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Generalized shear deformation theory for thermo elastic analyses of the Functionally Graded Cylindrical shells

M. Arefi*

Department of Solid Mechanic, Faculty of Mechanical Engineering, University of Kashan, Kashan 87317-51167, Iran

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Abstract. The present paper addresses a general formulation for the thermo elastic analysis of a functionally graded cylindrical shell subjected to external loads. The shear deformation theory and energy method is employed for this purpose. This method presents the final relations by using a set of second order differential equations in terms of integral of material properties along the thickness direction. The proposed formulation can be considered for every distribution of material properties, whether functional or non functional. The obtained formulation can be used for manufactured materials or structures with numerical distribution of material properties which are obtained by using the experiments. The governing differential equation is applied for two well-known functionalities and some previous results are corrected with present true results.

Keywords: shear deformation theory; thermo elastic; shell; temperature; cylinder; energy

1. Introduction

Functionally graded materials have been created for the first time in laboratory by a Japanese group of material scientist. The properties of these materials can change gradually and continuously along the coordinate system. These variable properties can be simulated using an analytical function. This functional distribution of material properties can aid researchers in order to control the distribution of displacement or stress in a solid structure. It is reported various methods of production for manufacturing the functionally graded materials. It can be investigated that the manufacturing of a functionally graded materials with predefined functionality is very difficult. It is observed a deviation from first goal in manufactured structure due to unwanted conditions of production.

Lame (1976) studied the exact solution of a thick walled cylinder under inner and outer pressures. It was supposed that the cylinder to be axisymmetric and isotropic. This solution was applicable for simple and quick solution of the pressure vessels. Naghdi and Cooper (1956) considered the effect of lateral shear and consequently, constitutes the theory of shear deformation. Mirsky and Hermann (1958) employed the first order shear deformation theory for analysis of an isotropic cylinder.

*Corresponding author, Assistant Professor, E-mail: aref@kashanu.ac.ir; arefi63@gmail.com

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In the beginning of decade 1980, one Japanese group of material scientist, has created new class of materials. Properties of this material vary continuously along the thickness of the structure. In the beginning of decade 1990, it was started researches about the mechanical, thermal and vibration analysis of FG materials (1990). Tutuncu and Ozturk (2001) presented exact solution of FG spherical and cylindrical pressure vessels. Jabbari *et al*. (2002) analyzed the thermo elastic analysis of a FG cylinder under thermal and mechanical loads. It was supposed that the material properties vary as a power function in terms of radial coordinate system. By substituting the derived temperature field into navier equation, governing differential equation was solved, analytically.

Shao (2005) investigated thermo elastic analysis of a thick walled cylinder under mechanical and thermal loads. The cylinder was divided into many annular sub cylinders. It was assumed that every sub cylinders to be isotropic. Based on this assumption, the heat transfer and equilibrium equations were employed for every sub cylinder, individually. Tutuncu (2007) used the exponential varying for material distribution of a functionally graded cylinder. Jabbari *et al*. (2009) investigated the thermo elastic behavior of a FG cylinder under thermal and mechanical loads. Primarily, they employed two-dimensional differential equation of heat transfer for different boundary conditions. By regarding two differential equations of equilibrium in cylindrical coordinate system and imposing the temperature distribution, they obtained two navier equations in terms of two unknown components of displacement.

The power function distribution has been considered as a well-known model for simulation of variable material properties. The exponential distribution has been considered as the other model for simulation of variable material properties. The solution procedure and method for one functionality differs from the other, basically. This difference can be justified with reviewing two most visited articles in context of functionally graded structures. The first paper devotes to Jabbari *et al*. (2009) work where the thermo elastic analysis of a functionally graded cylindrical shell under pressure has been developed. The functionality has been simulated by using a simple power function. The problem has been lead to a Cauchy-Euler differential equation and the obtained differential equation has been solved with the simple power solution. The other work has been performed by Tutuncu and ozturk (2007). The exponential varying has been employed for simulation of variable material properties. The obtained differential equation has not been any analytical solution. Due to this problem, the differential equation has been solved by using the Frobenius method, where considered a series solution for evaluation of the results. It is obvious that for other functional or non functional distribution, the problem must be solved using the numerical methods. For example, for a sinusoidal distribution of material properties, the designer must rewrite the differential equation, fundamentally. After derivation of differential equation, there is no confidence guarantee for straightforward solution of the obtained differential equations. This incompleteness has forced designer and researchers for performing the individual analyses for every distribution of material properties.

Due to this incompleteness, the present paper proposes an analytical method for solution of functionally graded structures with every functional or non functional distribution of material properties based on the shear deformation theory. A comprehensive reviewing the literature containing the pressure vessel problems and furthermore works in literature review about the functionally graded materials justified presentation of this study, carefully.

The present paper develops the shear deformation theory in order to present the comprehensive methodology and formulation for thermo elastic analysis of a functionally graded cylindrical shell. Derived differential equations are containing numerous coefficients in terms of integration of variable properties along the thickness direction. These formulations have many advantageous for application in various problems with different functional or non functional distribution of material properties.

Some application of shear deformation theory for thermo elastic and electro elastic analysis of functionally graded and functionally graded piezoelectric materials have been performed by Arefi and Rahimi (2012).

