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Improved stress recovery for elements at boundaries

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Abstract. Patch recovery attempts to derive a more accurate stress filed over a particular element than the finite element shape function used for that particular element. Elements that have a free edge being the boundary to the structure have particular stress relationship that can be incorporated to the stress field to improve the accuracy of the approximation.

Key words: finite element; stress; errors.

1. Introduction

The first work done on the patch type recovery to improve the overall stress field approximation by the finite element method could be attributed to Hinton and Campbell (1974). These researchers noted that the stress fields could be better approximated by averaging the results over stress discontinuities in the finite element model.

The identification by Barlow (1976) that the finite element analysis produces stresses that are more accurate at or near the Gauss points of a finite element with respect to other locations in the element has assisted in improving the stress approximations by the finite element method. Zienkiewicz and Zhu (1992) used the Gauss point stresses and a least squares function to improve the stress approximation at the nodes of a finite element analysis. Lee and Lo (1993) are other researchers who have had published success with this method.

Wieberg, Abdulwahab (1993) and Wieberg and Li (1993, 1994) use a patch of elements around a central element as opposed to patches of elements around a central node used by previous authors, to estimate an improved stress field through out the central element rather than improving the point wise stress quantity at the node. The different patch techniques are shown in Fig. 1. This paper is based on the stress field over the entire element, hence the central element type patch recovery technique was used in this work.

2. Patch recovery technique

Based on the work by Barlow (1976), the optimal locations for accurate stress quantities in

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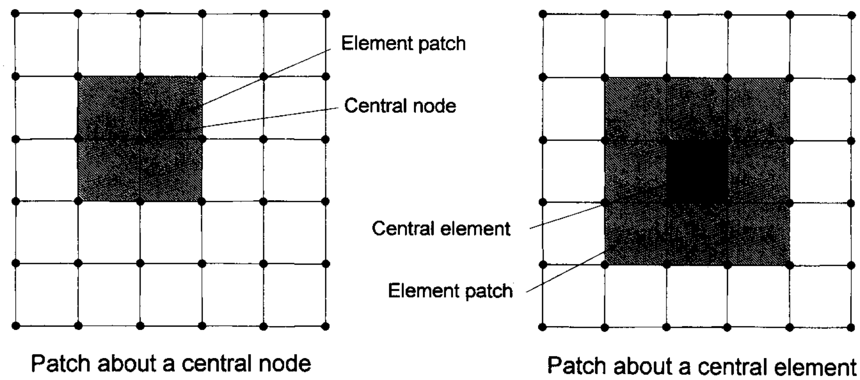

Fig. 1 Patch types about central node and central element.

a finite element, are at the Gauss points required for the integration of a function one order lower than the shape function used for the element. Hence for an eight node membrane element having a shape function with a complete polynomial of order two, there are four locations where the derived stresses are most accurate, where the values of the isoparametric constants ξ and η can take a value of $\pm\frac{1}{\sqrt{3}}$.

Both the central node and central element type patch recovery technique take the stresses derived at the Gauss points surrounding the central node or element respectively. The central element type also includes the stresses obtained at the Gauss points of the central element.

A set of polynomials are used to approximate the stress field of all elements sampled in the patch. There are three polynomials used to represent the stress field for the plane stress or plane strain case. Two polynomials used to represent the two axial stresses and one for the shear stress. The polynomials are selected based on a least squares fit to the Gauss point stresses in the samples. The order of the polynomials used can be chosen arbitrarily, but with the limitations that the number of coefficients in the polynomials do not exceed the number of sampling points.

Finite element stresses for the two dimensional cases are obtained by combining the first differential of the finite element shape functions and a material property. As the convergence rate of the displacements due to the finite element shape function is $O(h^p)$, then as the stresses are derived from the differentiation of the shape function, the convergence rate for the stress field is expected to be $O(h^{p-1})$. The Gauss point stresses (Mackinnon and Carey 1989) have been proved to be superconvergent, that is the rate of convergence of these stresses is an increased rate than the convergence rate of the entire stress field.

