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Complementarity and nonlinear structural analysis of skeletal structures

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Abstract. This paper deals with the formulation and solution of a wide class of structures, in the presence of both geometric and material nonlinearities, as a particular mathematical programming problem. We first present key ideas for the nonholonomic (path dependent) rate formulation for a suitably discretized structural model before we develop its computationally advantageous stepwise holonomic (path independent) counterpart. A feature of the final mathematical programming problem, known as a nonlinear complementarity problem, is that the governing relations exhibit symmetry as a result of the introduction of so-called nonlinear "residuals". One advantage of this form is that it facilitates application of a particular iterative algorithm, in essence a predictor-corrector method, for the solution process. As an illustrative example, we specifically consider the simplest case of plane trusses and detail in particular the general methodology for establishing the static-kinematic relations in a dual format. Extension to other skeletal structures is conceptually transparent. Some numerical examples are presented to illustrate applicability of the procedure.

Key words: elastoplastic analysis; large displacement; mathematical programming; nonlinear complementarity problem; structural plasticity.

1. Introduction

Complementarity, namely the requirement that two sign-constrained vectors are orthogonal or perpendicular, is a typical and recurrent mathematical feature of many problems in nonlinear mechanics, primarily those involving traditional plasticity (see e.g., Maier and Munro 1982, Maier and Lloyd Smith 1986, Lloyd Smith 1990) or some form of contact conditions (e.g., Klarbring 1993. Bolzon, *et al.* 1994, 1995). This was first recognized by Maier in the late 1960s (e.g., Maier 1970) and has since played a unifying role to such broad classes of structural mechanics problems as the above-mentioned (see e.g., Maier and Nappi 1984 for an insightful overview of the framework provided by mathematical programming, in particular complementarity, to discrete plasticity).

It is notable that engineering (and other) applications of complementarity (Ferris and Pang 1995) have not escaped the attention of the mathematical programming community. After all, after more than three decades of research, the subject of complementarity systems has become a well-established and fruitful discipline of its own, rather than being motivated solely, as in its origins, by the analysis of stationary points for optimization problems. To engineers, the study of complementarity problems has a two fold appeal: a refined mathematical formalism rich in useful theoretical results and a wealth of efficient and robust numerical algorithms.

This paper considers the general formulation and numerical solution of a wide class of structures in the presence of both material and geometrical nonlinearities. In order to be more specific, we specialize to the case of skeletal structures with piecewise linear constitutive laws in our elucidation of static-kinematic duality and in our illustrative examples. The formulation makes direct use of the vector-matrix analytical dress provided by Maier's mathematical programming framework for an exact formulation as a nonlinear complementarity problem (NCP) in rates. In view of the difficulties involved in solving the rate nonholonomic problem directly, an approximate and well-known stepwise holonomic representation is used (De Donato and Maier 1973) together with the artifice (De Freitas and Lloyd Smith 1984-85) of collecting terms that destroy the symmetry of key operators in so-called nonlinear "residuals". This finite incremental holonomic form of the formulation is particularly amenable for solution by an iterative predictor-corrector type scheme which is briefly described. Using the simple case of planar trusses, we show, in particular, how the static-kinematic relations are obtained in a systematic manner and also in a form that exhibits duality or contragredience. Such a property is particularly advantageous for extremum characterizations of solutions and quantifications regarding stability, existence and uniqueness. However, these and other theoretical considerations are not dealt with in this computation-oriented work.

The organization of the paper is as follows. In Section 2, we briefly review the governing relations for a wide class of finite element discretized structures. The computationally tractable, albeit approximate, stepwise holonomic counterpart is introduced in the following Section 3. We present in Section 4 the stepwise holonomic problem as an NCP which can clearly be solved by an iterative scheme involving a predictor-corrector type procedure, the main algorithmic steps of which are also described. In order to clarify the systematic way in which the key static-kinematic relations can be set up in the desirable form, namely one exhibiting duality and symmetry, we detail this in Section 5 using the simplest plane truss case. Combination of statics, kinematics and the assumed piecewise linear constitutive laws, leading to expressions for the key structural operators, is also briefly given. We then present some examples to illustrate application of the method in Section 6, before concluding with some general remarks.

A word regarding notation is in order. We do not use any special convention to distinguish between scalars, vectors and matrices, and between functional dependence and multiplication; these should be clear from the context. Vectors are assumed to be column vectors. Transpose is indicated by the superscript T, the inverse of a matrix by the superscript -1 and a superimposed dot represents a derivative. The complementarity relation between nonnegative vectors w and z is written as $w^T z=0$ implying the componentwise condition $w_i z_i=0$ for all i.

2. Nonholonomic formulation

Consider a structural system discretized, as is typical, into an aggregate of finite elements (bars, frame members, etc.). It is assumed that the material behaviour is directly reflected by the element behaviour (e.g., Corradi 1978) and can be obtained from the classical flow theory of plasticity.

