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Path-dependent three-dimensional constitutive laws of reinforced concrete —formulation and experimental verifications—

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Abstract. A three-dimensional constitutive modeling for reinforced concrete is presented for finite element nonlinear analysis of reinforced concrete. The targets of interest to the authors are columns confined by lateral steel hoops, RC thin shells subjected to combined in-plane and out-of-plane actions and massive structures of three-dimensional (3D) extent in shear. The elasto-plastic and continuum fracture law is applied to pre-cracked solid concrete. For post cracking formulation, fixed multi-directional smeared crack model is adopted for RC domains of 3D geometry subjected to monotonic and reversed cyclic actions. The authors propose a new scheme of decomposing stress strain fields into sub-planes on which 2D constitutive laws can be applied. The proposed model for 3D reinforced concrete is experimentally verified in both member and structural levels under cyclic actions.

Key words: constitutive law; smeared crack; nonlinear analysis; confinement; cyclic action.

1. Introduction

The extension of material models to generic three-dimensional (3D) stress fields is aimed as a recent topic of structural mechanics. For computing ductility of seismic resistant members laterally reinforced by transverse hoops, tri-axial constitutive laws of concrete play a crucial role (Maekawa, *et al.* 1993). The 3D analysis of RC shells subjected to cyclic in-plane and out-of-plane actions accompanying multi-directional cracking is expected to rationalize seismic design of underground RC tanks (Shawky and Maekawa 1996) for energy facilities. After the Hanshin-Awaji Earthquake (Kobe) in 1995, retrofitting of existing RC structures having poor seismic

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performance is regarded as an urgent matter for engineers. Here, quantitative assessment of steel jacketing and non-metal fibers to enhance the ductility owing to 3D confinement is required.

Owing to the construction efficiency of RC and economical features, size of structures tends to be enlarging for public works. Here, accurate evaluation of shear capacity of structures with complex geometry is required in safety design (Sonobe, *et al.* 1994) since loading tests for examining capacity can be hardly conducted. These engineering topics can not be solved in use of two-dimensional analysis. For meeting the challenge, a project of developing full 3D RC behavioral simulator coded COM3 was initiated in 1989 by the authors' research circle in the University of Tokyo. In this paper, on-going project for developing versatile analytical tools being used for examining overall structural functions is reported and the scheme of future performance-based design is briefly stated in line with 3D constitutive law for reinforced concrete under alternate actions.

2. Elasto-plastic continuum fracture model

Pre-cracked concrete

Mechanical behaviors of concrete can be idealized as combined plasticity and continuum fracturing which identifies the loss of elastic energy absorption capability. In tri-axial stress states, no continuum damage in isotropic elasticity and confinement-dependent fracturing in deviatoric elasticity were experimentally shown (Maekawa, *et al.* 1993). An explicit confinement-dependent relation between plasticity in shear and deviatoric elasticity is not observed in the nonlinear mechanics of concrete. But the dilatancy on isotropic plasticity which is dependent on the elastic shear is experimentally verified and formulated as,

$$I_1=3K_oI_{1e},\ J_2=2G_oK(I_{1e},\ J_{2e},\ J_{3e})\cdot J_{2e},\ J_{2p}=H(J_{2e}),\ \frac{dI_{1p}}{dJ_{2p}}=D(K,\ I_{1e}) \tag{1}$$

where, I_1, J_2, I_{1e}, J_{2e}, J_{3e}, I_{1p}, and J_{2p} are stress, elasticity and plasticity invariants of isotropy and deviator, the functional K and H rule the evolution on fracturing and plasticity, D is the dilatancy derivative dependent on confinement level denoted by the first invariant of elasticity. The functions were identified through cyclic compressive experiments with steel casing (Maekawa, *et al.* 1993) and verified by numerous reports on tri-axial loading to solid concrete as,

$$K=\exp\left[-\frac{F}{3.25}\left\{1-\exp\left(-\frac{F}{0.8}\right)\right\}\right],\ F=\frac{\sqrt{2}\,J_{2e}}{0.23\varepsilon_0-\sqrt{3}\,I_{1e}}\cdot\frac{1}{5}\left\{\frac{3\sqrt{3}}{2}\left(\frac{J_{3e}}{J_{2e}}\right)^3+6\right\} \quad \text{fracturing (2-1)}$$

$$H=\frac{9}{10}\varepsilon_0\cdot\left(\frac{J_{2e}}{\varepsilon_0}\right)^3 \quad \text{plasticity (2-2)}$$

$$D=D_0K^2+D_1(1-K)^2,\ D_0=\frac{(-1+2\upsilon_0)}{\sqrt{3}(1+\upsilon_0)}\ \text{ and }\ D_1=\frac{\sqrt{2}\,I_{1e}+0.38\varepsilon_0}{0.28\varepsilon_0} \quad \text{dilatancy (2-3)}$$

