

V-notched elements under mode II loading conditions

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Abstract. We apply the Finite Fracture Mechanics criterion to address the problem of a V-notched structure subjected to mode II loading, i.e., we provide a way to determine the direction and the load at which a crack propagates from the notch tip and express the critical conditions in terms of the generalized stress intensity factor. Weight functions for V-notch emanated cracks available in the literature allow us to implement the fracture criterion proposed in an almost completely analytical manner: the determination of the critical load and the direction of crack growth is reduced to a stationary point problem. A comparison with experimental data presented in the Literature concludes the paper.

Keywords: V-notches; Finite Fracture Mechanics; mode II

1. Introduction

The development of suitable fracture criteria for brittle (isotropic or orthotropic) materials containing V-notches or multi-material interfaces is a problem of primary concern in order to control fracture onset phenomena taking place in mechanical components, composite materials, and electronic devices. As well-known, the singularity of the stress field in the vicinity of the notch tip makes the problem non-trivial.

Concerning re-entrant corners in homogeneous media subjected to mode I loading, since the pioneering paper by Carpinteri (1987), a good correlation has been found between the critical value of the generalized stress intensity factor (i.e., the generalized fracture toughness) and the failure loads. Theoretical models relating the generalized fracture toughness to material tensile strength, fracture toughness and re-entrant corner amplitude have been widely formulated (Seweryn 1994, Lazzarin and Zambardi 2001, Leguillon 2002, Carpinteri *et al.* 2008, 2009, 2011).

Fewer contributions are available for what concerns the case of mixed mode loading conditions (Seweryn *et al.* 1998, Lazzarin and Zambardi 2001, Yosibash *et al.* 2006, Gomez *et al.* 2009), which has been recently faced by a Finite Fracture Mechanics (FFM) criterion (Cornetti *et al.* 2013, Sapora *et al.* 2013). The proposed approach (as well as the ones previously cited) is based on the assumption that the region around the corner dominated by the singular stress field is large compared to intrinsic flaw sizes, as well as to inelastic zone or fracture process zone sizes. This hypothesis is the analogous of small-scale yielding in Linear Elastic Fracture Mechanics (LEFM).

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The comparison between theoretical FFM predictions and experimental data has been found satisfactory, both for what concerns the failure load and the crack propagation angle.

On the other hand, according to the terminology adopted in (Yosibash *et al.* 2006, Sapora *et al.* 2013), the mode II loading situation has to be treated separately and was not considered in the investigations. Indeed, this drawback can be overcome thanks to the new formalism introduced in (Hills and Dini 2011, Cornetti *et al.* 2013).

In the present paper, the FFM criterion is experimentally validated for what concerns pure mode II loading conditions, by considering some data available in the Literature. The analysis is limited to sharp notch opening angles lower than 102.6° , i.e., to the case where mode II is singular. As a matter of fact, for amplitudes greater than 102.6° , the corner is no longer a stress concentration point. Observe that only the framework of two-dimensional elasticity will be investigated: indeed, some interesting works have recently been proposed also for what concerns out-of-plane effects (see, for instance, (Berto *et al.* 2011, Kotousov *et al.* 2013) and related references).

The study can be useful also for other different structural problems, as, for instance, to analyse the competition between bulk and interface crack propagation at V-notches where an adhesive joint is present (Garcia and Leguillon 2012). Eventually, it is worth noting that, in case of interface crack propagation, the present approach complies with the one proposed by Cornetti *et al.* (2012).

