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Optimal reinforcement design of structures under the buckling load using the homogenization design method

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Abstract. The material-based homogenization design method generates arbitrary topologies of initial structural design as well as reinforcement structural design by controlling the amount of material available. However, if a small volume constraint is specified in the design of lightweight structures, thin and slender structures are usually obtained. For these structures stability becomes one of the most important requirements. Thus, to prevent overall buckling (that is, to increase stability), the objective of the design is to maximize the buckling load of a structure. In this paper, the buckling analysis is restricted to the linear buckling behavior of a structure. The global stability requirement is defined as a stiffness constraint, and determined by solving the eigenvalue problem. The optimality conditions to update the design variables are derived based on the sequential convex approximation method and the dual method. Illustrated examples are presented to validate the feasibility of this method in the design of structures.

Key words: structural optimization; topology design; homogenization design method; buckling problem; reinforcement structure.

1. Introduction

Material-based topology optimization using the homogenization method – homogenization design method – has been studied quite extensively for continuum structural design in recent years (Bendsφe and Mota Soares 1993). Since the homogenization design method is applied to the generalized shape and topology design of optimum structures, different design criteria are considered depending on the design application. Such criteria include minimizing a mean compliance in the static problem (Bendsφe and Kikuchi 1988, Suzuki and Kikuchi 1991), maximizing eigenfrequencies in the dynamic problem (Díaz and Kikuchi 1992, Ma, *et al.* 1995), and maximizing absorbed energy in the crashworthiness problem (Mayer, *et al.* 1996). The structural layout of these design problems is controlled by the admissible amount of material within a specified design domain. It is observed that with a low volume constraint the homogenization design method usually results in a thin and slender structure which can be identified as a combination of trusses, frames, and beams. For these structural elements, the stability requirement is one of the most important measures that the structural designer must take into consideration.

As far as the optimum structural design for stability is concerned, the buckling load is consid-

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ered as the stability constraint for minimum weight design in the traditional sizing optimization problem (Knot, *et al.* 1976). In this work, the generalized eigenvalue for linear stability is defined and the optimal design is obtained by using optimality criteria based on an iterative scheme. An alternative formulation, maximizing the buckling load for constant volume of material, is proposed to optimize a two-bar and a four-bar shallow truss for nonlinear stability (Kamat, *et al.* 1984). However, it is known that in these previous works, only the physical dimensions of structural members are allowed to change.

Meanwhile, a significant weight reduction can be achieved by dealing with topology—the number of holes and connectivity—of a structure. Using material-based topology optimization, Neves, *et al.* (1995) presented a computational model to design a two-dimensional structure with a buckling load criterion. In their work, the design domain is composed of a porous material with square holes, which represents only isotropic material behavior. A single loading condition is considered to calculate buckling load, and generalized gradients are employed to deal with the nonsmooth optimization problem.

In this paper, an optimum reinforced structural design method considering a stability criterion is proposed to generate a structure with a much higher buckling load under multiple load cases. To evaluate orthotropic material properties, rectangular holes and orientation are introduced to micro-structures in the design domain. The linearized buckling problem with a finite element approxima-tion is formulated to analyze buckling behavior. The optimality criteria are presented by the convex approximation method and the dual method with the Lagrangian function of the optimization model. Finally, this method is applied to a two-dimensional as well as a three-dimensional design problem.

2. Homogenization design method

The homogenization design method entails finding the optimal material distribution within the elastic design domain while the criterion and constraints are satisfied. As shown in Fig. 1, the design domain Ω is composed of a porous material containing infinitely many microstructures, and the amount of material available is specified. In the design domain, boundary conditions are given and loading conditions, including the body force f and the traction t on the boundary Γ_h are applied. The porosity of a microstructure is represented by a rectangular hole in a two-dimensional (2D) structure and a body hole in a three-dimensional (3D) solid structure as shown in Fig. 2. Microstructure is classified as the void which contains no material (hole size=1), the solid medium which contains isotropic material (hole size=0), and the generalized porous medium which contains orthotropic material (0<hole size<1). The volume of a microstructure is defined as $\rho=\rho_0(a+b-ab)$ for a 2D structure and $\rho=\rho_0[1-(1-a)\ (1-b)\ (1-c)]$ for a 3D structure, where ρ_0 is the mass density.