The constitutive model formulated above is applied to non-cracked concrete volume in finite elements. For verification of 3D constitutive laws for pre-cracked continuum concrete, comparison under non-uniform stress states was made with laterally reinforced concrete columns (Pallewatta, *et al.* 1995) under axial compression. Discrete reinforcing bars are modeled as beam elements with flexural and shear stiffness, and fiber stress strain relation of mild steel is defined with

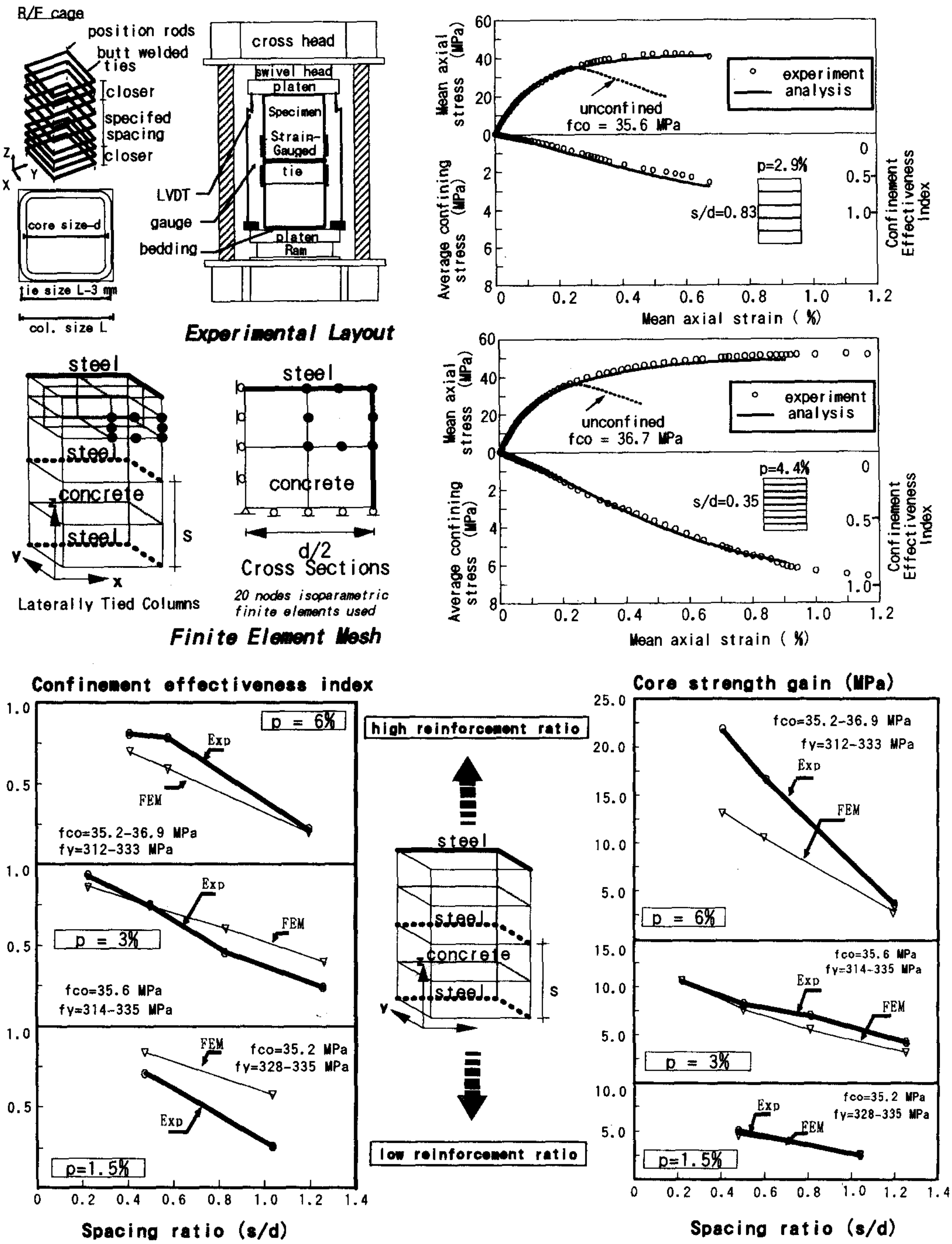

Fig. 1 Laterally confined reinforced concrete columns subjected to axial actions.