Following well-known notation and description (Maier 1970) in terms of natural (unaffected by rigid body motions) generalized quantities, we can compactly describe the general nonholonomic rate problem governing the response of the elastoplastic structure under large displacements by the following set of relations which makes use of the three key ingredients of compatibility, equilibrium and constitution. Vector and matrix quantities represent the unassembled contributions of corresponding elemental entities, namely of concatenated vectors and block diagonal matrices, respectively.

$$q = q(u), \tag{1}$$

$$F = C^T Q,\ C = \frac{\partial q}{\partial u}, \tag{2}$$

$$q = e + p, \tag{3}$$

$$Q = Se + R_s, \tag{4}$$

$$\dot{p} = N\dot{\lambda},\ N = \frac{\partial \phi^T}{\partial Q} \tag{5}$$

$$\phi = \phi(Q, \lambda) \tag{6}$$

$$\phi \leq 0,\ \dot{\lambda} \geq 0,\ \phi^T \dot{\lambda} = 0. \tag{7}$$

Eq. (1) represents compatibility involving a highly nonlinear dependence of strains q on the nodal displacements (degrees of fredom) u; an explicit example for plane frames is given by Tin-Loi and Misa (1996). Equilibrium between the nodal load vector F and the generalized stresses Q is given by Eqs. (2) and involves the compatibility matrix C. Relations (3)-(7) embody the nonholonomic constitutive laws expressed in rate form, in the spirit of the flow theory of plasticity. In particular, total strains q are given as the sum of elastic e and plastic p components in Eq. (3). Elasticity is described in a Lagrangian form by Eq. (4), where S is a symmetric (not necessarily positive definite) matrix of unassembled element stiffnesses with the residual R_s collecting terms that would otherwise destroy the reciprocity of the elastic causality operators (De Freitas and Lloyd Smith 1984-85). The plastic strain rates $\dot{p}$ are defined in Eq. (5) by an associated flow rule and expressed as functions of the plastic multiplier rates $\dot{\lambda}$ through the matrix of unit outward normal vectors N to the yield surface. The generally nonlinear yield functions ϕ are specified in Eq. (6); this includes Maier's remarkable piecewise linear representations as special cases (Maier 1970). Finally, a complementarity relationship between the sign-constrained vectors ϕ and $\dot{\lambda}$ establishes the nonholonomic nature of plasticity. The mechanical interpretation of this condition, for the y-th yield mode, is that (a) if $\phi_y<0$ (no yielding) then $\dot{\lambda}_y=0$ (no plastic flow) or (b) if $\phi_y=0$ (active yield mode) then either $\dot{\lambda}_y>0$ (plastic flow) or $\dot{\lambda}_y=0$ (elastic unloading with $\dot{\phi}_y<0$ or neutral state with $\dot{\phi}_y=0$).

3. Stepwise holonomic formulation

Since the nonholonomic problem, as written in Eqs. (1)-(7), is difficult to solve directly, the entire structural response evolution is best approximated by a sequence of finite incremental problems, each concerning a configuration change caused by a finite increment of load step, from a previously known state. The nonholonomic constitutive laws are simply transformed through an implicit backward difference integration scheme into a stepwise holonomic format (De Donato and Maier 1973); the penalty, although acceptable in practice, is an approximate representation. Of course, it is still possible to capture exactly, through some iterative scheme, events such as plastic activation and elastic unloading, but this may often lead to an unnecessary computational burden with little gain in accuracy, when numerous activations/unstressing occur, as is often the case.

We adopt the notation that any such finite increment is denoted by Δ and symbols with and without hats represent known and unknown values, respectively; e.g., $x=\hat{x}+\Delta x$. The finite incremental counterpart of the nonholonomic problem then becomes

$$\Delta q=q(\hat{u}+\Delta u)-q(\hat{u}), \tag{8}$$

$$F=\hat{F}+\Delta F=C^T(\hat{Q}+\Delta Q), \tag{9}$$

$$\Delta q=\Delta e+\Delta p, \tag{10}$$

$$\Delta Q=\hat{S}_\Delta \Delta e+\Delta R_s, \tag{11}$$

$$\Delta p=N\Delta\lambda, \tag{12}$$

$$\phi=\hat{\phi}+\Delta\phi=\phi(\hat{Q}+\Delta Q,\ \hat{\lambda}+\Delta\lambda), \tag{13}$$

$$\phi\leq 0,\ \Delta\lambda\geq 0,\ \phi^T\Delta\lambda=0. \tag{14}$$

This approximate set of relations requires that any unloading be accounted for at the beginning of each holonomic step and not within each step. It should also be noted that S_Δ represents the incremental form of the unassembled stiffness matrix, and matrices C and N are evaluated at the end of each step.

4. Numerical algorithm for stepwise holonomic problem

The stepwise holonomic problem, as written in Eqs. (8)-(14), is not easily amenable to numerical computation. Also, there does not appear to be a simple way of writing this problem in the form of a standard NCP for direct solution by one of the many algorithms that exist at present (e.g., Ferris and Pang 1995). We therefore propose an extension of the classical initial stress algorithm (e.g., Franchi and Genna 1984) for the numerical solution process.