Since the porosity is different over the design domain, the theory of homogenization is employed to evaluate equivalent elastic material properties of microstructures (Guedes and Kikuchi 1990). In homogenization theory, a structure is assumed to be composed of periodic microstructures, and the equivalent material properties are estimated by a limiting process that involves diminishing the microscopic size. In addition, the orientation of material axes must be considered to define mate-rial properties. A rotation angle θ is used in a 2D structure and the Euler angle $\{\theta, \psi, \varphi\}$ is used in a 3D structure to represent the orientation. Thus, elastic material properties

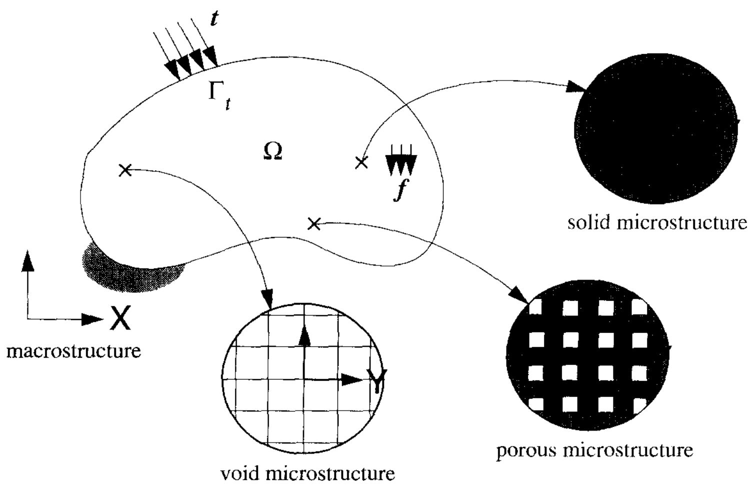


Fig. 1 Design domain and microstructures.

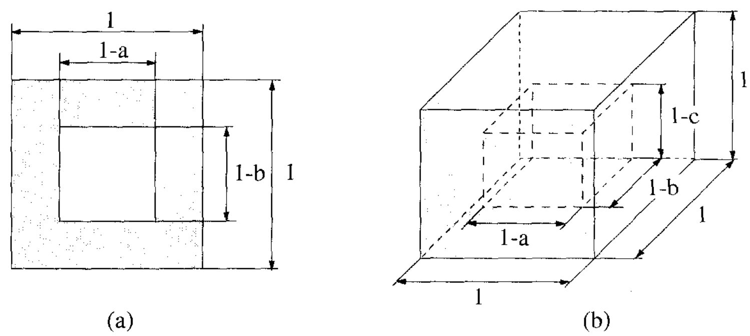


Fig. 2 Microstructures used in the homogenization design method; (a) 2D structure, and (b) 3D structure.

of a structure can be defined by the dimensions and orientation of microstructural holes.

During the optimization process, microstructures are changed between the void and the solid. This implies that material can be moved from one part of the structure to another if the total amount of material available is specified. Thus, the optimal shape and topology design of structures can be regarded as finding an optimal material distribution within a prescribed admissible structural domain.

3. Problem formulation

In a typical design problem maximizing the stiffness of a structure, the design domain is discretized into finite elements with the location of supports and applied loads as shown in Fig. 3(a), and the optimal structure is obtained by using a prescribed amount of material. As the amount of material is reduced, the homogenization design method tends to generate a thinner and more slender structure as in Fig. 3(b). The post-processed structure shown in Fig. 3(c) is considered as a two-bar truss structure. For this kind of structure, stability becomes one of the most important requirements. Thus, the post-processed structure, called a core structure, is used

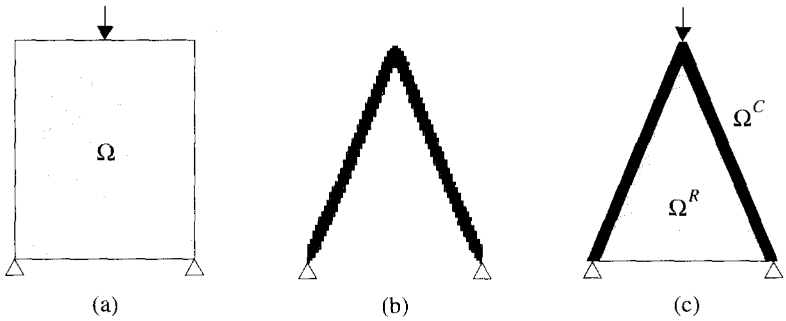


Fig. 3 Structural design using the homogenization design method; (a) stiffness problem, (b) opti-mal stiff structure, and (c) buckling problem.

as the starting point for reinforcement structural design subject to the buckling load. The core structure Ω^C is filled with a solid medium of isotropic material and Ω^R is the new design domain where reinforcement material is distributed. The buckling load and mode shape of the core structure are modified by adding reinforcement material. Thus, this approach results in finding the optimal material distribution of reinforcement in Ω^R that maximizes the buckling load of a structure.