reversed cyclic feature defined by Kato (Kato, *et al.* 1979). Fig. 1 shows the axial mean stress versus mean strain relations, and laterally induced average stress σ_y, that is defined by Eq. (3-1) and experimentally obtained, is attached with analytical results as,

$$\sigma_v = \frac{1}{V_c}\int_{V_c}\frac{\sigma_{c,xx}+\sigma_{c,yy}}{2}\,dV = \frac{1}{2V_c}\oint \sigma_s \cdot A_s dS \tag{3-1}$$

where, $\sigma_{c,xx}$, $\sigma_{c,yy}$ and σ_s are sectional lateral stresses of concrete and steel stress along a re-bar axis denoted by S, and V_c is the total volume of concrete in the control domain. The microscopic local stresses are not uniform and associated evolution of plasticity and continuum fracture is different in space. In experiment, just averaged axial stress and mean lateral confinement of engineering concern can be obtained, but in analysis we can have a local point-wise information concerning solid mechanics. Thus, the laterally confined columns subjected to axial forces offer appropriate verification of 3D constitutive laws at the member level. The full plasticity of lateral ties would yield,

$$\sigma_{v,full} = \frac{f_y V_s}{2V_c} = \frac{1}{2}p \cdot f_y \tag{3-2}$$

where, f_y and p are yield strength of ties and volumetric reinforcement ratio. The confinement effectiveness is defined as $\sigma_v/\sigma_{v,full}$. FEM analysis brings about an accurate prediction of column strength gain and ductility enhanced by discretely arranged lateral ties (See Fig. 1). It is shown that the lateral tie does not reach full yielding (i.e., it means confinement effectiveness less than unity) at the ultimate condition when a square section with larger spacing of ties is assumed.

3. Three-dimensional reinforced concrete shells

Degenerated smeared crack modeling

It is possible to reduce 3D stress fields developing in thin shells by neglecting the stress component normal to the thickness. Here, layer integration scheme of RC plates is adopted (Polak 1992). Two-dimensional in-plane constitutive law for reinforced concrete is specified in each layer though 3D stress field would be spatially assumed. In this simplification, three-dimensional confinement effect to concrete is neglected since the free stress boundary can be set toward the thickness direction. The authors installed the multi-directional fixed smeared crack modeling of concrete with two-way reinforced concrete membrane into the scheme of Mindlin plates which allow out-of-plane shear deformation. The RC model assuming mean stress-strain relation of embedded reinforcement is size-independent (Okamura and Maekawa 1991, 1994), and its stiffness is positive definite until the compression failure of concrete occurs as shown in Fig. 2.

For computing the average yield strength of reinforcement (Tamai, *et al.* 1987, Hsu 1993), the whole section was assumed fully effective no matter how large curvature would be introduced. If the nominal reinforcement ratio defined in each layer would be used for post yield averaging of steel, it brings about overestimation of mean elongation of reinforcement by cracked concrete through bond, because adjacent concrete strips can work well as confinement agent against the free deformation of bare bars.

For dealing with reversed cyclic in-plane as well as out-of-plane stress paths, multi-directional crack and their interaction have to be computationally drafted. Under in-plane stress field, *active crack method* was proposed for two way cracks having arbitrary crossing angles (Okamura and Maekawa 1991). In general, overall non-linearity of RC panels subjected to cyclic shear is governed by an active crack deformation and the rest of crack is dormant. Here, nonlinear path-dependent

