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Optimal locations of point supports in laminated rectangular plates for maximum fundamental frequency

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Abstract. This paper investigates the optimal locations of internal point supports in a symmetric cross-ply laminated rectangular plate for maximum fundamental frequency of vibration. The method used for solving this optimization problem involves the Rayleigh-Ritz method for the vibration analysis and the simplex method of Nelder and Mead for the iterative search of the optimum support locations. Being a continuum method, the Rayleigh-Ritz method allows easy handling of the changing point support locations during the optimization search. Rectangular plates of various boundary conditions, aspect ratios, composed of different numbers of layers, and with one, two and three internal point supports are analysed. The interesting results on the optimal locations of the point supports showed that (a) there are multiple solutions; (b) the locations are dependent on both the plate aspect ratios and the number of layers; (c) the fundamental frequency may be raised significantly with appropriate positioning of the point supports.

Key words: laminated plates; optimization; point supports; Rayleigh-Ritz method; vibration.

1. Introduction

Laminated plates are extensively used in the aeronautical and aerospace industry and in other fields of modern technology. Comparing with traditional materials (such as steel and metal alloys), laminated materials have many advantages such as high strength to weight ratio, high stiffness, increased service life, good thermal insulation and non-magnetic properties. Moreover, laminates can be tailored to provide mechanical properties to meet design requirements by careful selection of the matrix material, fibre orientation and fibre volume fraction.

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When using thin laminated plates, it is sometimes necessary to enhance the stiffness of the plates against deflection, buckling and vibration. One way to do this is to introduce internal supports for the plates. Recently, Wang, *et al.* (1992, 1993), Xiang, *et al.* (1996a, b) have shown that by making a paradigm shift in conventional design to allow for the location of internal supports in isotropic plates to be optimally designed, plate buckling capacities and frequencies can be considerably raised. This paper extends the aforementioned studies to thin symmetric cross-ply laminated rectangular plates with internal rigid point supports. These simple point supports impose a zero transverse deflection at their positions. The locations of the point supports are to be optimally selected so as to maximize the fundamental frequency of the laminated plate.

The computational method proposed in the earlier studies (Xiang, *et al.* 1996b) is used herein for determining the point support locations in the rectangular laminated plates. It involves the Rayleigh-Ritz method for the vibration analysis and the Nelder and Mead simplex method for the direct optimization search of the optimal support locations. The advantage of using the Ritz method as opposed to discretization methods is that it does away with the need to coincide the nodal points with the changing positions of the point supports during the search iterations.

The study considers rectangular plates of various boundary conditions, aspect ratios, number of layers and number of internal point supports. In addition to determining the optimal locations of the point supports for the variety of rectangular plate designs, the sensitivity of the fundamental frequency with respect to the support locations is also examined.

2. Rayleigh-Ritz method for vibration analysis

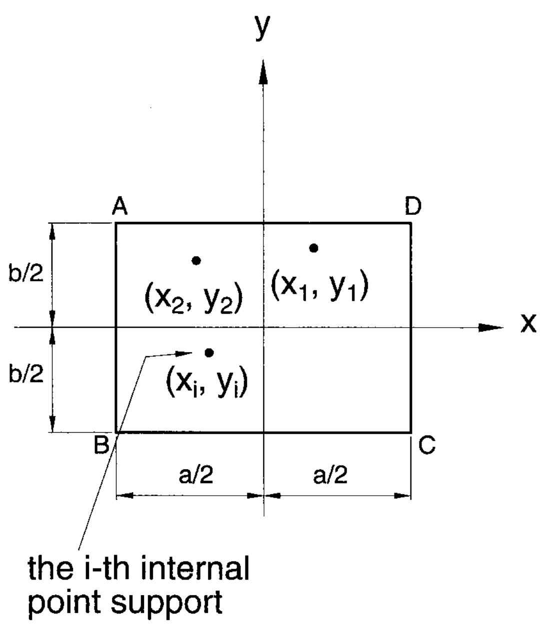


Fig. 1 Geometry and coordinate system of a symmetric cross-ply laminated plate with internal point supports.

Consider a flat, symmetric cross-ply laminated rectangular plate of thickness h, length a, width b, and number of layers l as shown in Fig. 1. The plate may have any combination of edge boundary conditions and is further supported by M number of internal point supports that impose a zero transverse deflection constraint at their locations. The problem is to determine the optimal locations of the internal point supports for maximum fundamental frequency of vibration.