It is often observed that the structure under the quasi−static loading responds linearly until the critical load at which buckling occurs. Thus, the buckling problem can be defined as determining the critical load associated with structural instability for prescribed loading conditions. There are two distinct instabilities which include the local buckling for an individual structure member and the overall buckling for the whole structure. Only the overall buckling problem (that is, the geometrically nonlinear problem) is considered in this work, and it is also restricted to the linear buckling problem where small prebuckling displacements are assumed and equilibrium in the initial state is of interest.

The method for analyzing the linear buckling problem is based on the displacement method of the finite element model (Zienkiewicz and Taylor 1989). Introducing Green-Lagrange strain, which is valid whether displacements or strains are large or small, the general strain can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\left(\dfrac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\dfrac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\dfrac{\partial u}{\partial y}\right)^2 + \frac{1}{2}\left(\dfrac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \frac{1}{2}\left(\dfrac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\dfrac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\dfrac{\partial w}{\partial z}\right)^2 \\ \left(\dfrac{\partial u}{\partial y}\right)\left(\dfrac{\partial u}{\partial z}\right) + \left(\dfrac{\partial v}{\partial y}\right)\left(\dfrac{\partial v}{\partial z}\right) + \left(\dfrac{\partial w}{\partial y}\right)\left(\dfrac{\partial w}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial z}\right)\left(\dfrac{\partial u}{\partial x}\right) + \left(\dfrac{\partial v}{\partial z}\right)\left(\dfrac{\partial v}{\partial x}\right) + \left(\dfrac{\partial w}{\partial z}\right)\left(\dfrac{\partial w}{\partial x}\right) \\ \left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right) + \left(\dfrac{\partial v}{\partial x}\right)\left(\dfrac{\partial v}{\partial y}\right) + \left(\dfrac{\partial w}{\partial x}\right)\left(\dfrac{\partial w}{\partial y}\right) \end{bmatrix} \tag{1}$$

where all derivatives of displacements u, v, w are computed in the fixed cartesian system of coordinates x, y, z. By substituting the nonlinear strain-displacement relation into the principle of virtual work, the discretized system equations neglecting the large displacement stiffness matrix can be written as

$$(\boldsymbol{K}+\boldsymbol{K}_\sigma)\phi=\boldsymbol{F} \tag{2}$$

where ϕ and $\boldsymbol{F}$ stand for the system degrees of freedom and the applied load, respectively. $\boldsymbol{K}$ represents the usual, small displacement stiffness matrix which is independent of the applied force and is unique for a given structure. $\boldsymbol{K}_\sigma$ is known as the stress (or geometric) stiffness matrix defined as

$$\boldsymbol{K}_\sigma=\int_V \boldsymbol{G}^T\begin{bmatrix}\sigma & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \sigma\end{bmatrix}\boldsymbol{G}dV \tag{3}$$

where $\boldsymbol{G}$ is obtained from the following relation between appropriate differentiation of shape functions and nodal displacements ϕ,

$$\left\{\frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial u}{\partial z}\ \frac{\partial v}{\partial x}\ \frac{\partial v}{\partial y}\ \frac{\partial v}{\partial z}\ \frac{\partial w}{\partial x}\ \frac{\partial w}{\partial y}\ \frac{\partial w}{\partial z}\right\}^T=\boldsymbol{G}\phi \tag{4}$$

and the stress matrix, which depends on the stress level,

$$\sigma=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ & \sigma_y & \tau_{yz}\\ \text{Sym} & & \sigma_z\end{bmatrix} \tag{5}$$

is determined by equilibrium equations of the multiple loading problem written as

$$\boldsymbol{K}^i\phi^i=\boldsymbol{F}^i \qquad i=1,\ 2,\ \ldots,\ p \tag{6}$$

where $\boldsymbol{F}^i$ is applied load vectors, ϕ^i is the corresponding deformation vectors, and p is the number of loading cases. Here the stress level is taken to be the maximum value (that is, $\max\limits_{1\le i\le p}\ \sigma^i$) among stresses evaluated in each loading case.

When a reference level of loading $\boldsymbol{F}_{ref}$ is applied, the corresponding stress stiffness matrix $(\boldsymbol{K}_\sigma)_{ref}$ can be obtained by carrying out linear static analysis. For another load level $\lambda\boldsymbol{F}_{ref}$ where λ is a scalar multiplier, the stress stiffness can be defined as $\lambda(\boldsymbol{K}_\sigma)_{ref}$ where the distribution of stresses is not changed. As buckling displacement $d\phi$ takes place from a reference configuration ϕ, there is no change in the externally applied load. Accordingly, when the onset of overall buckling stability occurs, the relation is given by

$$[\boldsymbol{K}+\lambda(\boldsymbol{K}_\sigma)_{ref}]\ \phi=[\boldsymbol{K}+\lambda(\boldsymbol{K}_\sigma)_{ref}]\ (\phi+d\phi)=\lambda\boldsymbol{F}_{ref} \tag{7}$$