2. Formulation

In order to use shear deformation theory, the author must reminder the Lame's solution for a cylindrical pressure vessel. The radial distribution of the radial displacement based on the mentioned solution is (Arefi and Rahimi 2012)

$$u = c_1 r + \frac{c_2}{r} \tag{1}$$

where r is the radial component of coordinate system (Fig. 1). The radial component can be decomposed into radius of mid-plane *R* and distance of every layer with respect to mid-plane *z* as follows (Arefi and Rahimi 2012)

$$\mathrm{r{=}R{+}z} \tag{2}$$

By substitution of *r* from Eq. (2) into Lame's solution (Eq. (1)) and applying the Taylor expansion, Eq. (1) may be rewritten as a function of *z* as follows

$$u = c_1(\mathrm{R+z}) + \frac{c_2}{\mathrm{R+z}} = c_0' + c_1'z + c_2'z^2 + c_3'z^3 + \ldots + c_n'z^n \tag{3}$$

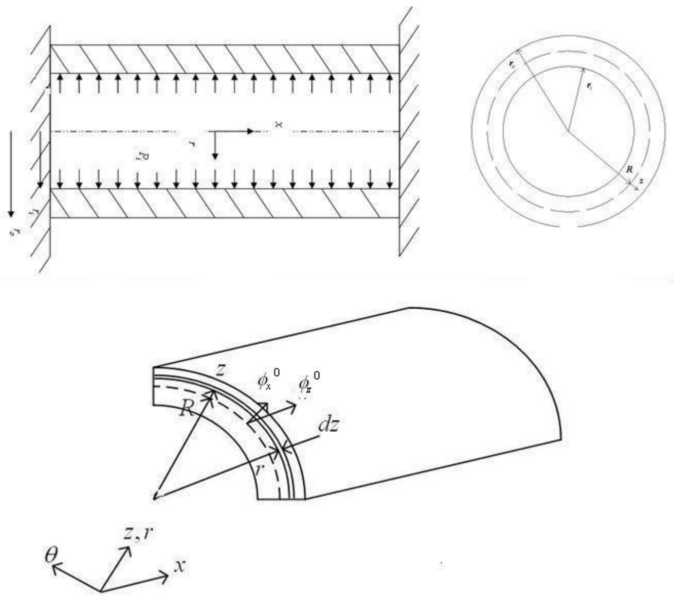

Fig. 1 The schematic figure of a pressurized FG cylindrical shell

This formulation (Eq. (3)) is known as the shear deformation theory (SDT). By setting n=1, first order (FSDT), by setting n=2, second order (SSDT) shear deformation and by setting n=3, third order (TSDT) shear deformation theories are employed. For a symmetric cylindrical shell and by using Eq. (3), the radial and axial components of deformation may be considered as follows

$$\begin{Bmatrix} u_x \\ w_z \end{Bmatrix} = \begin{Bmatrix} \phi_x{}^0 \\ \phi_z{}^0 \end{Bmatrix} + z \begin{Bmatrix} \phi_x{}^1 \\ \phi_z{}^1 \end{Bmatrix} + z^2 \begin{Bmatrix} \phi_x{}^2 \\ \phi_z{}^2 \end{Bmatrix} + ... + z^n \begin{Bmatrix} \phi_x{}^n \\ \phi_z{}^n \end{Bmatrix} \tag{4}$$

where (u_x, w_z) are displacement vector containing the axial and radial component, respectively. $\phi_x{}^0$, $\phi_z{}^0$, $\phi_x{}^1$, $\phi_z{}^1$ are functions of axial component of coordinate system (x), only. The strain component can be obtained by using Eq. (4) as follows

$$\begin{cases} \varepsilon_x = \dfrac{\partial u_x}{\partial x} = \dfrac{\partial \phi_x{}^0}{\partial x} + z \dfrac{\partial \phi_x{}^1}{\partial x} + z^2 \dfrac{\partial \phi_x{}^2}{\partial x} + ... + z^n \dfrac{\partial \phi_x{}^n}{\partial x} \\ \varepsilon_z = \dfrac{\partial w_z}{\partial z} = \phi_z{}^1 + 2z\,\phi_z{}^2 + 3z^2 \phi_z{}^3 + ... + nz^{n-1} \phi_z{}^n \\ \varepsilon_t = \dfrac{w_z}{r} = \dfrac{\phi_z{}^0 + z\,\phi_z{}^1 + z^2 \phi_z{}^2 + ... + z^n \phi_z{}^n}{R+z} \\ \gamma_{xz} = 2 \times \varepsilon_{xz} = \dfrac{\partial u_x}{\partial z} + \dfrac{\partial w_z}{\partial x} = \phi_x{}^1 + 2z\,\phi_x{}^2 + ... + nz^{n-1} \phi_x{}^n \\ + \dfrac{\partial \phi_z{}^0}{\partial x} + z \dfrac{\partial \phi_z{}^1}{\partial x} + z^2 \dfrac{\partial \phi_z{}^2}{\partial x} + ... + z^n \dfrac{\partial \phi_z{}^n}{\partial x} = \\ \phi_x{}^1 + \dfrac{\partial \phi_z{}^0}{\partial x} + z\,[2\phi_x{}^2 + \dfrac{\partial \phi_z{}^1}{\partial x}] + z^2 [3\phi_x{}^3 + \dfrac{\partial \phi_z{}^2}{\partial x}] \\ + ... + z^n [(n+1)\phi_x{}^{n+1} + \dfrac{\partial \phi_z{}^n}{\partial x}] \end{cases} \tag{5}$$

where, ε_x, ε_z, ε_t are the axial, radial and circumferential strain component, respectively and ε_{xz} is the shear component of strain. By having the strain component, the constitutive relations can be derived as follows

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x - v(\sigma_t + \sigma_z)}{E} + \alpha T \\ \varepsilon_z = \dfrac{\sigma_z - v(\sigma_t + \sigma_x)}{E} + \alpha T \\ \varepsilon_t = \dfrac{\sigma_t - v(\sigma_x + \sigma_z)}{E} + \alpha T \\ \gamma_{xz} = 2\varepsilon_{xz} = \dfrac{\tau_{xz}}{G} \end{cases} \tag{6}$$