As the convergence rate of the entire stress field is lower than displacement convergence rate, the lowest order polynomial used to represent the stress field on the patch was chosen to be the same order as the order of the highest complete polynomial finite element shape function. The convergence rate of the superconvergent Gauss point stresses is not expected to be at an increased rate to the convergence rate of the nodal displacements, hence this polynomial order is valid to obtain the optimal rate of convergence for the patch recovered stresses. A higher order polynomial used than that for the superconvergence rate would not increase the rate of convergence of the stress result.

The stress field is required for the central element in the patch, thus weighting the least squares

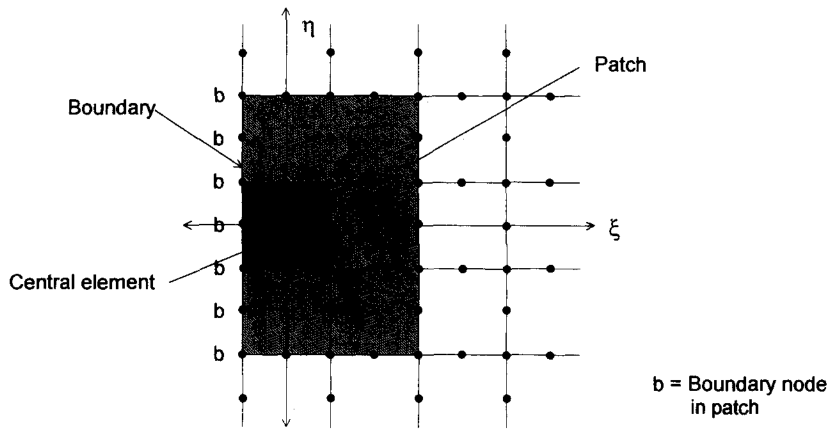

Fig. 2 Central element patch at boundary elements.

interpolation function such that the stress field is more accurate at the Gauss point stresses in the central patch element than the surrounding elements has been found to be beneficial. This weighting has been found optimal when it is calculated based on the squared inverse of the distance from the stress sampling points to the centroid of the central element in the patch.

3. Patch recovery for elements at boundaries

At a free edge in a structure, where the external load is zero or is applied at a predetermined quantity, the normal and shear stress to the free edge is known to be zero or the equal to the applied loading. This can easily be accounted for in the patch recovery technique by including the boundary nodes in the least squares interpolation of the polynomial function to represent the normal stress to the boundary and the shear stress. The direct stress tangential to the boundary is left unchanged.

A patch of elements around the central element on the boundary is shown in Fig. 2. This example contains a total of six elements in the patch including the central element. There are four optimal Gauss point stress associated with each element resulting in a total of 24 sampling points for the least squares interpolation function. In addition, for the boundary patch, there are five boundary nodes where an additional five stress points can be included in the least squares interpolation for the normal axial stress and the shear stress. The number of tangential axial stress locations does not increase as the stress at the boundary nodes is unknown.

The stresses are maintained in a local axis system for the calculation of the least squares interpolation polynomial. If the boundary is curved then the stress for the local axis system would follow the curvature of the boundary.

4. Example

The work by Zienkiewicz and Zhu (1992) on stress recovery examines the stresses obtained

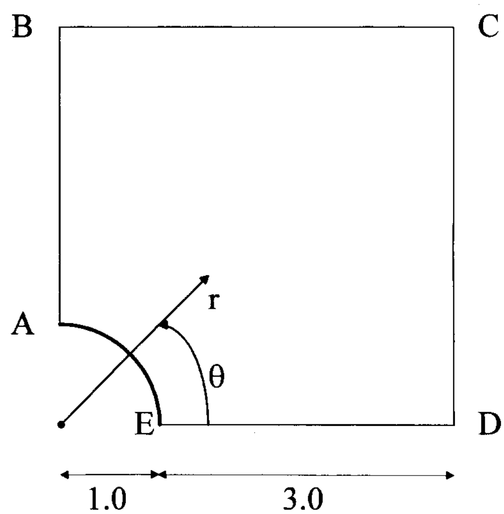

Fig. 3 Dimensions for the finite element model of a hole in an infinite plate.