By subtracting the first equation from the second equation, Eq. (7) yields the following eigenvalue problem:

$$[\boldsymbol{K}+\lambda(\boldsymbol{K}_\sigma)_{ref}]\ d\phi=0 \tag{8}$$

where eigenvalues λ give critical load factors and eigenvectors $d\phi$ associated with λ represent buckling modes. Among eigenvalues, only the smallest positive eigenvalue λ_1 is of interest, and hence, the buckling load of a structure is given by $\lambda_1\boldsymbol{F}$.

The formulation of this eigenvalue problem is similar to that of the free vibration eigenvalue problem. The difference is that only the fundamental buckling mode is usually of concern in the linear buckling problem whereas all of the modes within a certain frequency range is of concern in the vibration problem. In addition, the stress stiffness matrix $\boldsymbol{K}_\sigma$ in the linear buckling problem is not always positive definite whereas the mass matrix $\boldsymbol{M}$ in the vibration problem is usually of full rank and positive definite.

4. Optimization problem

The design domain, including a core structure and a reinforcement region, is discretized by N finite elements. 4-node (2D) and 8-node (3D) nonconforming elements (Hughes 1987) are utilized to attain improved behavior in bending situations. Thus, the design variables are hole dimension $x_i\,(i=1, \ldots, n_d)$ and orientation $\Theta_i\,(i=1, \ldots, n_\Theta)$ of elements, where n_d represents the number of size design variables ($n_d=2N$ for the 2D problem, $n_d=3N$ for the 3D problem) and n_Θ the number of orientation design variables ($n_\Theta=N$ for the 2D problem, $n_\Theta=3N$ for the 3D problem).

The design goal is to find the best distribution of material that avoids overall buckling. To maximize the buckling load, the optimization problem can be described in a discretized form as follows: Determine $\boldsymbol{x}$ and $\boldsymbol{\Theta}$ for a prescribed amount of material V_p that

$$\underset{\boldsymbol{x},\ \boldsymbol{\Theta}}{\text{maximize}}\ \lambda_1 \quad \text{or} \quad \underset{\boldsymbol{x},\ \boldsymbol{\Theta}}{\text{mimimize}}\ (-\lambda_1) \tag{9}$$

subject to

$$\sum_{i=1}^{n} \rho_i \leq V_p \tag{10}$$

and

$$0 < x_i^l \leq x_i \leq x_i^u \leq 1 \tag{11}$$

including equilibrium equations and the eigenvalue problem, where ρ_i is the element volume, and x_i^l and x_i^u represent the lower and the upper values of the size design variable, respectively. The Lagrangian function associated with the optimization problem is given by

$$L = -\lambda_1 + \kappa\left(\sum_{i=1}^{N} \rho_i - V_p\right) + \sum_{i=1}^{n_d} [\alpha_i^l(x_i^l - x_i) + \alpha_i^u(x_i - x_i^u)] \tag{12}$$

where Λ, α_i^l, and α_i^u are Lagrangian multipliers with $\kappa \geq 0$, $\alpha_i^l \geq 0$ and $\alpha_i^u \geq 0$. Here, to make the optimization problem convex, a shift parameter μ is introduced into the objective function and the Lagrangian function can be written as

$$L = -\lambda_1 - \mu\left(\sum_{i=1}^{N} \rho_i - V_p\right) + \Lambda\left(\sum_{i=1}^{N} \rho_i - V_p\right) + \sum_{i=1}^{n_d} [\alpha_i^l(x_i^l - x_i) + \alpha_i^u(x_i - x_i^u)] \tag{13}$$

where $\Lambda = \kappa + \mu$. Using a generalized reciprocal approximation, an intermediate variable is suggested as the ξth power of a design variable and then Lagrangian function is linearized with a sequential approximate approach in the κth iteration as

$$L^k(\boldsymbol{x},\ \Lambda,\ \alpha) = l_0^k + \sum_{i=1}^{N} \left[\frac{1}{\xi}\left(\frac{\partial \lambda_1}{\partial x_i} + \mu \frac{\partial \rho_i}{\partial x_i}\right)\Bigg|_k (x_i^k)^{\xi+1} x_i^{-\xi} + (\Lambda + \mu) \frac{\partial \rho_i}{\partial x_i}\Bigg|_k x_i \right]$$
$$+ \sum_{i=1}^{n_d} [\alpha_i^l(x_i^l - x_i) + \alpha_i^u(x_i - x_i^u)] \tag{14}$$