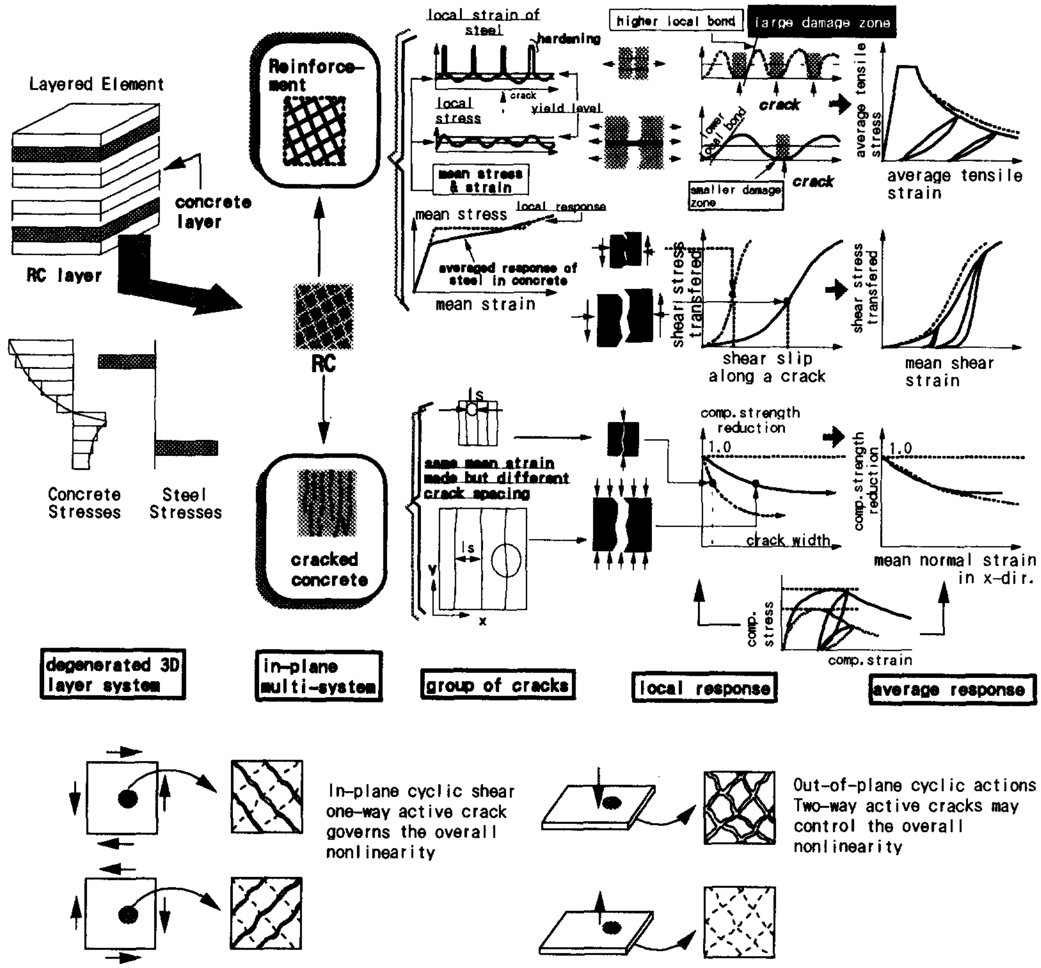

Fig. 2 Layer integration scheme and in-plane constitutive modeling for reinforced concrete.

parameters such as plasticity and past maximum strains are renewed on the loading history along the specified active crack and no evolution of non-linearity on the dormant one is assumed. This is due to the fact that transverse shear stiffness of either of two-way cracks is extensively large when non-linearity is in progress. Two way cracks can be both activated with larger crack opening when absolute stress levels are quite small after experiencing higher non-linearity. But, in this case, less inelasticity proceeds since the stress level is not enough to drive the path-dependency in reinforecd concrete. Then, adopted simplicity of ignoring dormant cracks and just evaluating major source of non-linearity is successful for two dimensional problems. Fig. 3 shows the computed reversed cyclic responses of RC plates in shear (Stevens, *et al.* 1986, Okamura and Maekawa 1991).

When we apply the multi-directional smeared crack approach to 3D RC shells under combined in-plane and out-of-plane actions, two activated cracks which govern overall nonlinear behaviors may take place as shown in Fig. 2. Out-of-plane shear may occur peripherally when concentrated forces would be set on RC shells. Here, path-dependent parameters and mobilization of both crakcs should be considered. The authors extend the original active crack method to more generic

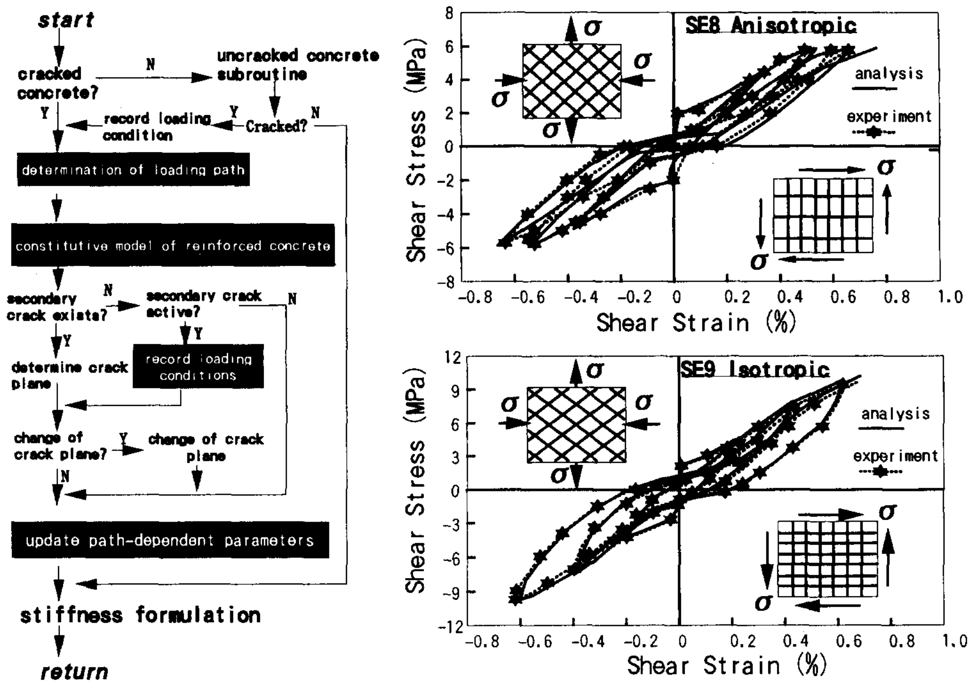

Fig. 3 Scheme of multi-directional cracks under in-plane reversed cyclic actions.