Conversely, the stress component in terms of strain components is derived by using Eq. (6) as

$$\begin{cases} \sigma_x = \dfrac{E}{(1+v)(1-2v)}\left\{(1-v)\varepsilon_x + v(\varepsilon_t + \varepsilon_z)\right\} - \dfrac{\alpha TE}{1-2v} \\ \sigma_t = \dfrac{E}{(1+v)(1-2v)}\left\{(1-v)\varepsilon_t + v(\varepsilon_x + \varepsilon_z)\right\} - \dfrac{\alpha TE}{1-2v} \\ \sigma_z = \dfrac{E}{(1+v)(1-2v)}\left\{(1-v)\varepsilon_z + v(\varepsilon_x + \varepsilon_t)\right\} - \dfrac{\alpha TE}{1-2v} \\ \tau_{xz} = \dfrac{E}{2(1+v)}\left\{\gamma_{xz}\right\} \end{cases} \tag{7}$$

To arrive the functional of the system, one can employ the energy method. The strain energy per unit volume of the structure is derived by using one half of inner product of stress components in the corresponding strain components. Recalling the Eqs. (5), (7), the strain energy per unit volume of the structure $\bar{u}$ is obtained as follows

$$\begin{aligned} &\bar{u} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\sigma_x\varepsilon_x + \sigma_z\varepsilon_z + \sigma_t\varepsilon_t + \tau_{xz}\gamma_{xz}\} \\ &\frac{1}{2}\frac{E}{(1+v)(1-2v)}\left[(1-v)(\varepsilon_x^{\,2} + \varepsilon_z^{\,2} + \varepsilon_t^{\,2}) + 2v(\varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_t + \varepsilon_t\varepsilon_z) + \frac{1-2v}{2}\gamma_{xz}^{\,2}\right] \\ &-\frac{\alpha TE}{2(1-2v)}(\varepsilon_x + \varepsilon_t + \varepsilon_z) \end{aligned} \tag{8}$$

By evaluation of the integration of the energy per unit volume of the structure $\bar{u}$ on the volume element of the cylinder $2\pi(R+z)dzdx$ and decomposing the obtained equation into mechanical and thermal strain energies, we have

$$\begin{aligned} &dV = 2\pi(R+z)dzdx \rightarrow U = \iiint_V \bar{u}dV \\ &U = \pi\int_0^l \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{E}{(1+v)(1-2v)}\left[(1-v)(\varepsilon_x^{\,2} + \varepsilon_z^{\,2} + \varepsilon_t^{\,2}) + 2v(\varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_t + \varepsilon_t\varepsilon_z) + \frac{1-2v}{2}\gamma_{xz}^{\,2}\right](R+z)dzdx \\ &-\pi\int_0^l \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TE}{1-2v}(\varepsilon_x + \varepsilon_t + \varepsilon_z)(R+z)dzdx = \int_0^1 [\mathrm{U_S}(x) - U_T(\mathrm{x})]dx \end{aligned} \tag{9}$$

where, $U_S(x)$ is the mechanical strain energy and $U_T(x)$is the thermal strain energy and can be obtained as follows

$$
\begin{aligned}
U_S &= \sum_{j=1}^{3}\sum_{i=1}^{n} A_i^{\,j}(x) f_i^{\,j}(x) \rightarrow f_i(x) = f_i(u, w, \phi_x^{\,1}, \phi_z^{\,1}, \ldots, \phi_x^{\,n}, \phi_z^{\,n}) \\
U_T &= \sum_{j=1}^{2}\sum_{i=1}^{n} B_i(x) g_i(x) \rightarrow g_i(x) = g_i(u, w, \phi_x^{\,1}, \phi_z^{\,1}, \ldots,, \phi_x^{\,n}, \phi_z^{\,n})
\end{aligned}
\tag{10}
$$

where, $A_i(x)$, $f_i(x)$, $B_i(x)$, $g_i(x)$ are expressed in appendix A, B. By definition of $\lambda = \dfrac{\pi}{(1+v)(1-2v)}$ and substitution of Eq. (5) into Eq. (9), we have:

$$
\mathrm{U_S}(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \lambda E(R+z)\left[(1-v)(\varepsilon_x^{\,2}+\varepsilon_z^{\,2}+\varepsilon_t^{\,2}) + 2v(\varepsilon_x\varepsilon_z + \varepsilon_x\varepsilon_t + \varepsilon_t\varepsilon_z) + \frac{1-2v}{2}\gamma_{xz}^{\,2}\right]dz =
\tag{11}
$$