where l_0^k is a constant. The approximated optimization problem can be solved by using the dual method defined as

$$\max_{\Lambda,\ \alpha} \ \min_{\boldsymbol{x}} L^k(\boldsymbol{x}, \Lambda, \alpha) \quad \text{subject to } \Lambda \geq 0,\ \alpha \geq 0 \tag{15}$$

since the approximated Lagrangian function is convex and separable (Haftka and Gürdal 1992). Thus, the stationarity of L with respect to x_i leads to the following updating scheme of size design variables for the kth iteration as

$$x_i^{k+1} = \begin{cases} x_i^l & \text{if } i \in I_l^k = \{i \mid z_i^k x_i^k/(\Lambda^k)^\eta \leq x_i^l\} \\ z_i^k x_i^k/(\Lambda^k)^\eta & \text{if } i \in I^k = \{i \mid x_i^l \leq z_i^k x_i^k/(\Lambda^k)^\eta \leq x_i^u\} \\ x_i^u & \text{if } i \in I_u^k = \{i \mid x_i^u \leq z_i^k x_i^k/(\Lambda^k)^\eta\} \end{cases} \tag{16}$$

where η is a given parameter. The scale factor z_i^k and the Lagrangian multiplier Λ^k are defined as

$$z_i^k = \left(\mu^k + \left(\frac{\dfrac{\partial \lambda_1}{\partial x_i}}{\dfrac{\partial \rho_i}{\partial x_i}} \right) \Bigg|_k \right)^\eta, \quad \text{and} \quad (\Lambda^k)^\eta = -\frac{\sum_{i \in I^k} \dfrac{\partial \rho_i}{\partial x_i}\Big|_k z_i^k x_i^k}{\left(\sum_{i=1}^N \rho_i - V_p\right)\Big|_k + \sum_{i \in I_l^k} \dfrac{\partial \rho_i}{\partial x_i}\Big|_k x_i^l + \sum_{i \in I_u^k} \dfrac{\partial \rho_i}{\partial x_i}\Big|_k x_i^u} \tag{17}$$

In order to accommodate the updating scheme, it is necessary to evaluate the sensitivity of the objective function and the constraint. The differentiability of λ_1 must be guaranteed to obtain the sensitivity of the objective function with respect to design variable $\boldsymbol{x}$. For simplicity, the problem is restricted to the case in which the lowest eigenvalues are not repeated. Thus, the sensitivity of the objective function can be obtained as

$$\frac{\partial \lambda_1}{\partial \boldsymbol{x}} = -\frac{(d\phi)_1^T \left(\dfrac{\partial \boldsymbol{K}}{\partial \boldsymbol{x}} + \lambda_1 \dfrac{\partial \boldsymbol{K}_\sigma}{\partial \boldsymbol{x}} \right) (d\phi)_1}{(d\phi)_1^T \boldsymbol{K}_\sigma (d\phi)_1} \tag{18}$$

where $(d\phi)_1$ stands for the buckling mode corresponding to the buckling load. The stationarity of L with respect to Θ_i can be regarded as aligning the angle Θ_i to the principal stress direction (Pedersen 1989). The overall procedure for optimization is shown in Fig. 4.

5. Examples

In order to show the optimal reinforcement design subject to the buckling load, two examples are provided. The first example illustrates finding the optimal reinforcement configuration of a plane stress 2D structure, and the second is for general 3D reinforcement structure design. The isotropic material has the property of Young's modulus $E=100$, Poisson's ratio $\nu=0.3$, and mass density $\rho_0=7.56\times10^{-5}$.

5.1. 2D example

Fig. 5 (a) shows the square design domain for the stiffness problem under compressive multiple loadings (L_1 and L_2). With 1,600 (40×40) QUAD4 finite elements discretization and 30% volume

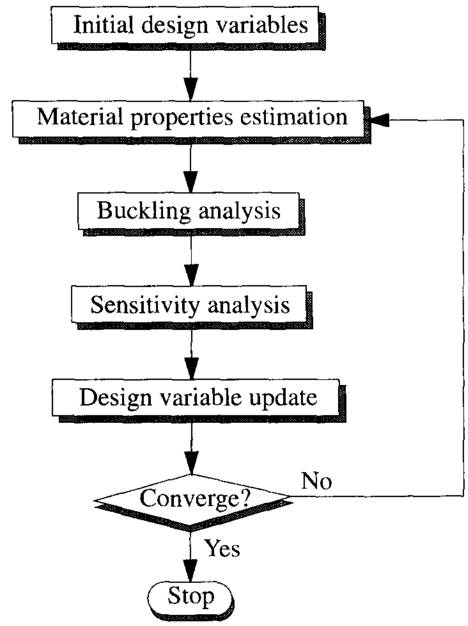


Fig. 4 Optimization procedure.