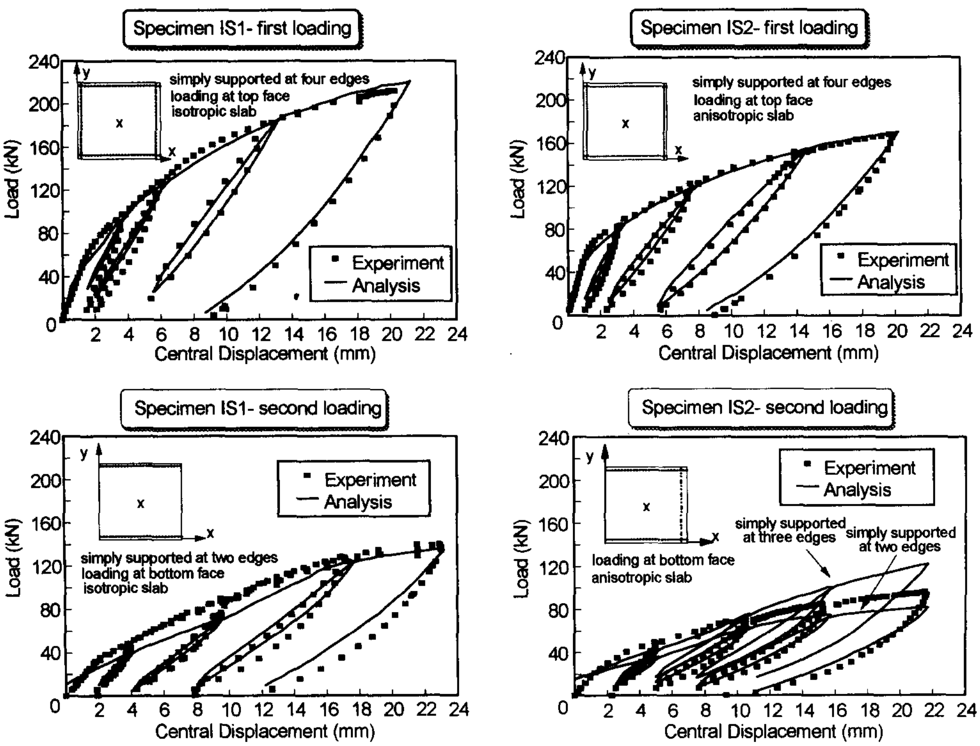

Fig. 4 Verification of RC shell models subjected to cyclic membrane and out-of-plane bending.

conditions as illustrated in Fig. 3, where the criterion of crack liveliness is made in terms of normal strain to each crack plane (Okamura and Maekawa 1991).

The experimental verification is shown in Fig. 4 which includes two RC square slabs under alternate loading. In order to introduce the load reversal, first, the single sided cyclic action was applied, and second, the slab was turned over upside down with different support conditions. Again, the single sided cyclic loading was conducted. Then, two-way cracking was observed in both faces of the slab, and bi-directional crack closure and opening were alternately brought about. As a whole, it is examined that 3D RC thin-plate constitutive law works well even under stress reversal. For the second loading case of Specimen IS2, the authors present analytical results of two different boundary conditions, since perfect reproduction of the specified support condition would be supposed difficult due to the residual deformation of the slab induced by the first loading and resultant imperfect touch between bearing supports and the specimen.

The path-dependence of bi-directional bending and shear can be simulated with membrane reinforced concrete model. Further verification of applicability was performed under combined membrane forces and bi-axial bending (Irawan and Maekawa 1995). The combined geometrical and material nonlinearity was also checked. However, the out-of-plane shear failure was not considered in the formulation. The punching shear fracture energy will be taken into account for versatile computational tool for 3D reinforced concrete shells in future.

4. Three-dimensional RC smeared crack solids

Full crack degree-of-freedom model

The in-plane membrane constitutive law of reinforced concrete is built by combining one-dimensional constitutive laws of compression, tension and shear (Collins and Vecchio 1982, Okamura and Maekawa 1991, Hsu 1993). With the same manner, it will be rational to compose the smeared crack modeling of three-dimension based on the cracked concrete in-plane constitutive law, since three-dimensional confinement effect is thought to be negligibly small owing to crack stress release. After closure of pre-cracking, the constitutive model is switched to elasto-plastic and fracturing continuum constitutive model, and tri-axial confinement effect on concrete in compression starts to be computed again.

Assume an orthogonal coordinate whose principal axis (1) is normal to the first crack plane and the rest of axes (2, 3) are within the crack reference plane. Here, we can define two-dimensional sub-spaces designated by axis (1, 2), (2, 3) and (3, 1) as shown in Fig. 5, and partial stresses rooted in the crack projection on (i, j) can be computed in use of in-plane RC constitutive law. Let $\sigma_{ij}^{(k,l)}$ denote the reduced component stress computed based on mean strain on (k, l) sub-space. When we simply assume the total load carrying mechanism as being composed of partial stresses on three sub-spaces, we have,

$$\sigma_{ii}=\frac{1}{2}\sum_{k\neq i}\sigma_{ii}^{(i,k)}(\varepsilon_{st}|_{s,t=i\vee k}),\qquad \sigma_{ij}|_{i\neq j}=\sigma_{ij}^{(i,j)}(\varepsilon_{st}|_{s,t=i\vee k})\tag{4}$$

where, in-plane membrane constitutive equation is used for computing component stresses $\sigma_{ij}^{(k,l)}$ from sub-space strains as $\varepsilon_{st(s,t\vee k,l)}$. Since the two-way cracks are considered in each sub-space, triple cracks are fictitiously considered in the scheme of smeared crack idealization.