in an infinite plate with a circular hole subject to a unidirectional load. Only a finite portion of this structure can be modelled. The portion of the structure that is examined is shown in Fig. 3.

This example was chosen due to the exact solution being in existence from Sokilnikoff (1956). The solution for the displacements and the stresses under a unit uniaxial stress in the x-direction are given by Eqs. (1) to (5).

$$u_r = \frac{1+v}{4Er}\left[(2-4v)r^2 + 2a^2 + 2\left[a^2(4-4v) + r^2 - \frac{a^2}{r^2}\right]\cos 2\theta\right] \tag{1}$$

$$u_\theta = -\frac{1+v}{2Er}\left[r^2 + a^2(2-4v) + \frac{a^2}{r^2}\right]\sin 2\theta \tag{2}$$

$$\sigma_x = 1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3a^4}{2r^4}\cos 4\theta \tag{3}$$

$$\sigma_y = -\frac{a^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3a^4}{2r^4}\cos 4\theta \tag{4}$$

$$\tau_{xy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta + \sin 4\theta\right) + \frac{3a^4}{2r^4}\sin 4\theta \tag{5}$$

The appropriate boundary conditions are applied to lines AB and ED to represent symmetry. Zienkiewicz uses edge tractions applied to lines BC and DC to simulate the analytical solution to the infinite plate. However, the finite element analysis package STRAND6.1 used in this work is capable of applying nodal constraints where the final displacement of a set of nodes can be entered into the analysis and the remainder of the nodes have their displacements calculated based on these constrained displacements.

This is a more suitable procedure to represent the continuation of the structure beyond its modelled volume. The perimeter nodes being constrained will exactly represent the effect of

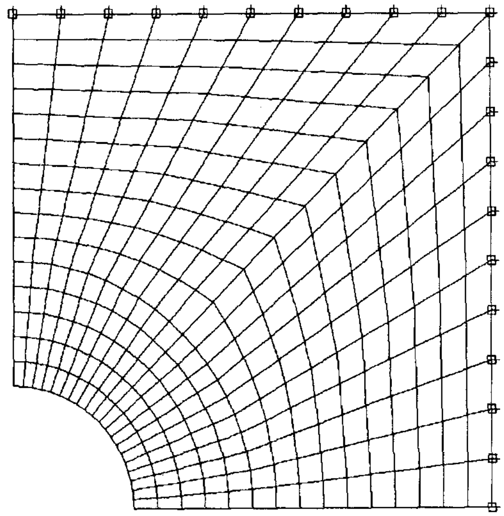

Fig. 4 Finite element mesh and constraints for the hole in the infinite plate model, the squares around the nodes represent constrained nodes.

the continuous structure. Surface tractions applied at the boundary of the model will not cause the boundary nodes to behave exactly as if there were structure beyond the model.

A number of meshes were used in the analysis of the structure consisting of eight node isoparametric elements. One such mesh is shown in Fig. 4.