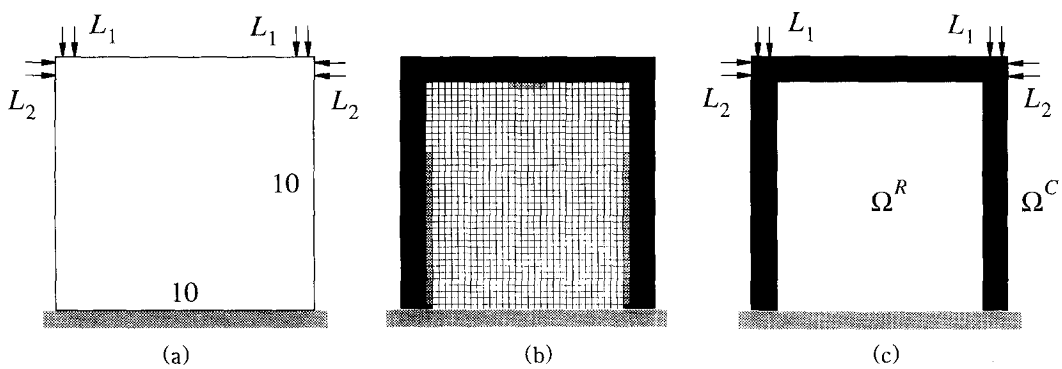


Fig. 5 Design domain of 2D problem; (a) stiffness problem, (b) optimal stiff structure, and (c) buckling problem.

constraint, the homogenization design method generates a frame–like structure, in which the stability criterion is important, as shown in Fig. 5 (b). In the buckling problem, the optimal stiff structure is regarded as a core structure Ω^C, which is not changed in the optimization process, and the inside domain of a core structure is specified as reinforcement region Ω^R in Fig. 5 (c). A uniformly perforated design domain ($a=b=0.2$ and $\theta=0.0$) is assumed as an initial configuration, and the volume constraint is given as 40% of total material including a core structure and reinforcement region. Material distributions during the optimization procedure are shown in Fig. 6, and the corresponding buckling mode shapes are illustrated. It is seen that reinforcement is added to resist the largest deformations in the buckling mode shape. Finally, the optimal reinforced structure is generated in Fig. 7 (a), and it can be interpreted as a combination of cross frames. Fig. 7 (b) shows the convergence history of the objective function and

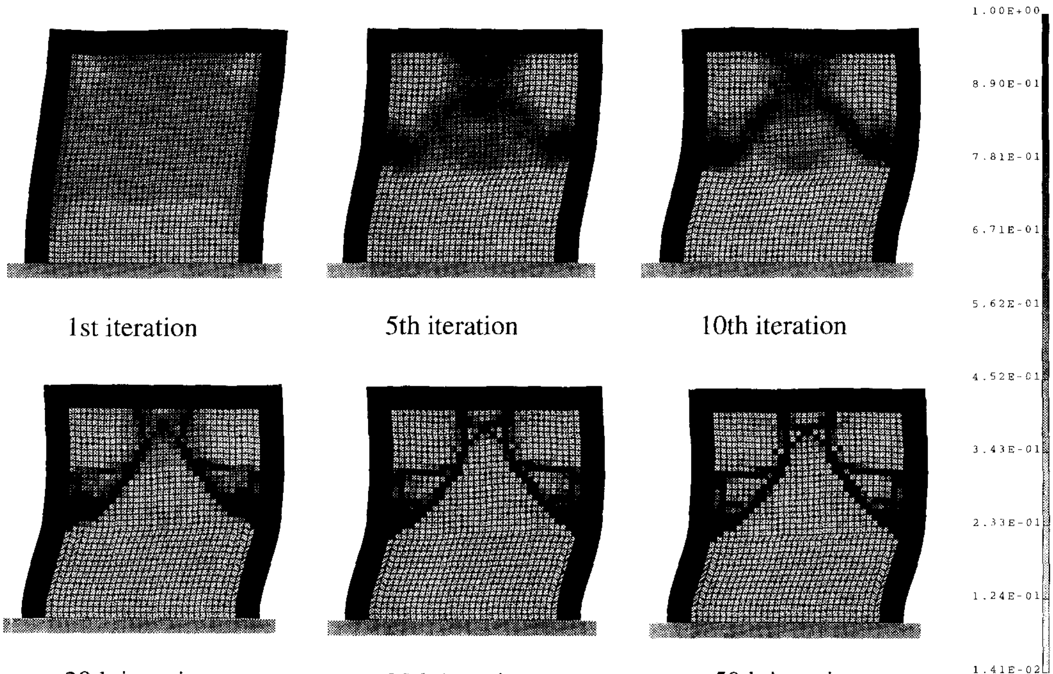


Fig. 6 Material distributions of 2D problem.