The reinforcement orthogonally arranged in space was assumed one-dimensional cord without

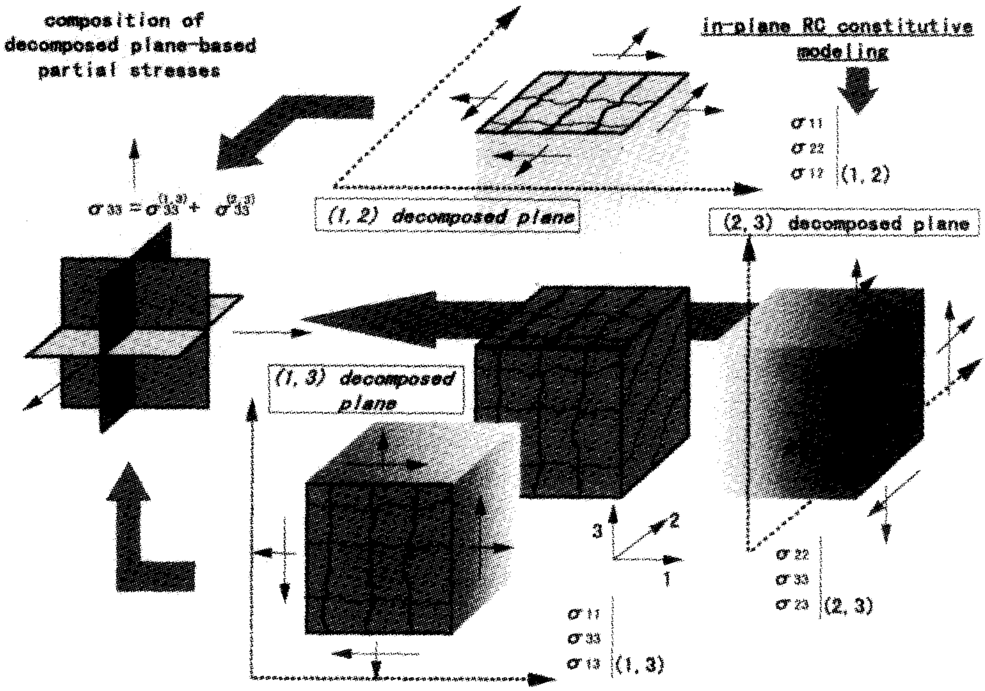

Fig. 5 Breakdown and composition of load carrying mechanism of 3D cracked solids of concrete.

dowel shear stiffness and its average stress-strain relation was defined with the same manner as 2D in-plane model (Tamai, *et al.* 1987). Here, the same tension-stiffness model (Tamai, *et al.* 1987) is used as that for in-plane stress states. But, for practical cases of RC members of large sizes with 3D extent, plenty of volume tends to be much under-reinforced because few amount of steel is enough to sustain flexure due to large effective distance among compression and tension resultants. Then, post crack tension-softening has to be considered in the light of fracture mechanics. The authors adopt coupled tension and shear softening model (An, *et al.* 1996) associated with size of smeared crack elements as,

$$\sigma_t = f_t\left(\frac{\varepsilon_{tu}}{\varepsilon_t}\right)^C \tag{5-1}$$

$$\int_0^\infty \sigma_t d\varepsilon_t = \frac{G_f}{l_r} \tag{5-2}$$

where, f_t; G_f; l_r are tensile strength and fracture energy of concrete and reference length of finite element. Eq. (5-1) has the same mathematical form as that for tension stiffness model used in RC in-plane constitutive model (Okamura and Maekawa 1991, Hsu 1993). In case of RC, factor C in Eq. (5-1) was verified constant (0.4 for deformed bar and 0.2 for wire fabricated mesh) and independent of the size of finite element provided that reinforcement not less than critical minimum reinforcement ratio would be placed in the volume concerned. For plain finite elements, the factor C in Eq. (5-1) is to be decided so that we satisfy Eq. (5-2) with respect to the fracture energy balance (An, *et al.* 1996).