This paper is dedicated to improving the stress field for the elements on the boundary. Only the first row of elements around the circle were considered in the error measure. The error measure used in this work is similar to that used by Stephen and Steven (1996) where the energy is calculated for the entire region for both the exact and the approximated stress fields and then a relative difference is used as the error measure. The error measure can be written as:

$$\varepsilon=\frac{\pi_h-\pi}{\pi} \tag{6}$$

where π and π_h represent the energy of the structure calculated from the exact and approximate stress fields respectively. The energy can be derived from the equation:

$$\pi=\frac{1}{2}\int_{\Omega}\{\sigma\}^T[D]^{-1}\{\sigma\}\,d\Omega \tag{7}$$

The matrix $[D]$ is the material property matrix and the vector $\{\sigma\}$ is the vector of exact stresses in the structure. For the hole in the infinite plate the material property matrix is that of a plane strain structure and can be written as:

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{8}$$

and the stress vector can be written as:

$$\{\sigma\}=\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{array}\right\} \tag{9}$$

The energy associated with the approximation stress field is similar to Eq. (7) where the stress vector represents the approximation stress field derived from the finite element analysis.

The integration of the energy function is done using the isoparametric constants ξ and η in the same manner as the finite element formulation. The energy in Eq. (7) can be rewritten in terms of global x and y coordinates and also in the isoparametric coordinates as follows:

$$\pi=\frac{1}{2}\int_{\Omega}\{\sigma\}^T[D]^{-1}\{\sigma\}\,d\Omega=\frac{1}{2}\iint\{\sigma\}^T[D]^{-1}\{\sigma\}\,dxdy=\frac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1}\{\sigma\}^T[D]^{-1}\{\sigma\}\,Jd\xi d\eta \tag{10}$$

The value of J is the determinate of the 4×4 Jacobian matrix for the eight node isoparametric element. It is also computationally easier to use the local axis stresses in Eq. (10).

To calculate the elastic stiffness matrix using a complete polynomial of order 2, a 2×2 Gaussian integration technique is required. Hence it is suitable to use the same order of numerical integration for the finite element derived stresses at the four Gauss points of the central patch elements as the convergence rate can at best only equal the order of the complete polynomial used as the finite element shape function. The exact stress field function has trigonometric and inverse distances to the fourth power. A 4×4 Gaussian integration technique was used for calculation of the energy associated with the exact stress field in the structure. A 4×4 Gaussian integration technique was adopted for the patch recovered stress fields although this high order of numerical integration was not of vital importance.

The energy errors were calculated for:

(1) The finite element analysis results using the central element Gauss point stresses.
(2) The standard patch using a quadratic interpolation function derived using a weighted least squares technique.
(3) The standard patch using a cubic interpolation function derived using a weighted least squares technique.
(4) A modified patch accounting for the boundary normal and shear stresses using a quadratic interpolation function derived using a weighted least squares technique.
(5) A modified patch accounting for the boundary normal and shear stresses using a cubic interpolation function derived using a weighted least squares technique.

The energy errors for the above cases are plotted in Fig. 5.

Both of the quadratic and cubic polynomials for the patches produce results that are superior than the finite element solution alone. With the boundary effects included in both the quadratic and cubic patches the energy errors are improved again on the standard patch results. Not only are the results using the patch recovery technique superior than the finite element analysis results, the patch recovered stresses converge to zero error at an increased rate than the finite element analysis solution.

The improvement in the energy error using the boundary effects does not appear to be of a great benefit in either the quadratic or cubic polynomial for the patch when viewed on a log-log graph in Fig. 5. However, when the results of the boundary effects being included in the interpolation function are examined by dividing the standard patch energy error by the energy error for the patch including the boundary effects, the boundary effects make a significant