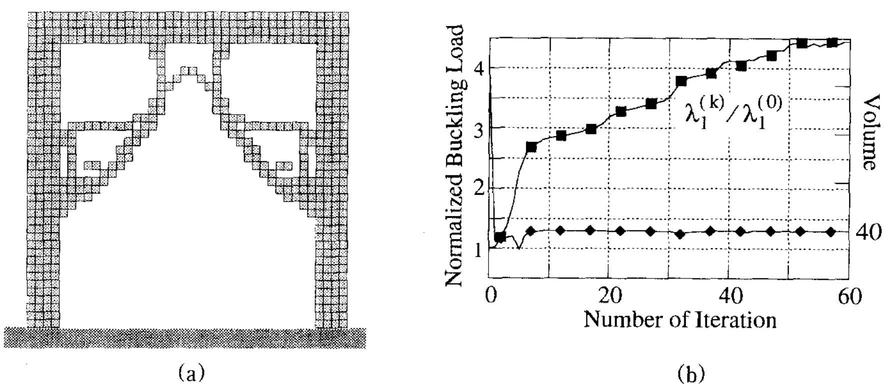


Fig. 7 Results of 2D problem; (a) optimal reinforced structure, and (b) convergence history.

the constraint function. The buckling load of the optimal reinforced structure is increased by about four times compared to the initial structure of uniform material distribution, and the volume constraint is satisfied.

5.2. 3D problem

A simple table-like structure is chosen as a core structure Ω^C at the beginning and the interior space is assigned as a reinforcement region Ω^R as shown in Fig. 8. The whole design domain is discretized by 1,000 (10×10×10) HEXA8 finite elements, and a distributed force is applied

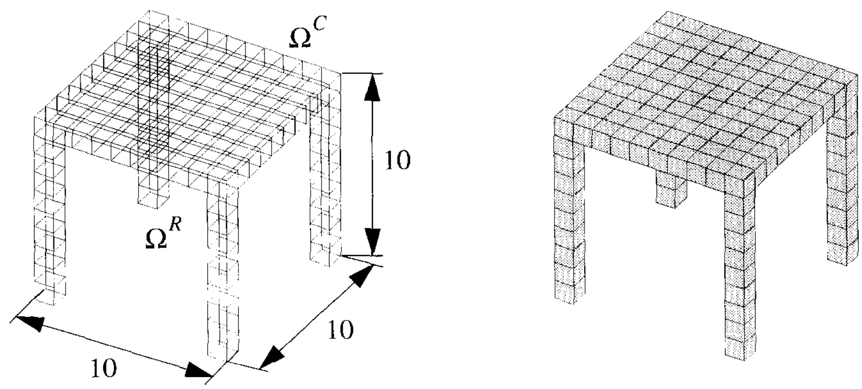


Fig. 8 Design domain of 3D problem.