Assuming harmonic and small amplitude vibrations, the bending strain energy U and the kinetic energy T for a symmetric cross-ply laminated plate are given by:

$$U=\frac{1}{2}\int_A\left\{D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2+4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}dA \tag{1}$$

$$T=\frac{1}{2}\rho h\omega^2\int_A w^2\,dA \tag{2}$$

in which x, y are the coordinates as shown in Fig. 1, A = plate area, w = transverse displacement at the midplane of the plate, ρ = plate density, ω = angular frequency of vibration and D_{ij} are the plate stiffness rigidities which are defined as

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{l}\bar{Q}_{ijk}(h_k^3-h_{k-1}^3) \qquad i,\ j=1,\ 2,\ 6 \tag{3}$$

where h_k is the distance from utmost surface of the k-th layer to the midsurface of the laminated plate and the reduced stiffnesses Q_{ijk} are given by

$$\bar{Q}_{11k}=Q_{11k}\ \cos^4\theta_k+Q_{22k}\ \sin^4\theta_k \tag{4a}$$

$$\bar{Q}_{12k}=Q_{12k}(\cos^4\theta_k+\sin^4\theta_k) \tag{4b}$$

$$\bar{Q}_{22k}=Q_{11k}\ \sin^4\theta_k+Q_{22k}\ \cos^4\theta_k \tag{4c}$$

$$\bar{Q}_{66k}=Q_{66k}(\cos^4\theta_k+\sin^4\theta_k) \tag{4d}$$

with θ_k (either $=0°$ or $90°$) being the fibre orientation for the k-th layer and

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}} \tag{5a}$$

$$Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \tag{5b}$$

$$Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}} \tag{5c}$$

$$Q_{66}=G_{12} \tag{5d}$$

The rigid point supports may be simulated by elastic lateral springs with sufficiently large spring constant S (say, 10^9). The strain energy R due to the deformation of the springs is given by

$$R=\sum_{i=1}^{M}\frac{1}{2}S_i[w(x_i,\ y_i)]^2 \tag{6}$$

where $w(x_i, y_i)$ is the transverse displacement of the plate at the i-th point support whose coordinates are (x_i, y_i).

Using the *pb*-2 Ritz method (Liew and Wang 1992, 1993), the transverse displacement w may be parameterized by

$$w(x, y)=\Phi(x, y) \sum_{q=0}^{p} \sum_{i=0}^{q} c_s \phi_s(x, y) \tag{7}$$

where p is the degree set of a complete two-dimensional polynomial, c_s the s-th Ritz coefficient and $\phi_s(x, y)$ the s-th term of the polynomial function given by

$$\phi_s(x, y)=x^i y^{q-i} \tag{8}$$

The subscript s is defined as

$$s=\frac{(q+1)(q+2)}{2}-i \tag{9}$$

The basic function $\Phi(x, y)$, in Eq. (7), ensures that the Ritz function satisfies the geometric boundary conditions. This function is defined by the product of the boundary equations of the plate raised to appropriate powers, i.e.,

$$\Phi(x, y)=\left(x+\frac{a}{2}\right)^{r1}\left(y+\frac{b}{2}\right)^{r2}\left(x-\frac{a}{2}\right)^{r3}\left(y-\frac{b}{2}\right)^{r4} \tag{10}$$

where the value of r_j depends on the support edge condition and is given by

$$r_j=0 \text{ if the } j\text{-th edge is free (F)} \tag{11a}$$

$$r_j=1 \text{ if the } j\text{-th edge is simply supported (S)} \tag{11b}$$

$$r_j=2 \text{ if the } j\text{-th edge is clamped (C)} \tag{11c}$$

By substituting Eq. (7) into Eqs. (1), (2) and (6) and then minimizing the total energy functional with respect to the Ritz coefficients, we obtain

$$\frac{\partial}{\partial c_s}=(U+R-T)=0 \text{ for } s=1, 2, ..., m \tag{12}$$

where $m=(p+1)(p+2)/2$ is the total number of polynomial terms in Eq. (7).

Eq. (12) is the governing eigenvalue equation for the free vibration of the laminated rectangular plate with internal point supports and may be written as:

$$([K]+[L]-\omega^2[M])\{c\}=\{0\} \tag{13}$$

where $\{c\}$ is a column matrix containing the unknown Ritz coefficients and the elements of the stiffness matrix $[K]$, spring stiffness matrix $[L]$ and the mass matrix $[M]$ are furnished by

$$K_{ij}=[D_{11}\Omega_{ij}^{2020}+D_{12}(\Omega_{ij}^{2002}+\Omega_{ij}^{0220})+D_{22}\Omega_{ij}^{0202}+4D_{66}\Omega_{ij}^{1111}] \tag{14}$$

$$L_{ij}=\sum_{p=1}^{M} S_p[\Phi(x_p, y_p)\phi_i(x_p, y_p)]\,[\Phi(x_p, y_p)\phi_j(x_p, y_p)] \tag{15}$$

$$M_{ij}=\rho h\Omega_{ij}^{0000} \tag{16}$$

$$\Omega_{ij}^{klmn}=\int_A \frac{\partial^{k+l}[\Phi(x,\ y)\phi_i(x,\ y)]}{\partial x^k \partial y^l}\frac{\partial^{m+n}[\Phi(x,\ y)\phi_j(x,\ y)]}{\partial x^m \partial y^n}\,dA \tag{17}$$

where $i,\ j=1,\ 2,\ ...,\ m$.