$$
\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \lambda E(R+z)
\left[
\begin{aligned}
&(1-v)\{[(\frac{\partial \phi_x^{\,0}}{\partial x})^2 + z^2(\frac{\partial \phi_x^{\,1}}{\partial x})^2 + \ldots + (\frac{\partial \phi_x^{\,n}}{\partial x})^2 z^{2n} + 2z\,\frac{\partial \phi_x^{\,0}}{\partial x}\frac{\partial \phi_x^{\,1}}{\partial x} + 2z^2[\frac{\partial \phi_x^{\,0}}{\partial x}\frac{\partial \phi_x^{\,2}}{\partial x}] \\
&+2z^3[\frac{\partial \phi_x^{\,0}}{\partial x}\frac{\partial \phi_x^{\,3}}{\partial x} + \frac{\partial \phi_x^{\,1}}{\partial x}\frac{\partial \phi_x^{\,2}}{\partial x}] + \ldots + 2z^i \sum_{p=0}^{[i/2]} \frac{\partial \phi_x^{\,p}}{\partial x}\frac{\partial \phi_x^{\,i-p}}{\partial x}] \\
&[(\phi_z^{\,1})^2 + 4z^2(\phi_z^{\,2})^2 + 16z^4(\phi_z^{\,3})^2 + \ldots + n^2 z^{2n-2}(\phi_z^{\,n})^2 + 4z\,\phi_z^{\,1}\phi_z^{\,2} + 2z^2[3\phi_z^{\,1}\phi_z^{\,3}] \\
&+2z^3[4\phi_z^{\,1}\phi_z^{\,4} + 2\times 3\times \phi_z^{\,2}\phi_z^{\,3}] + \ldots + 2z^i \sum_{p=0}^{[i/2]} p(i+2-p)\phi_z^{\,p}\phi_z^{\,i+2-p}] \\
&+(\frac{1}{R+z})^2[(\phi_z^{\,0})^2 + z^2(\phi_z^{\,1})^2 + z^4(\phi_z^{\,2})^2 + \ldots + z^{2n}(\phi_z^{\,n})^2 + 2z\,\phi_z^{\,0}\phi_z^{\,1} + 2z^2\phi_z^{\,0}\phi_z^{\,2} \\
&+2z^3[\phi_z^{\,0}\phi_z^{\,3} + \phi_z^{\,1}\phi_z^{\,2}] + \ldots + 2z^i \sum_{p=0}^{[i/2]} \phi_z^{\,p}\phi_z^{\,i-p}]\} \\
&2v\{\phi_z^{\,1}\frac{\partial \phi_x^{\,0}}{\partial x} + z[\phi_z^{\,1}\frac{\partial \phi_x^{\,1}}{\partial x} + 2\phi_z^{\,2}\frac{\partial \phi_x^{\,0}}{\partial x}] + z^2[\phi_z^{\,1}\frac{\partial \phi_x^{\,2}}{\partial x} + 2\phi_z^{\,2}\frac{\partial \phi_x^{\,1}}{\partial x} + 3\phi_z^{\,3}\frac{\partial \phi_x^{\,0}}{\partial x}] + \ldots \\
&+2z^i \sum_{p=0}^{i} [(i+1-p)\phi_z^{\,i+1-p}\frac{\partial \phi_x^{\,p}}{\partial x}] + nz^{2n-1}\frac{\partial \phi_x^{\,n}}{\partial x}\phi_z^{\,n}] \\
&+(\frac{1}{R+z})[\phi_z^{\,0}\frac{\partial \phi_x^{\,0}}{\partial x} + z[\phi_z^{\,0}\frac{\partial \phi_x^{\,1}}{\partial x} + \phi_z^{\,1}\frac{\partial \phi_x^{\,0}}{\partial x}] + z^2[\phi_z^{\,0}\frac{\partial \phi_x^{\,2}}{\partial x} + \phi_z^{\,1}\frac{\partial \phi_x^{\,1}}{\partial x} + \phi_z^{\,2}\frac{\partial \phi_x^{\,0}}{\partial x}] + \ldots \\
&+2z^i \sum_{p=0}^{i} [\phi_z^{\,i-p}\frac{\partial \phi_x^{\,p}}{\partial x}] + z^{2n}\frac{\partial \phi_x^{\,n}}{\partial x}\phi_z^{\,n}] \\
&+(\frac{1}{R+z})[\phi_z^{\,1}\phi_z^{\,0} + z[(\phi_z^{\,1})^2 + 2\phi_z^{\,0}\phi_z^{\,2}] + z^2[\phi_z^{\,0}\phi_z^{\,3} + 3\phi_z^{\,1}\phi_z^{\,2}] + \ldots
\end{aligned}
\right. dz
$$

$$
\left.\begin{array}{l}
+z^{i}\sum_{p=0}^{i}[(i-p+1)\phi_z^{\ p}\phi_z^{\ i+1-p}]+nz^{\ 2n-1}(\phi_z^{\ n})^2\} \\
+\frac{1-2\nu}{2}[(\phi_x^{\ 1})^2+(\frac{\partial\phi_z^{\ 0}}{\partial x})^2+2\phi_x^{\ 1}\frac{\partial\phi_z^{\ 0}}{\partial x}+z\,[2\phi_x^{\ 2}\phi_x^{\ 1}+2\phi_x^{\ 2}\frac{\partial\phi_z^{\ 0}}{\partial x}+\phi_x^{\ 1}\frac{\partial\phi_z^{\ 1}}{\partial x}+\frac{\partial\phi_z^{\ 0}}{\partial x}\frac{\partial\phi_z^{\ 1}}{\partial x}] \\
z^{\ 2}[4(\phi_x^{\ 2})^2+(\frac{\partial\phi_z^{\ 1}}{\partial x})^2+4\phi_x^{\ 2}\frac{\partial\phi_z^{\ 1}}{\partial x}]+z^{\ 3}[6\phi_x^{\ 2}\phi_x^{\ 3}+2\phi_x^{\ 2}\frac{\partial\phi_z^{\ 2}}{\partial x}+3\phi_x^{\ 3}\frac{\partial\phi_z^{\ 1}}{\partial x}+\frac{\partial\phi_z^{\ 1}}{\partial x}\frac{\partial\phi_z^{\ 2}}{\partial x}] \\
z^{\ 4}[9(\phi_x^{\ 3})^2+(\frac{\partial\phi_z^{\ 2}}{\partial x})^2+6\phi_x^{\ 3}\frac{\partial\phi_z^{\ 2}}{\partial x}]+ \\
z^{\ i}[(p+1)\phi_x^{\ p+1}\frac{\partial\phi_z^{\ i-p}}{\partial x}+\frac{\partial\phi_z^{\ p}}{\partial x}\frac{\partial\phi_z^{\ i-p}}{\partial x}+(p+1)(i-p+1)\phi_x^{\ i-p+1}\phi_x^{\ p+1} \\
+(i-p+1)\frac{\partial\phi_z^{\ p}}{\partial x}\phi_x^{\ i-p+1}]]
\end{array}\right] \tag{12}
$$