Let us assume that we can rewrite relations (Eqs. (8)-(14)) as a so-called "mixed" NCP to facilitate application of a simple iterative scheme. This problem has the basic form

$$\begin{bmatrix} \hat{K}_{uu} & \hat{K}_{u\lambda} \\ \hat{K}_{u\lambda}^T & \hat{K}_{\lambda\lambda} \end{bmatrix}\begin{bmatrix} \Delta u \\ \Delta\lambda \end{bmatrix}=\begin{bmatrix} \cdot \\ -\phi \end{bmatrix}+\begin{bmatrix} \Delta F+\Delta R_1 \\ \hat{\phi}+\Delta R_2 \end{bmatrix}, \tag{15}$$

$$\phi\leq 0, \quad \Delta\lambda\geq 0, \quad \phi^T\Delta\lambda=0 \tag{16}$$

where ΔR_1 and ΔR_2 are nonlinear residuals which have been forcibly introduced to preserve symmetry of $\hat{K}$. The retention of symmetry, as mentioned earlier, is particularly advantageous if physical extremum characterizations of the problem are required, as they often are.

From the computational viewpoint, it is clearly advantageous to have the governing relations in the form given by Eqs. (15)-(16). The large displacement problem is then almost identical to a small displacement elastoplastic analysis, with two major differences. Firstly, matrix $\hat{K}$ is no longer constant; it is now, in general, a function of the changes in geometry and stress state of the structure. Secondly, we note the presence of two extra terms representing the nonlinear, as yet unknown, residuals ΔR_1 and ΔR_2. Incidentally, the linear case can be easily recovered by setting all residuals to zero and using the appropriate small displacement (constant) $\hat{K}$ matrix. This observation suggests a simple iterative scheme which basically consists of a sequence of alternating predictions and corrections of u until the necessary convergence criteria are met.

In particular, the prediction step involves calculation of Δu by solving the first part of Eq. (15) for some assumed ΔR_1 and the correction step refines this estimation through a calculation of $\Delta\lambda$ from the solution of a simple linear complementarity problem (LCP) given by Eq. (16). This predictor-corrector type scheme is similar to that detailed in Comi and Maier (1990).

Denoting the iteration number by the superscript i, the algorithmic steps of the predictor-corrector solver can be summarized as follows:

Step 1 (Initialization)

- $i=1$, $\Delta\lambda^{i-1}=0$, $\Delta R_1^{i-1}=0$, set convergence tolerance ε (e.g., 10^{-4}).
- Calculate $\hat{K}_{uu}$, $\hat{K}_{u\lambda}$ and $\hat{K}_{\lambda\lambda}$.

Step 2 (Prediction)

- $\Delta u^i=\hat{K}_{uu}^{-1}(\Delta F-\hat{K}_{u\lambda}\Delta\lambda^{i-1}+\Delta R_1^{i-1})$.
- Calculate Δq^i, ΔQ^i, $\cdots$, ΔR_2^i.
- If Euclidean-norm of out-of-balance load $\|\hat{F}+\Delta F-(C^{i-1})^T Q^{i-1}\|\leq\varepsilon\|\Delta F\|$ then stop.

Step 3 (Correction)

- Solve the LCP

$$\hat{K}_{u\lambda}^T\Delta u^i+\hat{K}_{\lambda\lambda}\Delta\lambda^i=-\phi^i+\hat{\phi}+\Delta R_2^i, \tag{17}$$

$$\phi^i\leq 0,\ \Delta\lambda^i\geq 0,\quad \phi^{iT}\Delta\lambda^i=0. \tag{18}$$

- Calculate ΔR_1^i.
- $i=i+1$, go to Step 2.

The following additional remarks are worthy of note.

(a) To traverse critical points and trace unstable equilibrium paths, a standard arc-length procedure can be embedded within the algorithm. This is described in detail by Tin-Loi and Misa (1996) for the particular case of semirigid frames in which an explicit spherical arc-length constraint is used (Forde and Stiemer 1987).

(b) If $\hat{K}_{uu}$ is used to compute Δu for every iteration, convergence, although guaranteed, can be very slow. In our implementation for skeletal structures, we have used satisfactorily a predictor step based on the tangent stiffness and when convergence is not achieved after a preset number of iterations, the calculations are restarted with $\hat{K}_{uu}$. We have, however, found that it is best to retain $\hat{K}_{uu}$ for first iterations in order to control the magnitudes of displacement increments and to ensure better convergence.

(c) The LCP in the corrector phase consists of small-size, uncoupled and positive semi-definite matrices and therefore should be easy to solve. We recommend use of the standard Lemke's algorithm (e.g., Cottle, *et al.* 1992).

(d) If desired, critical events such as unloading and the activation of hinges can be captured exactly by iterating on the load steps.

(e) Explicit evaluation of closed-form expressions for the residuals can be reasonably easily obtained, as shown in the next section for the case of plane trusses. However, it is not absolutely necessary to do so as these residuals can be indirectly calculated (Tin-Loi and Misa 1996).

5. Systematic calculation of structural operators

Whilst it is conceptually easy to understand how the governing system (15)-(16) is arrived at, it is not so easy to develop systematically and in a unified manner a method for calculating explicitly the structural operators required for the analysis. In this section, we review a scheme, originally attributed to Denke (1960) and later popularized by Lloyd Smith and De Freitas (e.g., Lloyd Smith 1990), to facilitate the obtention of static-kinematic operators. We detail the methodology using as example a simple plane truss element and adopt, without undue loss of generality, piecewise linear plastic laws in the form pioneered by Maier (see Maier and Munro 1982, Maier and Lloyd Smith 1986 for key references) to produce the governing system for the nonlinear analysis.