The fundamental frequency ω_1 is obtained by solving the generalised eigenvalue equation [Eq. (13)] for the lowest positive value using the subroutine RSG of the EISPACK package (Smith, *et al.* 1976).

3. Optimization procedure

The problem at hand of seeking the optimal locations $(x_i,\ y_i)$, $i=1,\ 2,\ ...,\ M$ of M number of point supports in the laminated rectangular plate for maximum fundamental frequency may be stated as

$$\operatorname*{Max}_{x_i,\,y_i}\ \omega_1 \rightarrow \operatorname*{Min}_{x_i,\,y_i}\ -\omega_1 \tag{18}$$

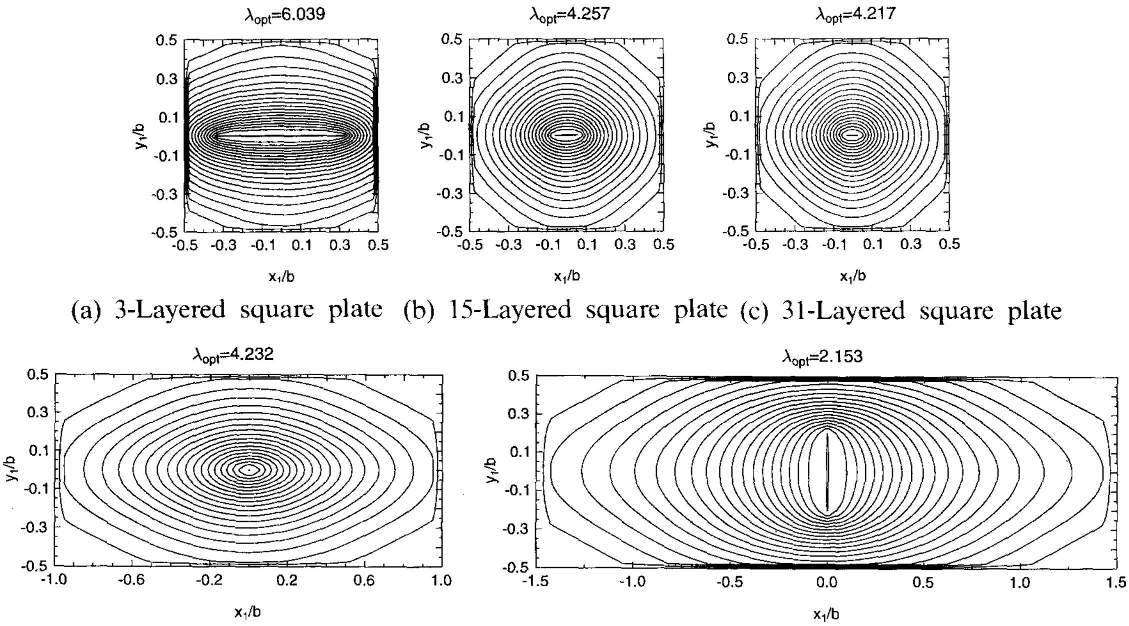
This unconstrained optimization problem may be solved using any standard direct search method that requires only the evaluation of function values. We have adopted the Nelder and Mead (1965) simplex method for this purpose.

Note that in the Rayleigh-Ritz method, a single integration operation for the stiffness and mass matrices defined by Eqs. (14) and (16) is only required. Subsequent iterations in the search for the optimal solution involve simple reformatting of the spring stiffness matrix $[L]$ in Eq. (15). Such a strategy enables a significant reduction in the computational effort for the optimization exercise. On the other hand, if one uses the finite element method, a sufficiently fine mesh design is needed so that the point supports can be located anywhere in the plate domain. Alternatively, a regeneration of an adopted coarser mesh must be done for each optimization iteration to cater for the changing positions of the point supports. Both these finite element strategies would require a considerable amount of computational effort when compared to the present Rayleigh-Ritz method.

4. Numerical results and discussions

The optimal locations of point supports for symmetric cross-ply laminated plates with one, two and three point supports are presented in this section. The plate is composed of an odd number of layers which have the same geometric and material properties and the stacking order is of (0°/90°/.../0°). The material properties of the plate are taken to be $E_1/E_2=40$, $G_{12}=0.6E_2$ and $\nu_{12}=0.25$. The fundamental frequency of the plate is expressed in a nondimensionalised form $\lambda=(\omega_1 b^2/\pi^2)\sqrt{\rho h/D_{22}}$. For convenience, a set of symbol is used to describe the boundary conditions of a plate. The symbol SFCF, for instance, represents simply supported, free, clamped and free conditions at the edges AB, BC, CD, and AD, respectively, as shown in Fig. 1.