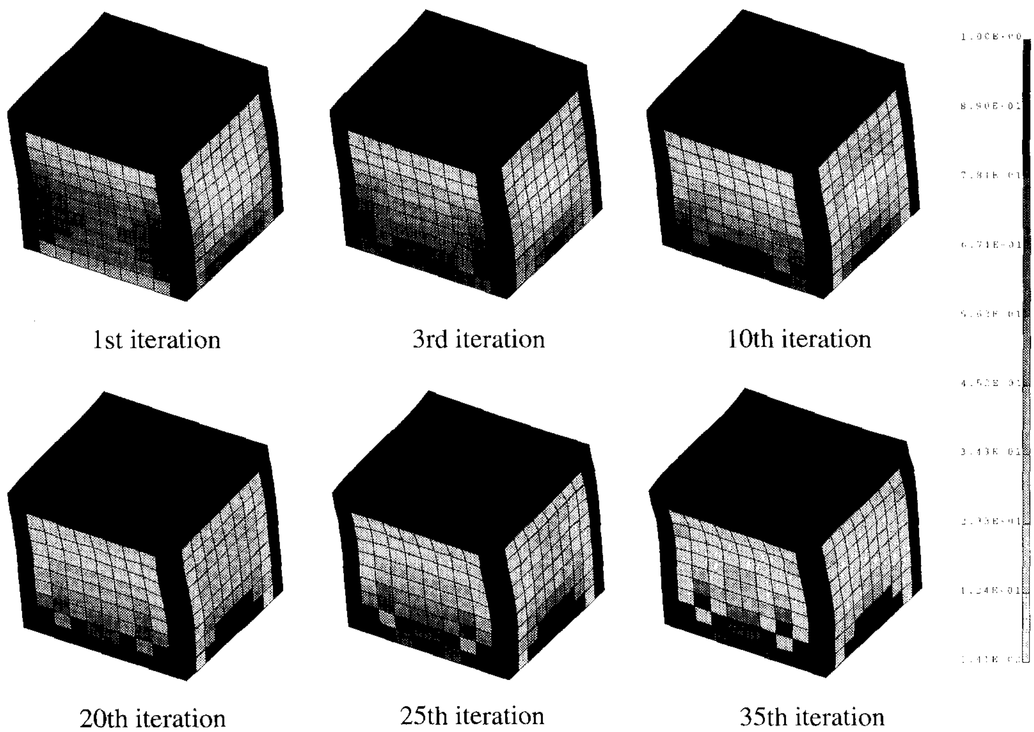
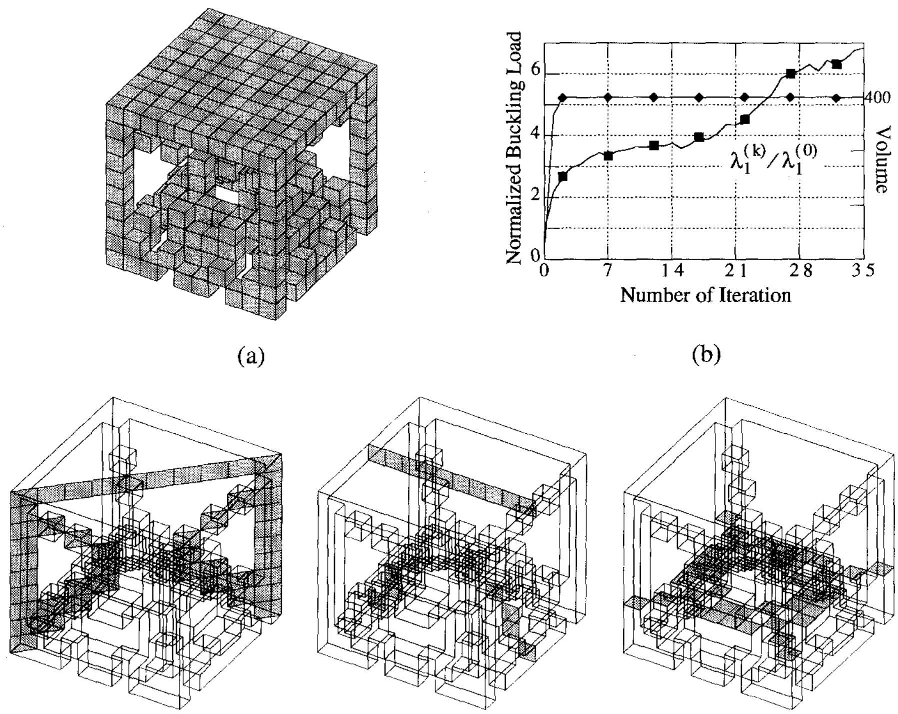


Fig. 9 Material distributions of 3D problem.

to the top surface of the structure. The prescribed amount of material, including a core and a reinforcement, is constrained with 40% of total material available, and the reinforcement design domain is initially considered as uniform material distribution in which size design variables are set to 0.1 and the Euler angles are assigned to zero. Fig. 9 illustrates the history of the structural generation for optimal reinforcement. The material is moved from one part of the reinforcement domain to another part, and converges to the final optimal reinforced structure—a supporting structure—shown in Fig. 10 (a). The convergence history of the objective function and the constraint function is presented in Fig. 10 (b). It shows that the proposed method generates



(c)

Fig. 10 Results of 3D problem; (a) optimal reinforced structure, and (b) convergence history, and (c) cross sections of the reinforced structure.

the reinforced structure of a higher buckling load and satisfies the volume constraint. Several cross-sectional views in Fig. 10 (c) are provided to identify the reinforced structure.

6. Conclusions

The topology design methodology associated with a homogenization method is proposed to find the best configuration of structural reinforcement of a structure subject to the buckling load. To analyze the buckling problem, linear buckling behavior is assumed, and the optimization problem is formulated to maximize the buckling load while the amount of material is specified. The sequential convex approximation method is utilized to derive the optimality conditions. The results of the optimal reinforced structure design indicate that the buckling criterion can be incorporated into the homogenization design method. Even though the linearized buckling model has limitations in estimating the actual collapse of structures, the buckling mode can be utilized to detect the imperfection of structures. Therefore, this method can provide the structural designer with an effective tool for designing reinforcement structures.

Acknowledgements

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