It is attractive to develop large displacement formulations for nonlinear structural analysis based on a small displacement framework. Such was Denke's aim when he introduced the concept of additional or fictitious forces. This artifice, it will be seen, can also lead to the preservation of static-kinematic duality which can be so advantageous for theoretical developments.

We first develop the Lagrangian description of statics and kinematics.

Consider a plane truss as an aggregate of n finite elements. As shown in Fig. 1(a), let Q^m and q^m denote, respectively, the natural generalized stress (axial force) and strain resultants pertaining to a generic element m of length L^m in its undeformed configuration at some orientation specified by local axes 1-2, with respect to a global reference axis system. Further, let F^m and u^m represent, respectively, the vectors of unconstrained nodal forces and displacements.

The exact description of member equilibrium in its *deformed* configuration can be expressed in the form

$$F^m = [A^m \; A_\pi^m] \begin{bmatrix} Q^m \\ -\pi^m \end{bmatrix} \tag{19}$$

where π^m is a vector of additional nodal forces, as shown in Fig. 1(b), acting also on the undeformed member. The constant matrices A^m and A_π^m are defined in terms of the direction cosines l_i^m ($i=1$, 2) of the local axes i with respect to the Lagrangian axis system as follows:

$$A^m = \begin{bmatrix} -l_1^m \\ l_1^m \end{bmatrix}, \tag{20}$$

$$A_\pi^m = \begin{bmatrix} -l_1^m & -l_2^m \\ l_1^m & l_2^m \end{bmatrix}. \tag{21}$$

In turn, the additional forces are defined by

$$\pi^m = Z^m Q^m \tag{22}$$

with

$$Z^{mT} = \begin{bmatrix} 1 - \dfrac{L^m + \delta_{\pi 1}^m}{L_c^m} & -\dfrac{\delta_{\pi 2}^m}{L_c^m} \end{bmatrix} \tag{23}$$

where L_c^m is the deformed member chord length and vector $\delta_\pi^{mT} = [\delta_{\pi 1}^m \; \delta_{\pi 2}^m]$ represents auxiliary displacements (Fig. 2) associated with the additional forces.

Static-kinematic duality (e.g., De Freitas and Lloyd Smith 1984-5) can be maintained by writing

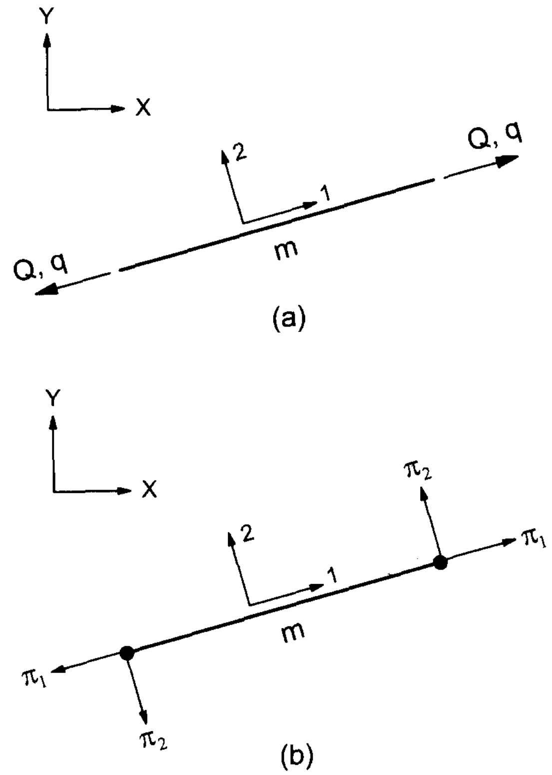

Fig. 1 Truss element: (a) natural stresses and strains; (b) fictitious forces.

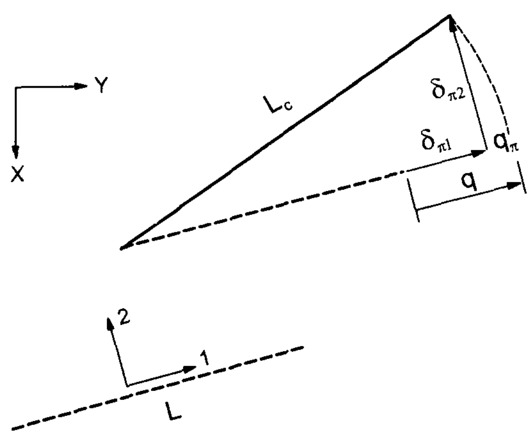

Fig. 2 Original and displaced configurations.