Note that convergence studies against the number of degrees of polynomials p in the Ritz function were carried out for plates having different combinations of edge conditions and point supports. It was found that a 14 degree (i.e., $p=14$) is adequate and thus all results presented herein are based on $p=14$.



(a) 3-Layered square plate (b) 15-Layered square plate (c) 31-Layered square plate

(d) 3-Layered plate of aspect ratio a/b=2 (e) 3-Layered plate of aspect ratio a/b=3

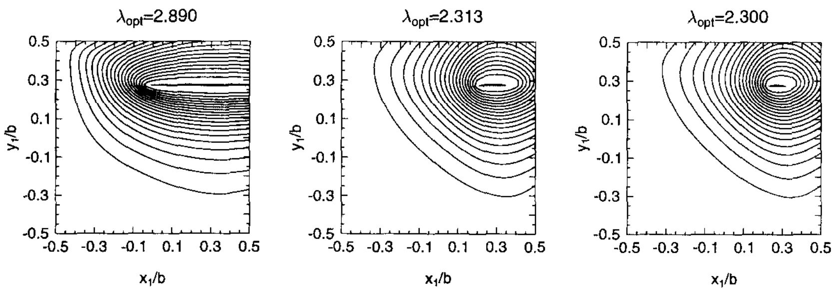
Fig. 2 Contour plots of fundamental frequency parameter, λ, with respect to the position of the point support for SSSS plates.

4.1. Laminated plate with a single point support

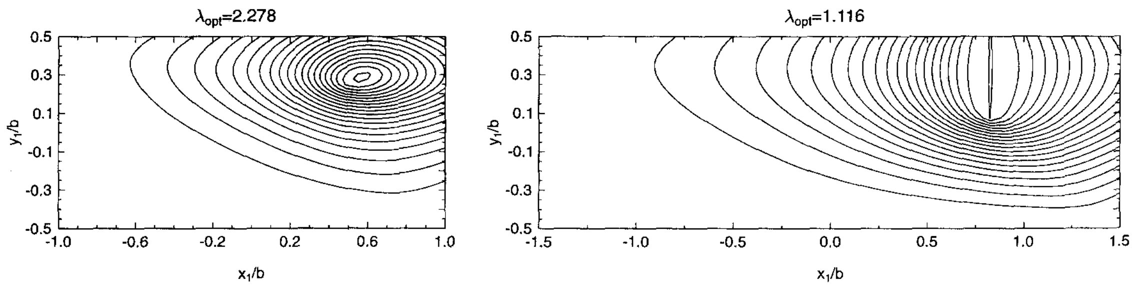
Fig. 2 shows the contour plots of the fundamental frequency parameter with respect to the position of a single point support $(x_1/b, y_1/b)$ in a simply supported (SSSS) plate. The frequency parameter λ_{opt} corresponding to the optimal locations of the point support for each case is also presented in the figure. For a three-layered square plate, the optimal location of the point support lies in the horizontal centreline of the plate for a certain range as shown in Fig. 2a. The reason for this range is that the laminated plate has the weaker stiffness in the vertical direction (y-direction). However, as the number of layers increases to 15 (see Fig. 2b) and to 31 (see Fig. 2c), this range of optimal support positions decreases because the difference in stiffnesses for both directions of the laminated plate is reduced. This shows that the optimal location of the point support is dependent on the number of layers. Also due to the cross-ply lamination, a range of support locations occurs whereas in the case of an isotropic plate, the optimal location of the point support for a square plate is found only at the plate midcentre.

By changing the aspect ratio of the three-layered rectangular plate, it can be seen from Figs. 2a, 2d and 2e that the horizontal range of optimal support locations first decreases to a small point when $a/b=2$ and then the location changes to that along the vertical centreline of the plate. This is due to the weaker stiffness direction being first in the vertical direction for a square plate and then switches to the horizontal direction for a long rectangular plate. The results show the effect of aspect ratio on the relative directional stiffness strength of the plate and thus the optimal location of the point support.

The optimal support locations in the case of clamped (CCCC) plates are similar to that in the simply supported plates.