$$
\mathrm{U_T}(x)=\int_{-\frac{h(x)}{2}}^{+\frac{h(x)}{2}}\frac{\alpha TE\,(R+z)}{1-2\nu}\left[\begin{array}{l}\frac{\partial\phi_x^{\ 0}}{\partial x}+\phi_z^{\ 1}+z\,\{\frac{\partial\phi_x^{\ 1}}{\partial x}+2\phi_z^{\ 2}\}+z^{\ 2}\{\frac{\partial\phi_x^{\ 2}}{\partial x}+3\phi_z^{\ 3}\}+\ldots+z^{\ n}\{\frac{\partial\phi_x^{\ n}}{\partial x}+(n+1)\phi_z^{\ n+1}\} \\ +\frac{\phi_z^{\ 0}+z\,\phi_z^{\ 1}+z^{\ 2}\phi_z^{\ 2}+\ldots+z^{\ n}\phi_z^{\ n}}{R+z}\end{array}\right] \tag{13}
$$

Now, we can complete the energy functional by using definition of external energies due to external works, magnetic or rotational loads. Energy of internal and external pressures is equal to multiplying the pressure in the radial deformation of the inner and the other surfaces of the cylinder, respectively. Inner pressure applies in the same direction of the deformation; conversely, outer pressure applies in the opposite direction of the deformation. Eq. (14) indicates work done by the internal and external pressure.

$$
\begin{aligned}
&W=2\pi\int_0^l[P_i(R-\frac{h}{2})w_{z=\frac{-h}{2}}-P_o(R+\frac{h}{2})w_{z=\frac{+h}{2}}]dx=\int_0^l[C_0\phi_z^{\ 0}+C_1\phi_z^{\ 1}+\ldots+C_n\phi_z^{\ n}]dx \\
&C_0=2\pi(P_i(R-\frac{h}{2})-P_0(R+\frac{h}{2})) \\
&C_1=2\pi\frac{h}{2}(-P_i(R-\frac{h}{2})-P_0(R+\frac{h}{2})) \\
&C_2=2\pi\frac{h^2}{4}(P_i(R-\frac{h}{2})-P_0(R+\frac{h}{2})) \\
&C_n=2\pi(\frac{h}{2})^n(P_i(-1)^n(R-\frac{h}{2})-P_0(R+\frac{h}{2}))
\end{aligned} \tag{14}
$$

Total energy of the system must be obtained by subtraction of Eq. (14) from Eq. (9) as follows

$$\begin{aligned}
&U=\int_0^l (U_S-U_T)dx-\int_0^l W dx=\int_0^l F(\phi_x{}^0,\phi_z{}^0,\phi_x{}^1,\phi_z{}^1,...,\phi_x{}^n,\phi_z{}^n,x)dx\\
&U_S=\sum_{i=1}^{9} A_i(x)f_i(x),\quad U_T=\sum_{i=1}^{4} B_i(x)g_i(x)\\
&\mathrm{W}=C_0\phi_z{}^0+C_1\phi_z{}^1+...+C_n\phi_z{}^n,\\
&C_1=2\pi(P_i(R-\frac{h}{2})-P_0(R+\frac{h}{2}))\ ,\\
&C_2=2\pi\frac{h}{2}(-P_i(R-\frac{h}{2})-P_0(R+\frac{h}{2}))\\
&C_n=2\pi(\frac{h}{2})^n(P_i(-1)^n(R-\frac{h}{2})-P_0(R+\frac{h}{2}))
\end{aligned} \tag{15}$$

where, $A_i(x)$, $f_i(x)$, $B_i(x)$, $g_i(x)$ are presented in Appendix A, B. As mentioned above, Eqs. (12), (13) includes $2n$ variables that n describes the order of employed shear deformation theory. Using Euler equation, variation of Eqs. (12), (13) can be expressed as follows

$$\begin{cases}
\dfrac{\partial F}{\partial \phi_x{}^i}-\dfrac{\partial}{\partial x}\Big(\dfrac{\partial F}{\partial(\frac{\partial \phi_x{}^i}{\partial x})}\Big)=0 & \mathrm{i}=0,1,...,\mathrm{n}\\
\dfrac{\partial F}{\partial \phi_z{}^i}-\dfrac{\partial}{\partial x}\Big(\dfrac{\partial F}{\partial(\frac{\partial \phi_z{}^i}{\partial x})}\Big)=0 & \mathrm{i}=0,1,...,\mathrm{n}
\end{cases} \tag{16}$$

where, $F(\phi_x{}^0,\phi_z{}^0,\phi_x{}^1,\phi_z{}^1,...,\phi_x{}^n,\phi_z{}^n,x)$ in Eq. (16) is functional of the system described as follows

$$F(\phi_x{}^0,\phi_z{}^0,\phi_x{}^1,\phi_z{}^1,...,\phi_x{}^n,\phi_z{}^n,x)=U_S-U_T-W \tag{17}$$

3. Results and discussion

Derived formulation in the previous section can be employed for obtaining the responses of the pressure vessels with various functionalities. In this work, two familiar functionalities are employed for investigation and comparison. Power and exponential varying are two well-known functionalities that have been employed in the previous works (Jabbari *et al*. 2002, Tutuncu 2007). First order shear deformation theory (FSDT) is applied in this research.