the compatibility equations in an explicitly linear format as follows:

$$\begin{bmatrix} q^m + q_\pi{}^m \\ \delta_\pi{}^m \end{bmatrix} = \begin{bmatrix} A^{mT} \\ A_\pi{}^{mT} \end{bmatrix} u^m \tag{24}$$

where $q_\pi{}^m$ is an is an additional fictitious deformation defined, as is obvious from Fig. 2, by

$$q_\pi^m = \delta_{\pi 1}^m - q^m. \tag{25}$$

We can now combine Eqs. (19) and (24) to form the element static-kinematic relations

$$\begin{bmatrix} \cdot & A^m & A_\pi^m \\ A^{mT} & \cdot & \cdot \\ A_\pi^{mT} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u^m \\ Q^m \\ -\pi^m \end{bmatrix} = \begin{bmatrix} F^m \\ q^m + q_\pi^m \\ \delta_\pi^m \end{bmatrix} \tag{26}$$

which clearly exhibit a duality relationship. As with Eq. (22), it will be convenient to express q_π^m in terms of δ_π^m through matrix Z^m. Simple algebraic manipulations lead to

$$q_\pi^m = Z^{mT} \delta_\pi^m + R_{q\pi}^m \tag{27}$$

where

$$R_{q\pi}^m = L^m \left(1 - \frac{L^m + \delta_{\pi 1}^m}{L_c^m} \right) \tag{28}$$

Hence Eq. (26) can be simplified by eliminating π^m, δ_π^m and q_π^m to give

$$\begin{bmatrix} \cdot & C^{mT} \\ C^m & \cdot \end{bmatrix} \begin{bmatrix} u^m \\ Q^m \end{bmatrix} = \begin{bmatrix} F^m \\ q^m \end{bmatrix} + \begin{bmatrix} \cdot \\ R_{q\pi}^m \end{bmatrix} \tag{29}$$

where

$$C^{mT} = A^m - A_\pi^m Z^m. \tag{30}$$

The governing exact Lagrangian static-kinematic relations for the whole structure, covering all n elements then become

$$\begin{bmatrix} \cdot & C^T \\ C & \cdot \end{bmatrix} \begin{bmatrix} u \\ Q \end{bmatrix} = \begin{bmatrix} F \\ q \end{bmatrix} + \begin{bmatrix} \cdot \\ R_{q\pi} \end{bmatrix} \tag{31}$$

where u represents the vector of nodal displacements, F is the applied nodal load vector, and the indexless symbols have self-evident definitions associated with conventional finite element descriptions, e.g., $Q^T = [Q^1, \cdots, Q^n]$, $\pi^T = [\pi^{1T}, \cdots, \pi^{nT}]$, matrices A, A_π are assembled through appropriate incidence matrices and $Z = diag[Z^1, \cdots, Z^n]$. The term $R_{q\pi}$ is generally considered to be a vector of residuals which destroy the duality relationship given by Eq. (31).

We now develop the finite incremental counterpart of the static-kinematic relations.

For statics, the incremental version of Eq. (19) is

$$\Delta F^m = [A^m \quad A_\pi^m] \begin{bmatrix} \Delta Q^m \\ -\Delta \pi^m \end{bmatrix} \tag{32}$$

while for kinematics Eq. (24) is replaced by

$$\begin{bmatrix} \Delta q^m + \Delta q_\pi^m \\ \Delta \delta_\pi^m \end{bmatrix} = \begin{bmatrix} A^{mT} \\ A_\pi^{mT} \end{bmatrix} \Delta u^m. \tag{33}$$

Further, the incremental forms of Eqs. (22) and (25) are, respectively,

$$\Delta \pi^m = \hat{Z}^m \Delta Q^m + \hat{P}^m \Delta \delta_\pi^m + \Delta R_\pi^m, \tag{34}$$

$$\Delta q_\pi^m = \hat{Z}^m \Delta \delta_\pi^m + \Delta R_{q\pi}^m \tag{35}$$

where, with $\hat{Z}^{mT}=[\hat{Z}_1^m \;\; \hat{Z}_2^m]$,

$$\hat{P}^m=\frac{\hat{Q}^m}{L_c^m}\left[\begin{array}{cc} \hat{Z}_1^m\hat{Z}_1^m-2\hat{Z}_1^m & (\hat{Z}_1^m-1)\hat{Z}_2^m \\ (\hat{Z}_1^m-1)\hat{Z}_2^m & \hat{Z}_2^m\hat{Z}_2^m-1 \end{array}\right], \tag{36}$$

$$\Delta R_{q\pi}^m=\frac{1}{2\hat{L}_c^m}\;(\Delta q^m\Delta q^m-\Delta\delta_\pi^{mT}\Delta\delta_\pi^m), \tag{37}$$

$$\Delta R_\pi^m=\frac{\Delta Q^m}{\hat{Q}^m}\hat{P}^m\Delta\delta_\pi^m+\frac{\Delta R_{q\pi}^m}{\hat{L}_c^m}(\hat{Q}^m+\Delta Q^m)\left[\begin{array}{c} \hat{Z}_1^m-1 \\ \hat{Z}_2^m \end{array}\right]. \tag{38}$$