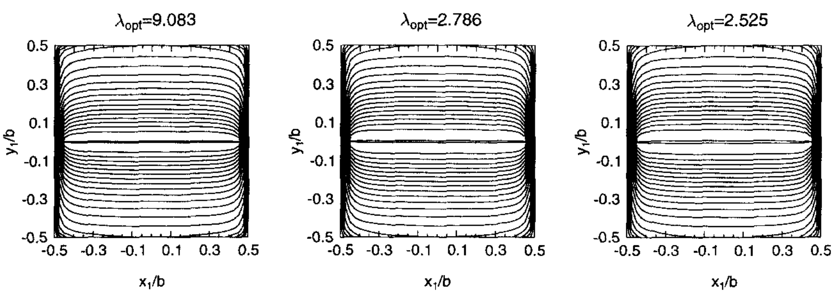


(a) 3-Layered square plate (b) 15-Layered square plate (c) 31-Layered square plate

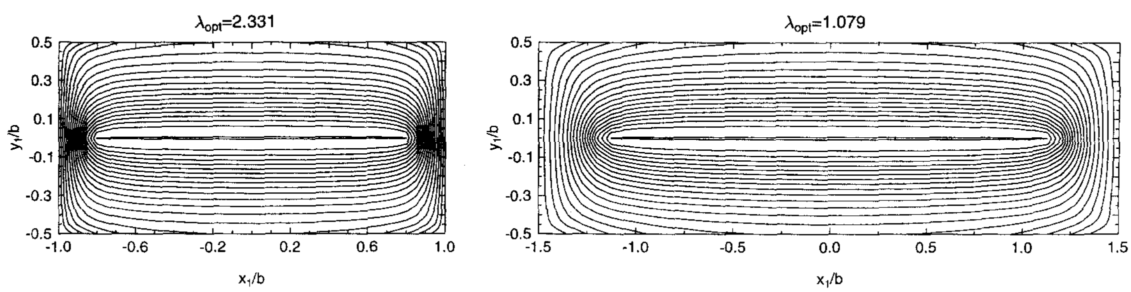


(d) 3-Layered plate of aspect ratio a/b=2 (e) 3-Layered plate of aspect ratio a/b=3

Fig. 3 Contour plots of fundamental frequency parameter, λ, with respect to the position of the point support for SSFF plates.

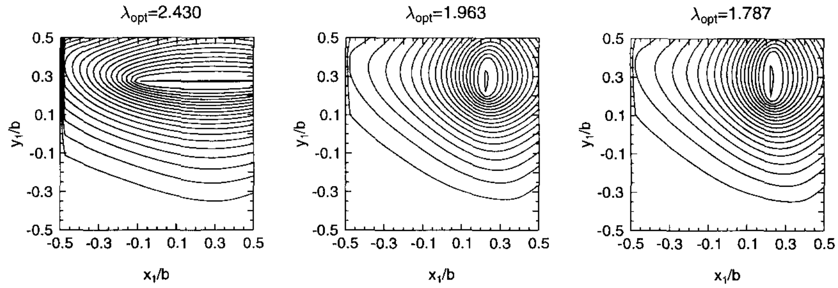


(a) 3-Layered square plate (b) 15-Layered square plate (c) 31-Layered square plate

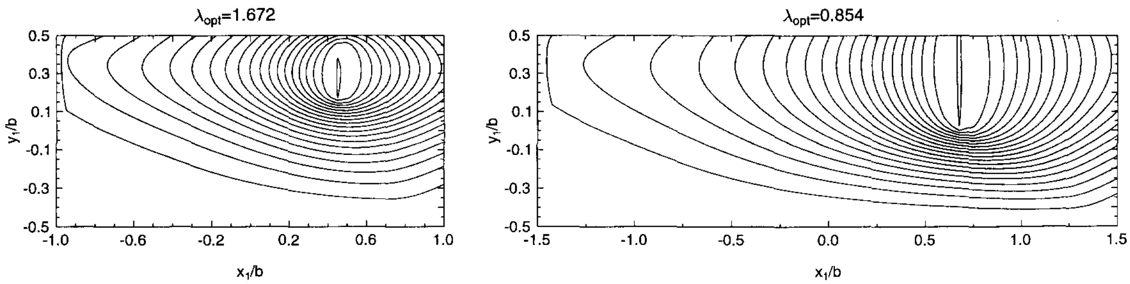


(d) 3-Layered plate of aspect ratio a/b=2 (e) 3-Layered plate of aspect ratio a/b=3

Fig. 4 Contour plots of fundamental frequency parameter, λ, with respect to the position of the point support for CFCF plates.



(a) 3-Layered square plate (b) 15-Layered square plate (c) 31-Layered square plate



(d) 3-Layered plate of aspect ratio a/b=2 (e) 3-Layered plate of aspectr ratio a/b=3

Fig. 5 Contour plots of fundamental frequency parameter, λ, with respect to the position of the point support for SCFF plates.

Figs. 3a to 3e show the contour plots of the fundamental frequency with respect to the support location (x_1/b, y_1/b) and also the number of layers (l=3, 15 and 31) for a SSFF plate. For the three-layered square plate, the optimal location lies in a narrow band parallel to the stronger stiffness direction and about a quarter length from the free edge as shown in Fig. 3a. This band shrinks to a smaller region with increasing number of layers as shown by Figs. 3b and 3c. This result is interesting as it differs significantly from the isotropic square plate case whose optimal support locations are found along the upper portion of the diagonal that connects the fixed corner point to the free corner (see Xiang, *et al.* 1996). The influence of the aspect ratios on the optimal support locations is shown by Figs. 3a, 3d and 3e for the three-layered plates. The band first contracts and then rotated to the vertical direction as the aspect ratio increases.