3.1 Power function

The power varying can be employed for derivation of the results. For power varying of the material properties, we suppose that the properties to be predefined in inner radius of the cylinder.

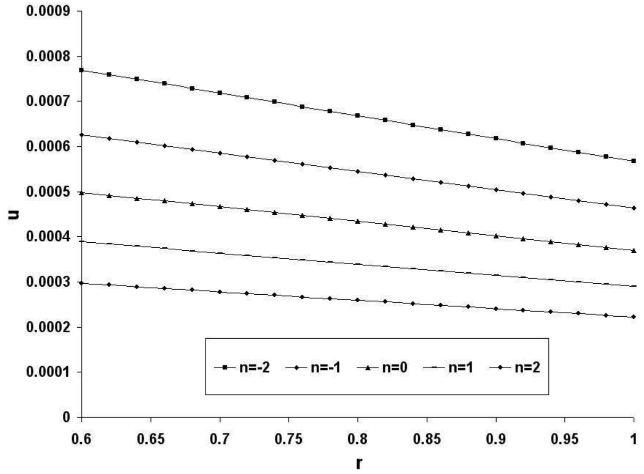

Fig. 2 The radial distribution of radial displacement for a FG cylinder with power function distribution of material properties

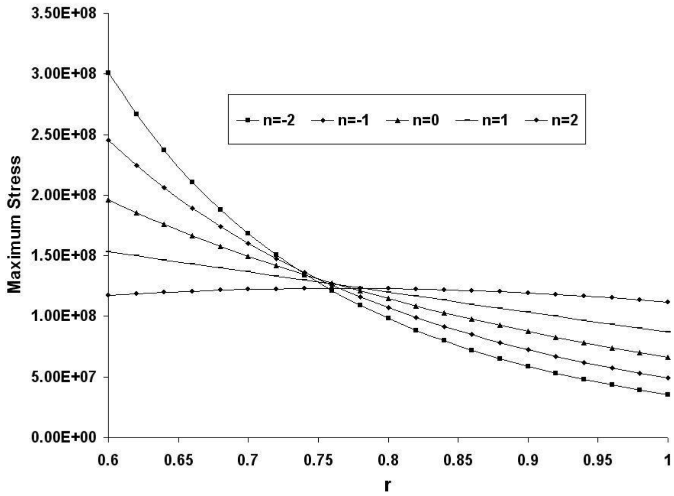

Fig. 3 The radial distribution of circumferential stress for a FG cylinder with power function distribution of material properties

Properties distribution along the thickness direction can be obtained based on the power function. The power distribution of material properties are defined along the thickness direction as follows

$$E(r) = E_{r_i} (\frac{r}{a})^n \tag{18}$$

where, $E(r), E_{r_i}$ are modulus elasticity distribution and modulus elasticity at inner radius, respectively. The redial distribution of radial displacement for different values of non homogenous index is depicted in Fig. 2. The FGM cylinder is loaded under 80 Mpa internal pressure at the inner surface. Inner and outer radii are considered 0.6 and 1 respectively.

It is observed that the values of radial displacement decreases with increasing the non

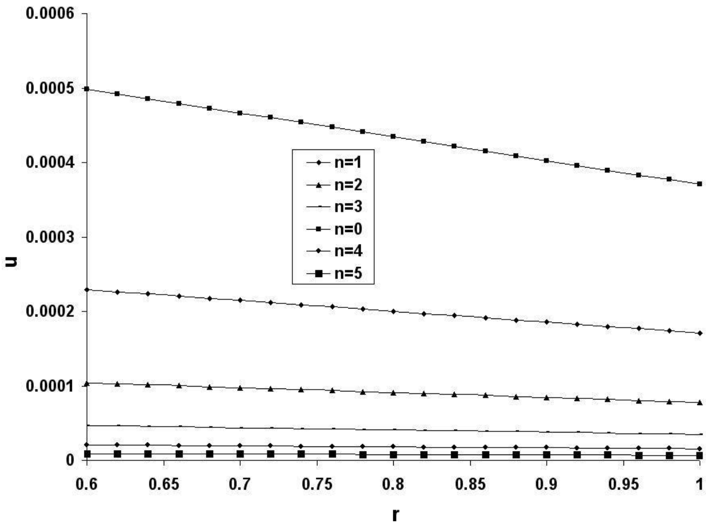

Fig. 4 The radial distribution of radial displacement along the thickness direction for exponential varying

homogenous index. This observation is due to increasing the stiffness of the cylinder when the non homogenous index increases.

The other component that is important in design calculation is maximum stress along the thickness direction. Circumferential stress is maximum stress component. The radial distribution of circumferential stress is presented in Fig. 3.

3.2 Exponential distribution

The effect of exponential distribution of material properties can be considered on the results.

$$E(r) = E_0 e^{nr} \tag{19}$$

By considering the above function for distribution of material properties, we can evaluate the responses of the functionally graded cylindrical shell under inner pressure for various values of non homogenous index.

Fig. 4 shows the radial distribution of radial displacement along the thickness direction for different values of non homogenous index.

This figure indicates that the radial displacement decreases with increasing the non homogenous index. This trend is compatible with supposed material properties as exponential varying. It is observed that with increasing the non homogenous index, the stiffness of material increases and consequently the radial displacement decreases uniformly.