The key incremental description of statics can be obtained by substituting Eq. (34) into Eq. (32) and using Eq. (33) to eliminate $\Delta\delta_\pi^m$. Similarly, for kinematics, we substitute Eq. (35) into the first part of Eq. (33) and then use the second part of Eq. (33) to eliminate $\Delta\delta_\pi^m$. The resulting static-kinematic relations then read

$$\left[\begin{array}{cc} \hat{K}_G^m & \hat{C}^{mT} \\ \hat{C}^m & \cdot \end{array}\right]\left[\begin{array}{c} \Delta u^m \\ \Delta Q^m \end{array}\right]=\left[\begin{array}{c} \Delta F^m \\ \Delta q^m \end{array}\right]+\left[\begin{array}{c} A_\pi^m\Delta R_\pi^m \\ \Delta R_{q\pi}^m \end{array}\right] \tag{39}$$

where

$$\hat{K}_G^m=-A_\pi^m\hat{P}^mA_\pi^{mT}, \tag{40}$$

$$\hat{C}^{mT}=A^m-A_\pi^m\hat{Z}^m. \tag{41}$$

At the structure level, the static-kinematic relations become

$$\left[\begin{array}{cc} \hat{K}_G & \hat{C}^T \\ \hat{C} & \cdot \end{array}\right]\left[\begin{array}{c} \Delta u \\ \Delta Q \end{array}\right]=\left[\begin{array}{c} \Delta F \\ \Delta q \end{array}\right]+\left[\begin{array}{c} A_\pi\Delta R_\pi \\ \Delta R_{q\pi} \end{array}\right] \tag{42}$$

by a familiar reinterpretation of the indexless form of Eq. (39), as was explained for the Lagrangian case.

Let us now adopt piecewise linear plastic laws (Maier 1970) so that Eq. (13) becomes

$$\phi=\hat{\phi}+\Delta\phi=N^T(\hat{Q}+\Delta Q)-H(\hat{\lambda}+\Delta\lambda)+R \tag{43}$$

where N is now constant, H is a hardening matrix and R is a vector of yield limits defined by the orthogonal distances from the origin to the various yield hyperplances in stress space.

We can easily manipulate Eqs. (10)-(14), (42), and (43) to give precisely the stepwise holonomic system (15)-(16) with

$$\hat{K}_{uu}=\hat{C}^T\hat{S}_\Delta\hat{C}+\hat{K}_G, \tag{44}$$

$$\hat{K}_{u\lambda}=-\hat{C}^T\hat{S}_\Delta N, \tag{45}$$

$$\hat{K}_{\lambda\lambda}=H+N^T\hat{S}_\Delta N, \tag{46}$$

$$\Delta R_1=-\hat{C}^T\Delta R_s+\hat{C}^T\hat{S}_\Delta\Delta R_{q\pi}+A_\pi\Delta R_\pi, \tag{47}$$

$$\Delta R_2=N^T\Delta R_s-N^T\hat{S}_\Delta\Delta R_{q\pi}. \tag{48}$$

6. Numerical examples

Numerous examples have been solved using the procedure described in the foregoing sections.

Only three are presented in this paper to illustrate application of method to skeletal structures. Some idea of the generality and robustness of the scheme can be obtained from the types of analyses involved.

Example 1 deals with a purely elastic shallow truss dome (Hangai and Kawamata 1973, De Freitas, *et al.* 1985) that is usually considered to be a benchmark problem for evaluation of geometrically nonlinear solution algorithms. Example 2 concerns the stiffening behaviour of a clamped elastoplastic beam at finite deformations, while Example 3 is a three storey one bay elastoplastic frame which exhibits an unstable post-collapse response.

6.1. Example 1

This example is concerned with the shallow truss dome shown in Fig. 3. Under the assumption of a linear elastic material with $A=1$ cm^2, we analysed it for two load cases: (a) a point load P applied vertically downwards at node 1 together with point loads of $2P$ applied also vertically downwards at each of nodes 2-7, and (b) the same pattern of loading as for the previous case but with an imperfection introduced into the structure consisting of reducing by 0.2 cm the vertical heights above the ground of nodes 2, and 5.

For these load cases, Figs. 4(a)-(c) show our results in the form of load versus vertical or radial deflections at the indicated nodes, solid circles refer to the original structure and crosses to the imperfection induced case. The results of De Freitas, *et al.* (1985), who used a work perturbation approach, are shown as solid lines for comparison. Very good agreement was obtained for both the original structure and for the topographically altered structure.

6.2. Example 2

The second example concerns a fixed ended beam under distributed loading. This example was experimentally investigated by Pang and Millar (1978) to study the effect of geometry change and axial restraint on post-elastic behaviour. It is a particularly challenging problem to simulate numerically as it obviously involves travelling hinges, which form and unload with increase of load.

The beam has a rectangular section 24 mm wide by 8 mm deep, made of an aluminium alloy with $E=68\times10^3$ MPa and yield stress of 310 MPa. As Pang and Millar (1978) did, we simulated the distributed load by ten equally spaced point loads. We analysed only half the beam because of symmetry. Two piecewise linear yield conditions, as shown in Fig. 5(a), were assumed; both are inscribed polygons to the classical yield locus of a rectangular section.

The graphs in Fig. 5(b) represent nondimensional load versus central deflection results; also shown are Pang and Millar's theoretical and experimental solutions. It should be noted that $w=P/40$ N/mm, the classical rigid-plastic collapse load $w_p=11.904$ N/mm and h is the beam depth. Inspite of the fact that slight end rotations might have occurred in the experiment, according to Pang and Millar (1978), the theoretical and experimental results are acceptably close. As expected, the three planes per quadrant idealization yields a better correlation with experimental results.

