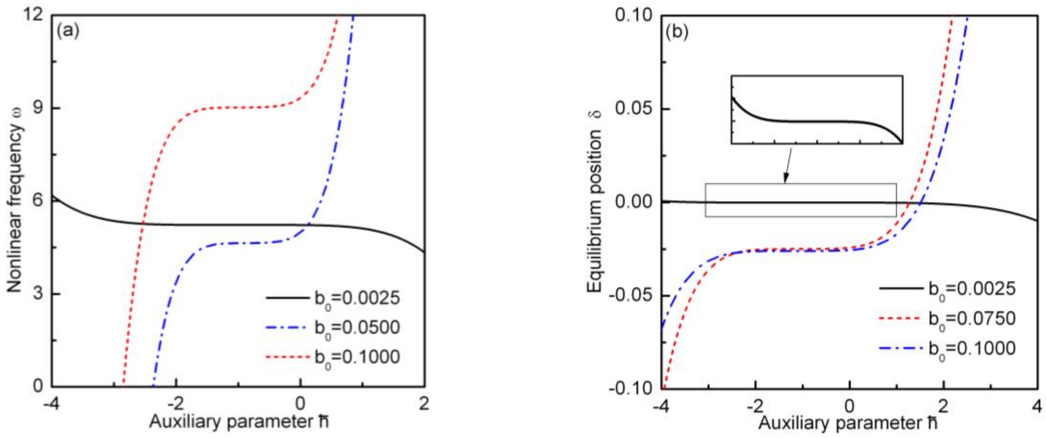
$$u(\tau)\approx u_0(\tau)+\sum_{m=1}^{m}u_m(\tau)\qquad \delta\approx\delta_0+\sum_{m=1}^{m}\delta_m\qquad \omega\approx\omega_0+\sum_{m=1}^{m}\omega_m \tag{44}$$

4. Numerical results and discussions

In this study, to represent the condition currently found in the civil and electric engineering, the following non-dimensional parameter values are chosen: $\alpha=EA/H$=198.25, f/L=0.042, with the associated value of the Irvine parameter: λ^2=22.563.The first eight symmetric and anti-symmetric natural frequencies and mode shapes are shown in Fig. 2.

It is certainly worthy to note that the series solutions contain the auxiliary parameter $\hbar$, which provides us a simple way to adjust and control the convergent regions of the solutions. Fig. 3 shows the influence of the auxiliary parameter $\hbar$ on the series solutions ω and δ by the 5th-order approximation in the case of three different initial conditions. The regions where the distribution of ω and δ versus $\hbar$ to different values of b_0 are horizontal lines which are so-called convergent regions for the corresponding functions. As shown in the figure, the initial condition b_0 is likely to have a quite significant effect on the region of convergence, and these regions vary as a result of the variation of the initial conditions b_0 for the same order approximations.

As the Fig. 3 shows, the nonlinear frequency ω does not increase as the rise in the value of



(a) nonlinear frequency ω (b) equilibrium position δ

Fig. 3 Influences of the auxiliary parameter on the approximate series solutions given by the 5th-order approximation in the case of three different initial conditions

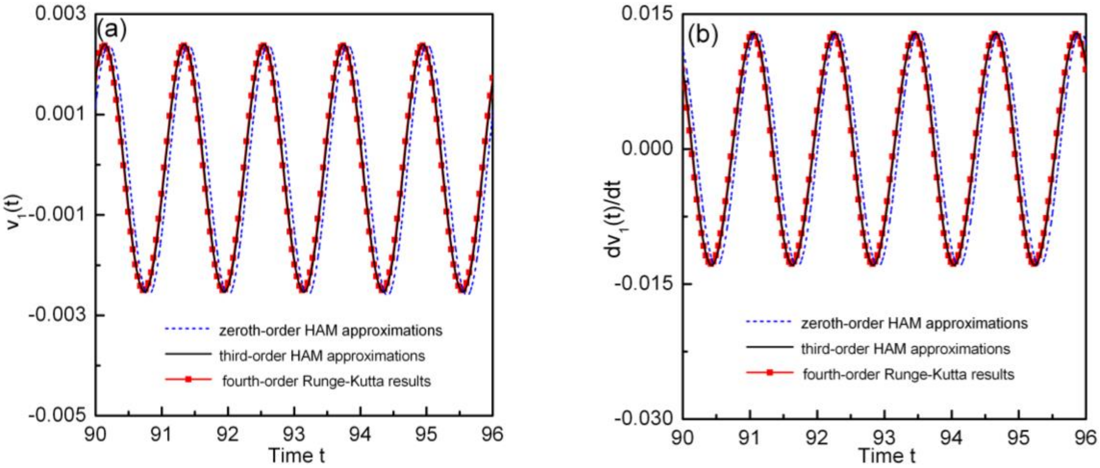
Table 1 Analytic approximations of the frequency and equilibrium position when b_0=0.0025 and $\hbar$=−1.0