The obtained results in this paper indicate that the received results in reference (Tutuncu 2007) are not true. Fig. 1 in the referred paper shows that the redial displacement increases with increasing the non homogenous index. The previous obtained results is in opposite with employed functionalities and the present obtained results. The reduction trend must be concluded for varying the non homogenous index with increasing manner.

The previous results related to employing the material with exponentially varying is presented as follows in Fig. 5.

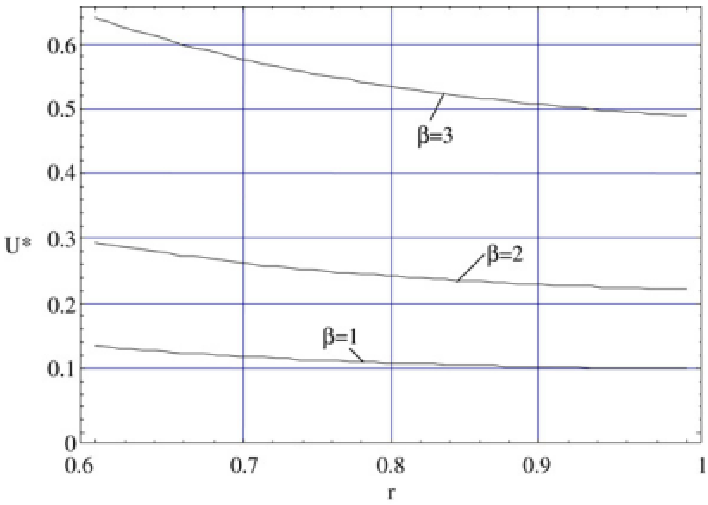

Fig. 5 The radial distribution of radial displacement along the thickness direction for exponential varying that has been presented in Reference (Tutuncu 2007)

where, β is non homogenous index in equation of non homogenous distribution of material properties $E(r)=E_0e^{Br}$.

4. Conclusions

The results that were extracted from the present paper are:

1. The shear deformation theory has been developed in this paper for thermo elastic analysis of a cylindrical pressure vessel made of functionally graded material. The obtained formulation shows application of shear deformation theory for different functionalities. The previous obtained differential equations have not adequate capability to simulate a functionally graded cylindrical pressure vessels with arbitrary functionality by using a unique and comprehensive method.

2. The obtained differential equations containing four equations of order two solved for two well-known models of material distributions. Power and exponential varying considered for presentation of capability of applied method. The presented formulation can free researchers for performing the individual methods for new created functionally graded.

3. The obtained results indicate that with increasing the non homogenous index, the radial displacement increases uniformly. This result is due to increasing the stiffness of structure that tends to decrease the radial displacement. This fact satisfied in the present paper in spite of the result of reference (Tutuncu 2007).

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Nomenclature

r	Radius of an arbitrary layer of cylinder	$\bar{u}$	Energy per unit volume
z	Coordinate of arbitrary layer of cylinder respect to mid-plane	dV	Element of volume
R	Radius of mid-plane of cylinder	$h(x)$	Local thickness of cylinder
u_x	Axial component of deformation	G	Shear modulus of elasticity
w_z	Radial components of deformation	E	Modulus of elasticity
$\phi_x{}^0$	Rotation component of axial deformation	P_i	Internal pressure
ϕ_z^0	Rotation component of radial deformation	P_0	External pressure
$\phi_x{}^n$	Component of axial deformation(order n, $n\geq 1$)	W	External work
$\phi_z{}^n$	Component of radial deformation (order n, $n\geq 1$)	C_n	General force and moment
ε_x	Axial strain	F	general potential function
ε_z	Radial strain	U_S	Strain energy

ε_t	Circumferential strain	U_T	Thermal energy
ε_{xz}	Shear strain in xz plane	f_i^j, g_i^j	the functions of variables
σ_x	Axial stress	A_i^j, B_i^j	the functions of variables
σ_z	Radial stress	E_i	Modulus of elasticity at the inner radius
σ_t	Circumferential stress	r_i	Inner radius
τ_{xz}	Shear stress	r_o	Outer radius
T	Temperature rising	n	Non homogenous index
α	Heat expansion coefficient	β	Non homogenous index
U	Total energy	G	Shear modulus of elasticity

Appendix A.

$$A_1^1(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda E(R+z)dz,\ \mathrm{f}_1^1=(1-\nu)[(\frac{\partial\phi_x{}^0}{\partial x})^2+(\phi_z{}^1)^2]+2\nu[\phi_z{}^1\frac{\partial\phi_x{}^0}{\partial x}]+\frac{1-2\nu}{2}[(\phi_x{}^1)^2+(\frac{\partial\phi_z{}^0}{\partial x})^2+2\phi_x{}^1\frac{\partial\phi_z{}^0}{\partial x}]$$

$$A_2^1(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez(R+z)dz,\ \mathrm{f}_2^1=(1-\nu)[2\frac{\partial\phi_x{}^0}{\partial x}\frac{\partial\phi_x{}^1}{\partial x}+4\phi_z{}^1\phi_z{}^2]+2\nu[\phi_z{}^1\frac{\partial\phi_x{}^1}{\partial x}+2\phi_z{}^2\frac{\partial\phi_x{}^0}{\partial x}]+\frac{1-2\nu}{2}[2\phi_x{}^2\phi_x{}^1$$

$$+2\phi_x{}^2\frac{\partial\phi_z{}^0}{\partial x}+\phi_x{}^1\frac{\partial\phi_z{}^1}{\partial x}+\frac{\partial\phi_z{}^0}{\partial x}\frac{\partial\phi_z{}^1}{\partial x}]$$