2. FFM criterion

By considering a polar coordinate system (r, θ) at the V-notch tip (Fig. 1), following the FFM approach proposed by Cornetti *et al.* (2013), a crack propagates by a finite crack extension Δ if the following two inequalities are satisfied

$$\begin{cases} \int_0^\Delta \sigma_{\theta\theta}(r, \theta) dr \geq \sigma_u \Delta \\ \int_0^\Delta \mathcal{G}(c, \theta) dc \geq \mathcal{G}_c \Delta \end{cases} \quad (1)$$

The former inequality requires that the average stress $\sigma_{\theta\theta}$ over a segment of length Δ must be

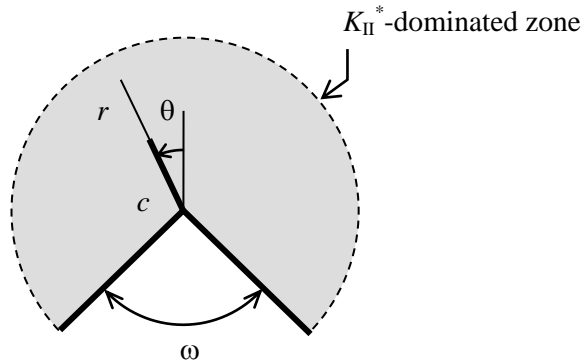


Fig. 1 V-notch and polar reference system

higher than the material tensile strength σ_u . The latter inequality imposes that the average energy available for a crack increment Δ (obtained by integrating the crack driving force \mathcal{G}) must exceed the energy necessary to create the new fracture surface, G_c . This quantity represents the fracture energy, related to the material fracture toughness by the well-known relation $G_c = K_{Ic}^2 / E'$, where $E' = E / (1 - \nu^2)$ in plain strain conditions, E being the Young's modulus and ν the Poisson's ratio. It is worth observing that, the strain energy release rate function $\mathcal{G}(c, \theta)$ is monotonically increasing with c for positive geometries (considered in the present analysis), while the stress $\sigma_{\theta\theta}(r, \theta)$ is monotonically decreasing with the distance r (as far as both the modes provide a stress singularity, i.e. for a notch opening angle less than 102.6°). This means that the lowest failure load (i.e., the actual one) is attained when the two inequalities are substituted by the two corresponding equations. Therefore, the system (1) reverts to a system of two equations in two unknowns: the crack advancement Δ and the corresponding (minimum) failure load, implicitly embedded in the functions $\sigma_{\theta\theta}$ and \mathcal{G} .

In case of mode II loading conditions, by assuming that the region around the corner dominated by the singular stress field is large compared to intrinsic flaw sizes, $\sigma_{\theta\theta}$ can be approximated by

$$\sigma_{\theta\theta} = \frac{K_{II}^*}{(2\pi r)^{1-\lambda_{II}}} f_{\theta\theta}^{II}(\theta, \omega) \quad (2)$$

where K_{II}^* is the generalized stress intensity factor (SIF), while λ_{II} and $f_{\theta\theta}^{II}$ represent the well-known solutions (eigenvalues and eigenvectors, respectively) to Williams' skew-symmetric problem. On the other hand, by means of Irwin's relationship, \mathcal{G} reads

$$\mathcal{G}(c, \theta) = \frac{K_I^2}{E'}(c, \theta) + \frac{K_{II}^2}{E'}(c, \theta) \quad (3)$$

where the SIFs for mode I and mode II, K_I and K_{II} , respectively, can be expressed as (Sapora *et al.* 2013)

$$K_I(c, \theta) = \mu_{12}(\theta, \omega) K_{II}^* c^{\lambda_{II}-1/2} \quad (4a)$$

$$K_{II}(c, \theta) = \mu_{22}(\theta, \omega) K_{II}^* c^{\lambda_{II}-1/2} \quad (4b)$$

The functions μ_{ij} can be evaluated from the best fit expressions provided in (Melin 1994) for $\omega=0^\circ$ (i.e., the crack case) and in (Beghini *et al.* 2007) for $\omega > 18^\circ$. It's interesting to observe that, for the pure mode II case, a contribution related to mode I (4a) must be taken into account (see also (Goldstein and Salganik 1974, Amestoy and Leblond 1985)).