Order of approximation	Frequency ω	Error: $\left\|\frac{\omega_n-\omega_{n-1}}{\omega_n}\right\|\times 100\%$	Equilibrium position δ	Error: $\left\|\frac{\delta_n-\delta_{n-1}}{\delta_n}\right\|\times 100\%$
1st	5.22787876	0.0660663%	-1.0218394×10^{-4}	2.8354553%
2nd	5.22778464	0.0018004%	-1.0227150×10^{-4}	0.0856098%
3rd	5.22778235	0.0000437%	-1.0227308×10^{-4}	0.0015467%
4th	5.22778268	0.0000063%	-1.0227279×10^{-4}	0.0002848%
5th	5.22778268	0.0000001%	-1.0227279×10^{-4}	0.0000069%

b_0($\omega_{0.1}\geq\omega_{0.0025}\geq\omega_{0.05}$), so in a large range of low values of δ_0, the dynamic behavior is generally softening while at high values it becomes hardening due to the definite prevailing of the cubic term. Correspondingly, the values of equilibrium position b_0 increase due to the rise in the values of b_0. Moreover, results show that, in order to determine the region of convergence, higher order approximation should be adopted provided that the initial condition b_0 is large. Hence, we can choose the appropriate value of $\hbar$ corresponding to different b_0 in order to ensure the convergence of all the series solutions. For the sake of convenience and simplicity, we select $\hbar$=−1.0 in this study.

In order to ascertain the order of the approximate solutions, different orders of the frequency and equilibrium position are given in Table. 1, and also the errors are listed. In this table, it is found that the series solutions converge very quickly, for the sake of simplicity, we could make the order of approximate series solutions equals third in our following computation.

By substituting the convergent analytic results of the equilibrium position δ into the initial conditions ($v_1(0)=b_0+\delta$, $\dot{v}_1(0)=0$), the fourth-order Runge-Kutta method is utilized to solve the ordinary differential equation of the suspended cable. Fig. 4 shows the solutions of displacement and velocity obtained by using the zeroth and third order HAM approximations and the Runge Kutta method. As shown in Fig. 4, the zeroth-order and third-order analytic solutions obtained by using the HAM are in line with the numerical results based on the Runge-Kutta method. The Fig. 5



(a) displacement $v_1(t)$ (b) velocity $\dot{v}_1(t)$

Fig. 4 Comparison of the HAM approximations and numerical solutions when $\hbar$=−1.0 and b_0=0.0025

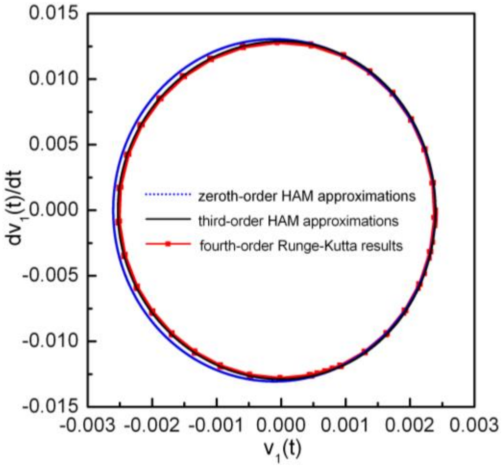


Fig. 5 Phase-space curves of the suspended cable by means of $\hbar$=−1.0 when b_0=0.0025