$$A_3^1(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez^2(R+z)dz,\ \mathrm{f}_3^1=(1-\nu)[(\frac{\partial\phi_x{}^1}{\partial x})^2+2\frac{\partial\phi_x{}^0}{\partial x}\frac{\partial\phi_x{}^2}{\partial x}+4(\phi_z{}^2)^2+2[3\phi_z{}^1\phi_z{}^3]+2\phi_z{}^0\phi_z{}^2]+2\nu[\phi_z{}^1\frac{\partial\phi_x{}^2}{\partial x}$$

$$+2\phi_z{}^2\frac{\partial\phi_x{}^1}{\partial x}+3\phi_z{}^3\frac{\partial\phi_x{}^0}{\partial x}]+\frac{1-2\nu}{2}[4(\phi_x{}^2)^2+(\frac{\partial\phi_z{}^1}{\partial x})^2+4\phi_x{}^2\frac{\partial\phi_z{}^1}{\partial x}]$$

$$A_4^1(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez^3(R+z)dz,\ \mathrm{f}_4^1=\frac{1-2\nu}{2}[6\phi_x{}^2\phi_x{}^3+2\phi_x{}^2\frac{\partial\phi_z{}^2}{\partial x}+3\phi_x{}^3\frac{\partial\phi_z{}^1}{\partial x}+\frac{\partial\phi_z{}^1}{\partial x}\frac{\partial\phi_z{}^2}{\partial x}]$$

$$A_i^1(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez^i(R+z)dz,\ \mathrm{f}_i^1=2(1-\nu)[\sum_{p=0}^{i-1}\frac{\partial\phi_x{}^p}{\partial x}\frac{\partial\phi_x{}^{i-p}}{\partial x}+\sum_{p=0}^{i-1}p(i+2-p)\phi_z{}^p\phi_z{}^{i+2-p}]+2\nu\sum_{p=0}^{i}[(i+1-p)\phi_z{}^{i+1-p}\frac{\partial\phi_x{}^p}{\partial x}]$$

$$\frac{1-2\nu}{2}[(p+1)\phi_x{}^{p+1}\frac{\partial\phi_z{}^{i-p}}{\partial x}+\frac{\partial\phi_z{}^p}{\partial x}\frac{\partial\phi_z{}^{i-p}}{\partial x}+(p+1)(i-p+1)\phi_x{}^{i-p+1}\phi_x{}^{p+1}+(i-p+1)\frac{\partial\phi_z{}^p}{\partial x}\phi_x{}^{i-p+1}]$$

$$A_1^2(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Edz,\ \mathrm{f}_1^2=2\nu[\phi_z{}^0\frac{\partial\phi_x{}^0}{\partial x}+\phi_z{}^1\phi_z{}^0],\quad A_2^2(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ezdz,\ \mathrm{f}_2^2=2\nu[\phi_z{}^0\frac{\partial\phi_x{}^1}{\partial x}+\phi_z{}^1\frac{\partial\phi_x{}^0}{\partial x}+(\phi_z{}^1)^2+2\phi_z{}^0\phi_z{}^2]$$

$$A_3^2(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez^2dz,\ \mathrm{f}_4^2=2\nu[\phi_z{}^0\frac{\partial\phi_x{}^2}{\partial x}+\phi_z{}^1\frac{\partial\phi_x{}^1}{\partial x}+\phi_z{}^2\frac{\partial\phi_x{}^0}{\partial x}+\phi_z{}^0\phi_z{}^3+3\phi_z{}^1\phi_z{}^2]$$

$$A_i^2(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\lambda Ez^idz,\ \mathrm{f}_i^2=2\nu[\sum_{p=0}^{i}[\phi_z{}^{i-p}\frac{\partial\phi_x{}^p}{\partial x}+\sum_{p=0}^{i}(i-p+1)\phi_z{}^p\phi_z{}^{i+1-p}]$$

$$A_1^3x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\frac{\lambda E}{R+z}dz,\ \mathrm{f}_1^3=(1-\nu)[(\phi_z{}^0)^2],\ A_2^3(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\frac{\lambda Ez}{R+z}dz,\ \mathrm{f}_2^3=(1-\nu)[2\phi_z{}^0\phi_z{}^1]$$

$$A_3^3(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\frac{\lambda Ez^2}{R+z}dz,\ \mathrm{f}_3^3=(1-\nu)[(\phi_z{}^1)^2],A_4^3(x)=\int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}}\frac{\lambda Ez^i}{R+z}dz,\ \mathrm{f}_4^3=(1-\nu)[2\sum_{p=0}^{i-1}\phi_z{}^p\phi_z{}^{i-p}]$$

Appendix B.

$$B_1^1(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TE(R+z)}{1-2\nu} dz, \mathrm{f}_1^1 = \frac{\partial \phi_x^{\,0}}{\partial x} + \phi_z^{\,1},$$

$$B_2^1(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TEz(R+z)}{1-2\nu} dz, \mathrm{f}_2^1 = \frac{\partial \phi_x^{\,1}}{\partial x} + 2\phi_z^{\,2},$$

...

$$B_i^1(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TEz^{i-1}(R+z)}{1-2\nu} dz, \mathrm{f}_\mathrm{i}^1 = \frac{\partial \phi_x^{\,i-1}}{\partial x} + i\,\phi_z^{\,i}$$

$$B_1^2(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TE}{1-2\nu} dz, \mathrm{f}_1^2 = \phi_z^{\,0},$$

$$B_2^2(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TEz}{1-2\nu} dz, \mathrm{f}_2^2 = \phi_z^{\,1},$$

...

$$B_i^2(x) = \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \frac{\alpha TEz^{i-1}}{1-2\nu} dz, \mathrm{f}_\mathrm{i}^2 = \phi_z^{\,i-1}$$