Figs. 4a to 4e present the contour plots for the fundamental frequency of a CFCF laminated plate with respect to the location of a point support (x_1/b, y_1/b). Both the number of layers and the aspect ratio have little effect on the optimal locations of the point support found at the plate centreline that is parallel to the free edges. Similar results are obtained for SFSF plates.

The contour plots for the case of a SCFF plate are given in Figs. 5a to 5e. The effects of the number of layers and aspect ratio on the optimal locations of the point support can be readily seen from these figures.

4.2. Laminated plates with two point supports

First we consider a simply supported (SSSS) square plate composed of three-layers with two internal point supports of coordinates (x_1/b, y_1/b) and (x_2/b, y_2/b). The optimization study showed

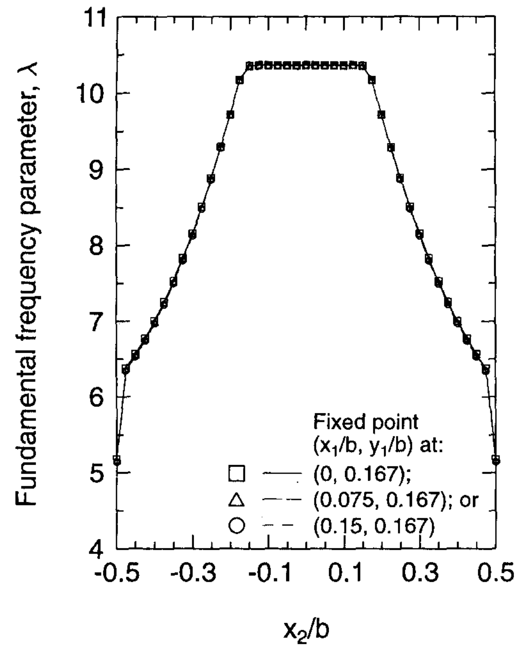


Fig. 6 Variation of fundamental frequency parameter, λ, of a three-layered SSSS square plate versus the varying x-coordinate of point support $(x_2/b, y_2/b)$ with point support $(x_1/b, y_1/b)$ fixed.

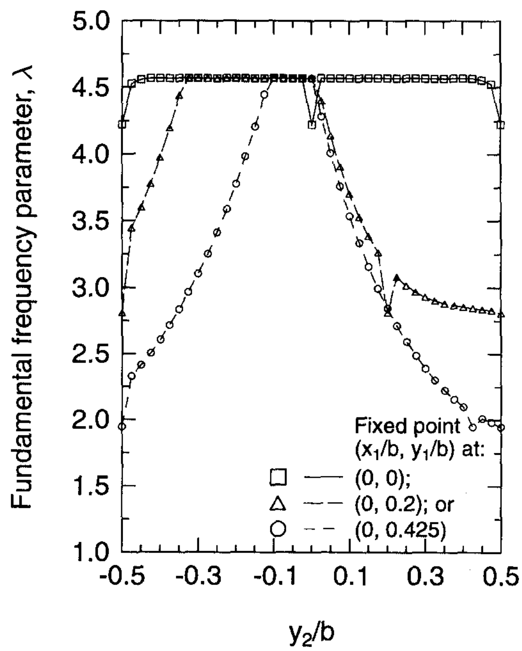
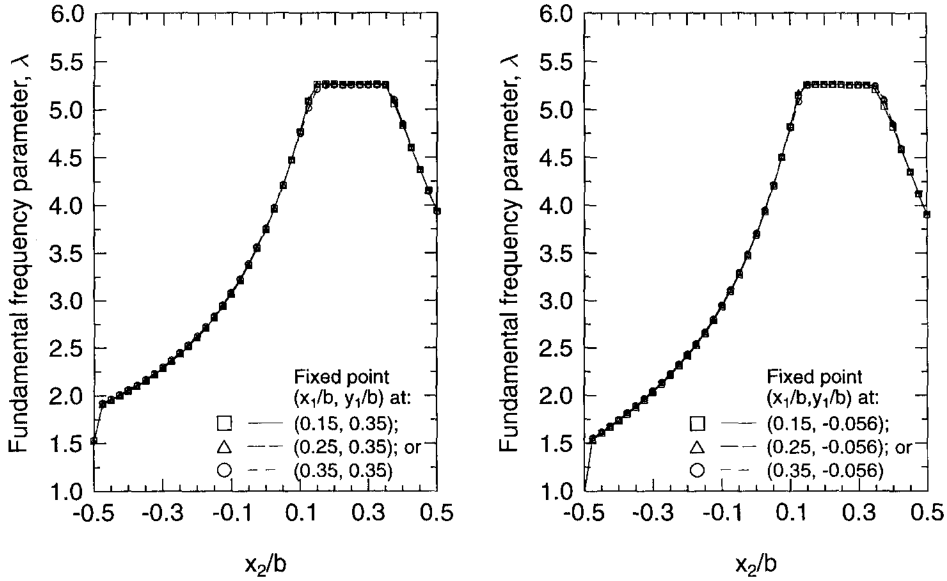