6.3. Example 3

In this example, a three storey one bay frame was analysed to compare our numerical results

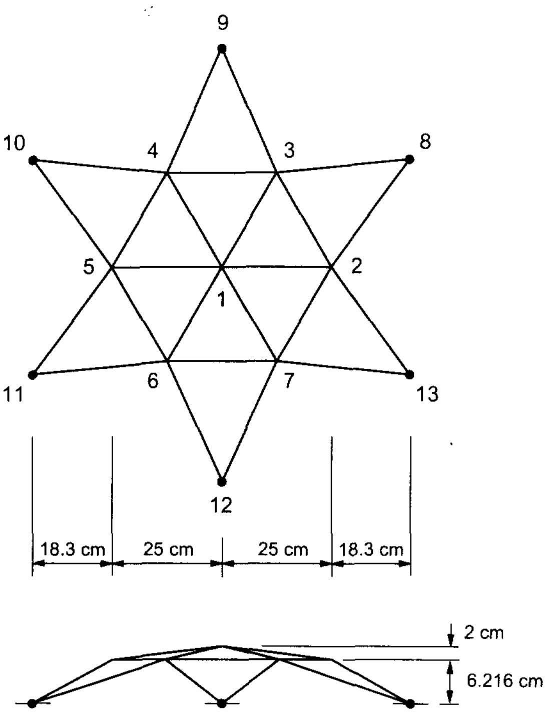

Fig. 3 Example 1: shallow truss dome.

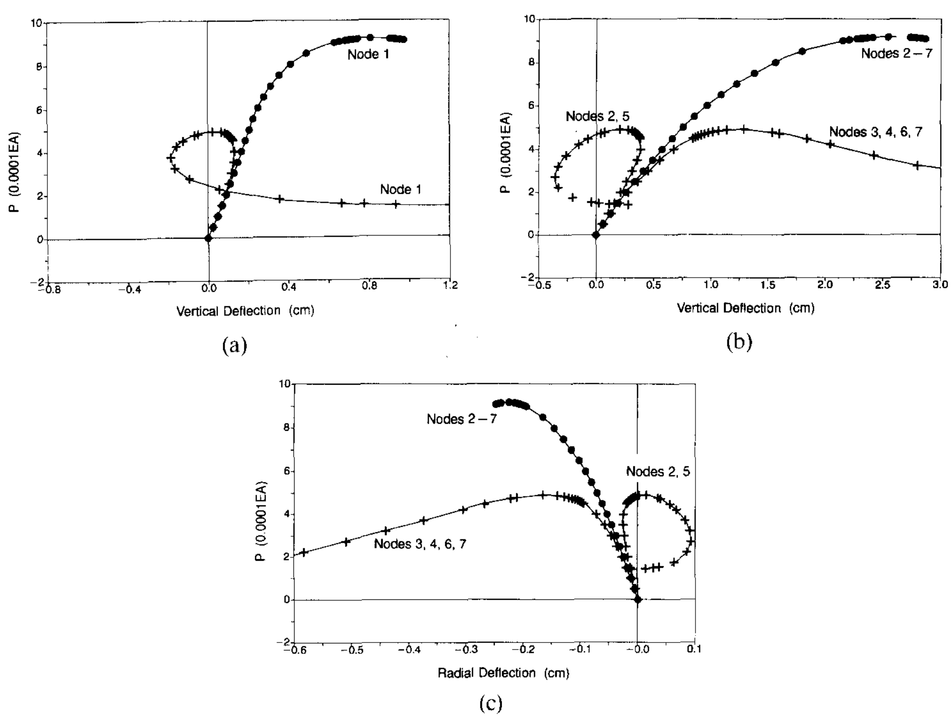

Fig. 4 Load-deflection results for Example 1: (a) vertical deflection at node 1; (b) vertical deflection at nodes 2-7; (c) radial deflection at nodes 2-7.

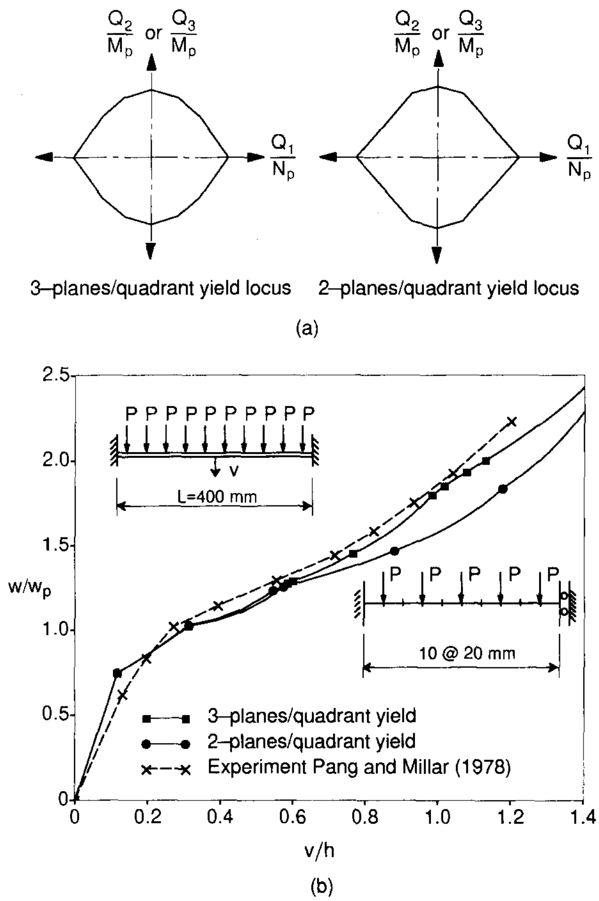

Fig. 5 Example 2: (a) yield polygons; (b) load-deflection results.

with those of a full-scale test. This frame was tested by Yarimci (1966) under nonproportional loading conditions.