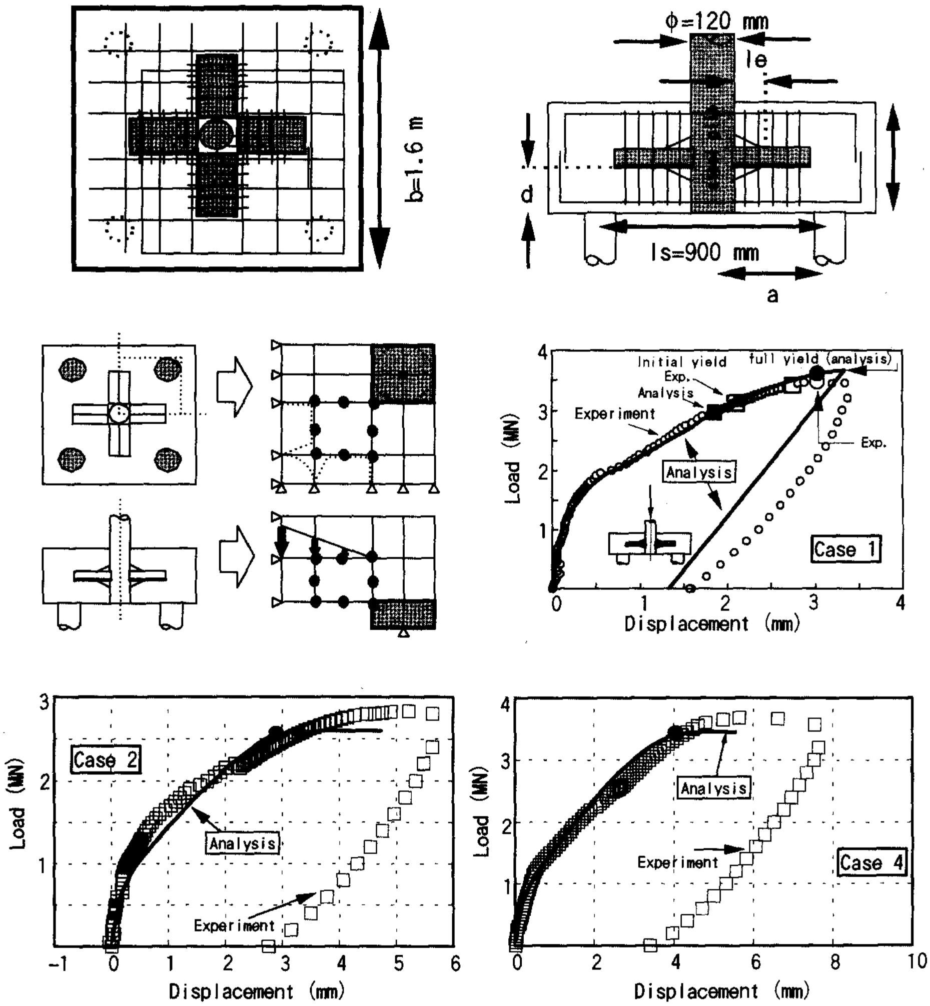

Fig. 6 Specimen details of footings and FE mesh; computed and experimental load-displacement relations.

Table 1. Specification and capacity of specimens

Specimen	case 1	case 2	case 3	case 4
main re-bars	D19	D22	D22	D22
web reinf.	D13	D13	D16	D13
slab thick (t:cm)	62	38	38	38
depth (d:cm)	26	13	13	13
diaph. (le:cm)	12	12	12	24
le/ϕ	1.0	1.0	1.0	2.0
a/d	1.9	3.9	3.9	3.9
ultimate P analy. (kN)	3607	2566	3260	3402
ultimate P exper. (kN)	3480	2826	3260	3680
initial yield analy. (kN)	2983	2011	no. yield	2527
initial yield exper. (kN)	3110	2050	no. yield	2750
entire yield analy. (kN)	3607	2566	no. yield	3402
entire yield exper. (kN)	3440	2600	no. yield	3680

The nonlinear 3D analysis with Eq. (4) was applied to the shear failure of reinforced concrete pile caps of larger size as shown in Fig. 6 and listed in Table 1. A single steel column is embedded in the concrete mass with anchorage frames. The concrete footing is supported by four piles. The strength of concrete ranges from 36 to 38 MPa. The yield strength of reinforcing bars is 376~396 MPa for D13 and D16, and 512~518 MPa for D19 an D22. The failure mode of the reality is shear with diagonal shear failure planes whose location and propagation path were indirectly sensored by measuring the local strain of reinforcing bars placed in the web zone and directly seen by cutting the large mass of concrete into smaller pieces after experiments.

The analytical and experimental results on ultimate capacity, initial yield and full yielding loads of web reinforcement are summarized in Table 1, and fair coincidence is obtained. The conical shaped diagonal cracks are formed inside the pile caps and finite element code named COM3 brought about the reasonably accurate estimation. The load-displacement relation of case 1, 2, 4 is shown in Fig. 6. Mesh sensitivity and convergence were checked in using doubly finer discretization. Although the axial load was transferred to concrete through 4 steel blades, the triangular distribution pattern of transferred bearing forces were assumed for simplicity.

The pile cap shear in Fig. 6 is useful for verification of 3D full smeared crack model for reinforced concrete with respect to 3D crack stress release of higher stability owing to sufficient amount of transverse shear reinforcement. But, the capability of simulating multi-directional cracks in 3D extent and unstable propagation of shear cracks can not be checked by the data in Fig. 6. Since shear failure simulation in 2D stress field was examined with the same scheme of smeared crack formulation (An, *et al.* 1996), similar cases in which principal stress direction does not turn in 3D but rotate just in a limited 2D stress field are not appropriate for verification of proposed 2D breakdown and 3D re-composition scheme as stated above. In considering this background, the authors select short RC columns under varying tri-directional forces in shear.