Fig. 7 Variation of fundamental frequency parameter, λ, of a 31-layered SSSS square plate versus the varying y-coordinate of point support $(x_2/b, y_2/b)$ with point support $(x_1/b, y_1/b)$ fixed.

multiple solutions with the locations of the point supports lying on the two horizontal lines with $y/b = -0.167$ and $y/b = 0.167$ for a certain range of x coordinates. To determine this range,



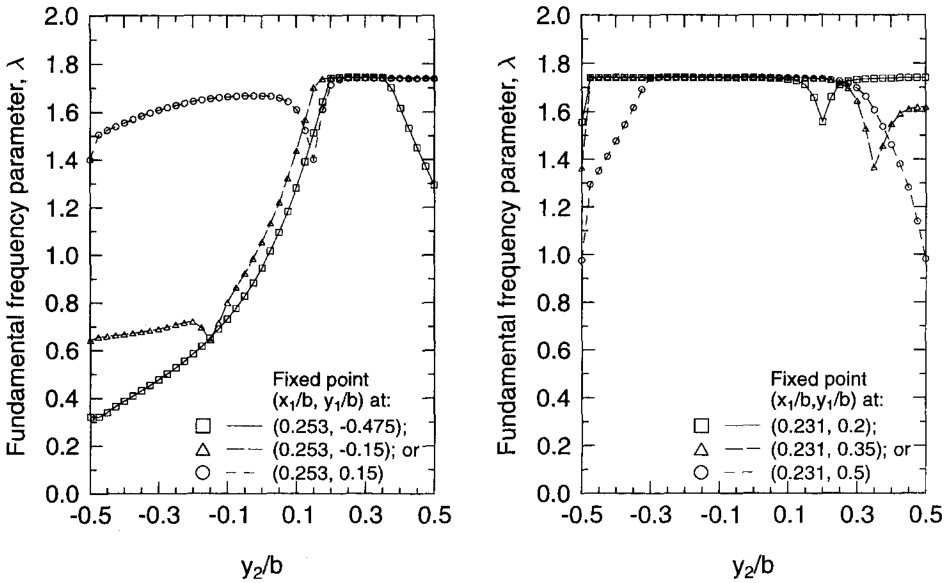
(a) Moving point along line $y_2/b = -0.056$ (b) Moving point along line $y_2/b = 0.35$

Fig. 8 Variation of fundamental frequency parameter, λ, of a three-layered SSFF square plate versus the varying x-coordinate of point support $(x_2/b,\ y_2/b)$ with point support $(x_1/b,\ y_1/b)$ fixed.

one point support $(x_1/b,\ y_1/b)$ was fixed at one of the horizontal line $y/b = 0.167$ while the other point support location $(x_2/b,\ y_2/b)$ was allowed to vary along the other horizontal line $y/b = 0.167$. Fig. 6 shows the variations of the fundamental frequency with respect to the varying internal support location. The figure shows clearly the range of x-coordinates for optimal locations of the point supports. So for maximum fundamental frequency, the two point supports must lie within the range of $-0.15 \leq x/b < 0.15$ and $|y/b| = 0.167$. Also, it can be seen that the frequency value is sensitive to the point support location outside the optimal regime.

Next, we study the effect of the number of layers of the plate on the above solution. The number of layers was increased to 31 and again multiple optimal locations of the point supports were found. However, the optimal locations of the two point supports, $(x_1/b,\ y_1/b)$ and $(x_2/b,\ y_2/b)$, lie on the vertical centreline of the plate (i.e., $x/b = 0$). To determine the range of the support locations along this line, the variation of the fundamental frequency with respect to the y-coordinate of one of the support $(x_2/b,\ y_2/b)$ was calculated while the other support $(x_1/b,\ y_1/b)$ is fixed at various values of $y_1/b = 0$, 0.2, 0.425. Fig. 7 shows that the range is large when the fixed support is placed at $y_1/b = 0$ and contracts to a smaller range when $y_1 b = 0.425$. This optimal results are certainly different from the 3-layered plate which goes to show that the optimal locations of point supports are highly dependent on the number of layers.