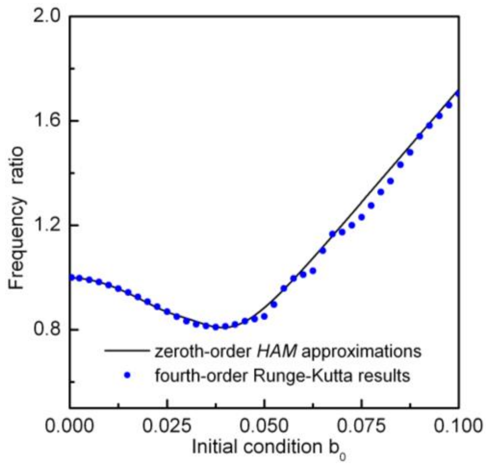


Fig. 6 The ratio between nonlinear and linear frequency versus initial condition b_0

shows the phase-space curves obtained with the HAM and numerical integrations. The comparison of the numerical integrations and the analytical series solutions show that using approximation of low order results in satisfactory accuracy. Furthermore, it is quite remarkable that the higher-order HAM approximate solutions are more accurate than the lower-order ones.

Fig. 6 illustrates the frequency ratio between the nonlinear and linear one in the case of different initial conditions. As shown in Fig. 6, it is interesting to find out that the response of the suspended cable is softening at low initial conditions due to the great value of the coefficient of the quadratic term, but it becomes hardening as the amplitude increases due to the large value of the coefficient of the cubic term. However, a limited range of initial conditions should be considered for the curve if the error must be taken under a prescribed value.

5. Conclusions

In this paper, an analytical technique, namely the homotopy analysis method (HAM) is implemented in nonlinear free oscillations of the suspended cable successfully. Particularly, the initial curvature leads to the quadratic nonlinear term in the equation of motion, so an equilibrium position is introduced to guarantee the rule of solution expression.

As long as the physical parameters of the suspended cable are given, the first eight in-plane symmetric and anti-symmetric mode shapes and natural frequencies can be achieved. The $\hbar$ -curve illustrates that the valid regions of convergence of the series solutions are related with the initial conditions. Moreover, as to the large value of the initial condition, higher order approximations should be adopted. The approximate series solutions of the corresponding frequency, equilibrium position, displacement and velocity are explicitly obtained, which agree well with the 4th-order Runge-Kutta numerical results. It is found that the series solutions converge very rapidly and the lower-order approximation achieve high degree of accuracy. Numerical results obtained by using the HAM show that the response of the suspended cable is softening at low initial conditions, but it becomes hardening as the amplitude increases.

Our experiences in the study and some other related researches reveal that the HAM is indeed a promising analytic technique for many types of nonlinear problems in engineering practices and theoretical studies, as long as the auxiliary linear and nonlinear operator, initial guess, auxiliary parameter and auxiliary function are properly chosen.

However, it should be pointed out that in this study, we only consider the first-order discrete model and just take into account the free vibrations of the system, so it is necessary to carry out further investigations and achieve more improvements in the future.

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Appendix

$$b_1 = -\int_0^1 \varphi_1''(x)\varphi_1(x)\,\mathrm{d}x - \int_0^1 \left[\alpha y''(x)\int_0^1 y'(x)\varphi_1'(x)\mathrm{d}x\right]\varphi_1(x)\,\mathrm{d}x \tag{1}$$

$$b_2 = -\int_0^1 \left[\alpha\varphi_1''(x)\int_0^1 y'(x)\varphi_1'(x)\mathrm{d}x\right]\varphi_1(x)\,\mathrm{d}x - \int_0^1 \left[\frac{1}{2}\alpha y''(x)\int_0^1 \varphi_1'^2(x)\mathrm{d}x\right]\varphi_1(x)\,\mathrm{d}x \tag{2}$$

$$b_3 = -\int_0^1 \left[\frac{1}{2}\alpha\varphi_1''(x)\int_0^1 \varphi_1'^2(x)\mathrm{d}x\right]\varphi_1(x)\,\mathrm{d}x \tag{3}$$