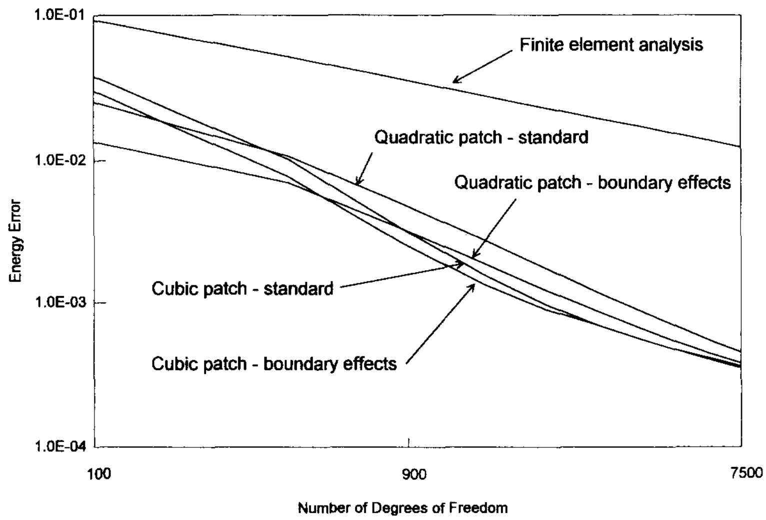

Fig. 5 Energy error measurements for the hole in the infinite plate on a log-log graph.

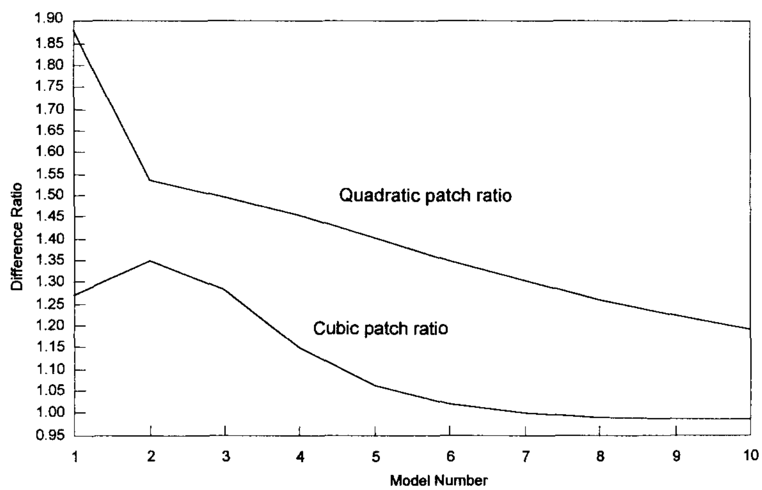

Fig. 6 Difference ratios for the standard and boundary effects for the infinite plate with a hole, the model number relates to the number of element subdivisions of the original model.

improvement in the overall results. This can be shown in Fig. 6 where the energy ratio is plotted against the model number. The model number corresponds to the square root of element subdivisions used from the original model. (The model number 1 relates to a 104 degree of freedom model an up to model number 10 relating to a 7520 degree of freedom structure).

The quadratic patch showed the best improvement for the use of the boundary effects in the patch. The coarsest model had an energy improvement of 88 per cent to a refined model having a 20 per cent improvement.

The cubic patch did not have as great an effect. The standard cubic patch produced superior results than the quadratic patch with and with out the boundary effects included for all models except for the very coarse ones.

For coarse models the boundary effects using a cubic patch produced up to 35 per cent improvement in the energy error which limited to having the same effect as the standard cubic patch.

5. Comments

This research is driven by the attempt to improve the estimation of the critical elastic buckling load using the finite element method. An accurate approximation for the stress field of a structure is vital in buckling analysis as the finite element method requires the stress field over the full body of the element rather than at individual points to calculate the geometric stiffness matrix.

Also at free edges many structures are likely to buckle does to reduced stiffnesses at these locations. Hence obtaining a more accurate stress field at this location is of major benefit.

6. Conclusions

It is a relatively simple procedure to include the known boundary stresses into the patch recovery technique as discussed earlier in this paper using the method of weighted least squares.

The example of the hole in the infinite plate examined in this paper showed not only an improved stress field approximation using the patch recovery technique, but the results using the patch converged to a zero energy error at a more rapid rate than the finite element solution alone.

For both the quadratic and the cubic polynomial interpolation function used in the patch recovery technique, the inclusion of the known boundary stress states has proved to be of great benefit in the assessment of a more accurate stress field.

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