A three-layered SSFF square plate with two point supports was analysed and the optimization study revealed that the optimal locations are found along two horizontal lines $y/b = -0.056$ and $y/b = 0.35$. Fig. 8 shows the variation of the fundamental frequency with respect to the varying x-coordinate of one of the support $(x_2/b, y_2/b)$ with the other support $(x_1/b, y_1/b)$ fixed. The optimal range of the x-coordiante is found to be $0.15 \leq x/b \leq 0.35$. This range is not sensitive to the location of the fixed point support as long as the fixed point support is located within the optimal range.



(a) Moving point along line $x_2/b=0.231$ (b) Moving point along line $x_2/b=0.253$

Fig. 9 Variation of fundamental frequency parameter, λ, of a 31-layered SSFF square plate versus the varying y-coordinate of point support $(x_2/b, y_2/b)$ with point support $(x_1/b, y_1/b)$ fixed.

For a 31-layered SSFF square plate, the optimal locations of the two point supports are found along two vertical lines $x/b=0.231$ and $x/b=0.253$. Fig. 9 presents the variation of the fundamental frequency with respect to the varying y-coordinate of the support $(x_2/b, y_2/b)$ with the other support $(x_1/b, y_1/b)$ fixed. Contrast to the three-layered square plate, the optimal range of the y-coordinate for the 31-layered plate is highly dependent on the location of the fixed point support.

4.3. Laminated plates with three point supports

We consider a simply supported (SSSS) square plate with three internal point supports. For a three-layered plate, the optimal locations of these supports and the corresponding plate frequencies are given in Table 1 based on 5 optimization runs using different sets of initial trial values for the support coordinates. It can be seen that the optimal locations of the supports lie on the vertical centreline of the plate with a small variation of the y-coordinate for each support.

Table 1 Optimal locations of three point supports for a three-layered SSSS square plate ($\lambda_{opt}=16.066$)

Optimization run	Coordinates of point supports		
	$(x_1/b, y_1/b)$	$(x_2/b, y_2/b)$	$(x_3/b, y_3/b)$
1	(0.000, −0.240)	(0.000, 0.029)	(0.000, 0.268)
2	(0.000, −0.252)	(0.000, 0.015)	(0.000, 0.285)
3	(0.000, −0.240)	(0.000, 0.016)	(0.000, 0.275)
4	(0.000, −0.237)	(0.000, 0.022)	(0.000, 0.282)
5	(0.000, −0.229)	(0.000, 0.000)	(0.000, 0.239)

Table 2 Optimal locations of three point supports for a three-layered CCCC square plate ($\lambda_{opt}=22.729$)

Optimization run	Coordinates of point supports		
	$(x_1/b, y_1/b)$	$(x_2/b, y_2/b)$	$(x_3/b, y_3/b)$
1	(−0.048, −0.222)	(0.051, 0.000)	(0.049, 0.222)
2	(−0.041, −0.222)	(0.031, 0.000)	(0.013, 0.222)
3	(−0.033, −0.222)	(0.045, 0.000)	(0.012, 0.222)
4	(−0.037, −0.222)	(0.052, 0.000)	(0.020, 0.222)
5	(−0.035, −0.222)	(0.056, 0.000)	(0.024, 0.222)

Table 3 Optimal locations of three point supports for a three-layered CFCF square plate ($\lambda_{opt}=11.601$)

Optimization run	Coordinates of point supports		
	$(x_1/b, y_1/b)$	$(x_2/b, y_2/b)$	$(x_3/b, y_3/b)$
1	(−0.078, −0.364)	(−0.094, 0.000)	(0.171, 0.364)
2	(−0.057, −0.364)	(−0.069, 0.000)	(0.172, 0.364)
3	(−0.252, −0.364)	(0.205, 0.000)	(0.085, 0.364)
4	(− .230, −0.364)	(0.239, 0.000)	(0.098, 0.364)
5	(−0.056, −0.364)	(−0.050, 0.000)	(0.182, 0.364)

For a three-layered clamped (CCCC) plate, the results are different from the simply supported case. Table 2 shows the results obtained from 5 different optimization runs. It can be observed that the three point supports lie on three different horizontal lines $y/b=-0.222$, $y/b=0$, $y/b=0.222$ with small variations in the x-coordinate for each support.

Table 3 gives the results of the optimal locations and the corresponding fundamental frequency for a three-layered CFCF square plate. Similar to the clamped plate, the three point supports lie on three different horizontal lines $y/b=-0.364$, $y/b=0$, $y/b=0.364$, but with a relatively larger range of x-coordinates for each support.

5. Concluding remarks

It is shown that the fundamental frequency can be significantly raised by the suitable positioning of the point supports and that multiple solutions often exist for the problem at hand. The optimal results obtained are interesting when compared to the case of isotropic plates as the optimal positions of the supports are not only dependent on the aspect ratio of the laminated rectangular plate but also on the number of layers.

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