The frame geometry and loading condition are shown in Fig. 6(a). Material and geometrical properties (kN, cm units) are: beams (10WF25) $E=2.0233\times10^4$, $A=49.10$, $I=5.994\times10^3$, $M_p=1.243\times10^4$; columns (5M18.9) $E=2.0844\times10^4$, $A=35.81$, $I=1.003\times10^3$, $M_p=4.519\times10^3$, $N_p=889.60$, where M_p and N_p represent the full plastic moment and axial capacities, respectively. The vertical loads $F_1=102.304$ kN and $F_2=88.96$ kN were first applied and maintained during subsequent application of the horizontal load H. In our analyses, we adopted rigid beam-to-column connections to model the fully welded connections. We further assumed that columns yielded under combined moment and axial force (using a piecewise linear yield locus with two hyperplanes per quadrant with a breakpoint at $M/M_p=1$, $N/N_p=0.15$) while beams yielded in pure bending.

The results of our analyses, both under the assumptions of a second-order theory and an exact large displacement formulation, are shown in Figs. 6(b) and 6(c). In Fig. 6(b), we show the hinge dispositions at the indicated limit loads. The test results gave a limit load of 7.17 kN. We compare, in Fig. 6(c), the load-displacement results of Yarimci (1966) with our computed responses; our exact large displacement analysis reproduces reasonably well the experimentally recorded response, even in the post-critical region.

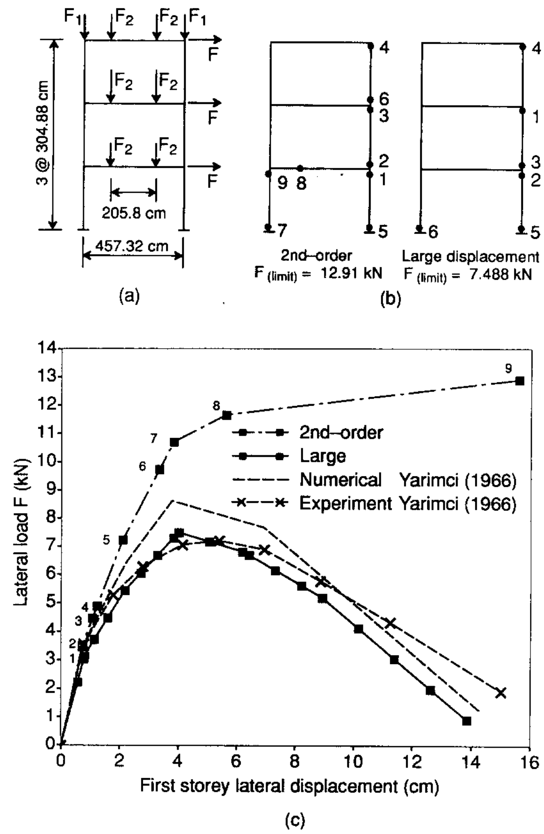

Fig. 6 Example 3: (a) structure and loading; (b) hinge dispositions at critical loads; (c) load-deflection results.

7. Concluding remarks

A general methodology for the formulation and solution of a large class of elastoplastic problems within the large displacement regime has been presented. Starting from the space discretized nonholonomic problem in rates, we show how the analysis can be transformed into its stepwise holonomic counterpart by an approximate backward difference scheme.

For computational (and theoretical) purposes, we indicate that it is desirable to cast the governing equations into a symmetric form which is very similar to that of a small displacement analysis except for the presence of some nonlinear residual terms. It is then not only easy to generate, if desired, any order analysis from such a format but, more importantly, we can apply known path-tracing solution algorithms for its numerical solution. In particular, we propose a robust variant of the classical predictor-corrector algorithm to solve the resulting mathematical programming problem known as a mixed NCP. Comments regarding measures to speed up convergence, to capture critical events and to trace any unstable equilibrium paths are made.

Using the case of a planar truss, we also reviewed in some detail a framework, popularized by Lloyd Smith and co-workers, for systematically obtaining the key structural operators required for the nonlinear analysis. In essence, the device of fictitious forces and deformations are used to set up static-kinematic duality. The resulting equilibrium and compatibility equations can then be easily combined with the assumed piecewise linear plasticity laws to give the final governing system.

Extensive computational testing, three examples of which we report herein, attests to the robustness, accuracy and relative efficiency of the proposed numerical scheme.

Current work is aimed at attempting to solve the relation set (8)-(14) directly by using the high-level mathematical programming modelling language GAMS (Brooke, *et al.* 1988), which has the ability to perform automatic differentiation and to transparently call powerful state-of-the-art NCP solvers such as PATH (Dirkse and Ferris 1995).

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