Fig. 7 illustrates a set-up of the short columns (Yoshimura 1996) used for verification of the model and the nonlinear analysis method. The columns are subjected to fixed vertical axial forces and constant horizontal shear whose direction is clamped in Y. After setting these forces, varying enforced displacement with orthogonal X-direction normal to the already applied shear is monotonically applied until failure and/or instability in which the force can not be kept constantly occurs. We recognize complexity of stress fields which accompanies tri-axially inclined cracks and varying principal stress directions. It means that spatial development of stress induced cracks is irregular unlike 2D RC in shear.

Analytically obtained shear failure interaction diagram is shown in Fig. 7. In computation, unstable crack propagation occurs during iterative nonlinear computation in the process of searching global equilibrium. The numerical instability point was defined as the computed prediction of ultimate capacity (An, *et al.* 1996) and the instability was checked not to be affected by load steps. Up to the peak of load displacement diagrams, full 3D constitutive model works well with reasonable accuracy for engineering purposes. But, still in the descending portions of the diagrams, computation is not successful. In fact, arch-length method of iteration with some restraint can not be adopted due to the horizontal load being kept constant. Here, strain localization in compression remains unsolved in full 3D finite element analysis. The authors regard this matter as a future topic of investigation.

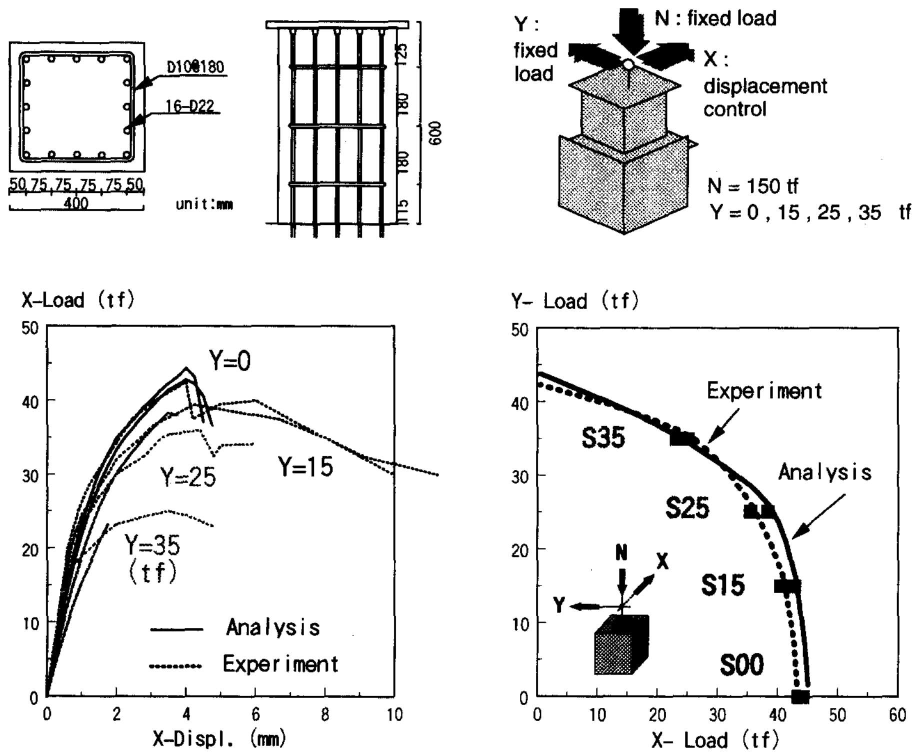

Fig. 7 Bi-directional shear failure of RC short columns and its simulation.

5. Conclusions

Both partially reduced and full 3D constitutive laws of reinforced concrete were proposed for RC thin shells and RC solids with multi-directional cracks, and their applicability under monotonic as well as reversed cyclic forces was examined in use of structural members of non-uniformity of stress states. For devising cyclic full 3D constitutive model, a simplified but reasonable accurate scheme of decomposing load carrying systems of cracked RC was newly presented. Development of three-dimensional constitutive laws is now regarded as a challenging topic of interest for creating a versatile computational tools for examining structural performances of reinforced concrete. It is clear that massive unsolved articles to improve the versatility of models remain, but the nonlinear behavioral technicality is evinced for an engineering platform in future.

The performance-based design scheme which will run as a future structural code and is discussed for international new scheme requires the verification routine of designed structures. Here, engineers are requested to simulate RC responses as it is to check the specified limit states in a general manner. Especially, when we examine seismic performance of structures, inelastic deformation and structural energy absorption has to be consistently predicted with reasonable linkage to the sound mechanics of materials and structures. General structural engineering tool is to be devised for the new system and order in the next century.

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