Upon substitution of Eq. (4) into Eq. (3), and of the corresponding relationship and Eq. (2) into the system (1), simple analytical manipulations yield

$$K_{II}^* = \frac{\sigma_u}{\bar{\mu}_{22}^{(1-\lambda_{II})} (f_{\theta\theta}^{II})^{2\lambda_{II}-1}} l_{ch}^{1-\lambda_{II}} \quad (5)$$

where the following quantities are introduced

Table 1 FFM parameters (crack propagation angles $|\theta_c|$ and functions g) dependent on the notch amplitude ω

ω (deg)	λ_{II}	$ \theta_c $ (deg)	g_K	g_Δ
0	0.5000	75.6	0.8110	0.5518
20	0.5620	71.2	0.8361	0.5294
30	0.5982	68.2	0.8572	0.5343
40	0.6382	65.7	0.8753	0.5369
50	0.6823	63.3	0.8902	0.5387
60	0.7309	60.9	0.9013	0.5399
70	0.7844	58.5	0.9078	0.5402
80	0.8434	56.3	0.9093	0.5395
90	0.9085	54.0	0.9054	0.5386
100	0.9805	51.8	0.8954	0.5373

$$\bar{\mu}_{22} = \frac{\mu_{12}^2 + \mu_{22}^2}{2\lambda_{II}} \quad (6a)$$

$$\bar{f}_{\theta\theta}^{II} = \frac{f_{\theta\theta}^{II}}{\lambda_{II}(2\pi)^{1-\lambda_{II}}} \quad (6b)$$

$$l_{ch} = \left(\frac{K_{Ic}}{\sigma_u} \right)^2 \quad (6c)$$

Since both functions $\bar{\mu}_{22}$ and $\bar{f}_{\theta\theta}^{II}$ in Eqs. (6) depend on θ , the critical generalized SIF K_{Ic}^* , corresponding to the minimum failure load, is obtained by setting the θ -derivative of the denominator in Eq. (5) equal to 0 (Cornetti *et al.* 2013)

$$\frac{d}{d\theta} \left[\bar{\mu}_{22}^{(1-\lambda_{II})} (\bar{f}_{\theta\theta}^{II})^{2\lambda_{II}-1} \right] = 0 \quad (7)$$

Condition (7) provides the critical crack propagation angle θ_c for each opening angle ω . Different values of θ_c , for $0 \leq \omega \leq 100^\circ$, are presented in Table 1.

Substituting θ_c into Eq. (5) gives the critical generalized SIF. Indeed, in critical conditions, a more convenient expression can be put forward:

$$K_{Ic}^* = g_K(\omega) \sigma_u l_{ch}^{1-\lambda_{II}} \quad (8)$$

where the function $g_K = \left(\frac{1}{\bar{\mu}_{22}^{(1-\lambda_{II})} (\bar{f}_{\theta\theta}^{II})^{2\lambda_{II}-1}} \right)_{\theta=\theta_c}$ has been evaluated in Table 1. It is worth

observing that, for the crack case, Eq. (8) reverts to the same expression provided by Griffith approach, i.e., Linear Elastic Fracture Mechanics (LEFM). In other words, fracture takes place according to the maximum strain energy direction, while the stress requirement in (1) limits to provide the value for the crack extension Δ .

Eventually, the other unknown of system (1) is obtained by means of

$$\Delta_c = \left(\frac{(\bar{f}_{\theta\theta}^{\text{II}})^2}{\bar{\mu}_{22}} \right)_{\theta=\theta_c} l_{ch} = g_{\Delta}(\omega) l_{ch} \quad (9)$$

which holds again for a fixed critical propagation angle θ_c , i.e., a fixed notch amplitude ω . Indeed, as can be evinced from Table 1, g_{Δ} slightly varies as ω varies.

In conclusion: the FFM criterion (1) can be regarded as a coupled Griffith-Rankine non-local failure criterion: fracture is energy driven, but a sufficiently high stress field must act at the crack tip to trigger crack propagation. By means of Eq. (7) it is possible to derive a fixed crack propagation angle for each notch amplitude. Through Eqs. (8) and (9) it is then possible to derive the critical generalized SIF (i.e., the critical failure load) and crack advance, respectively: Table 1 summarizes the parameters necessary for failure analysis.

3. Different approaches

Different approaches based on a critical distance and proposed in the Literature can be applied to mode II loading conditions. Among the others, let us cite the point stress (PS) criterion and the average stress (AS) criterion. The former is a generalization of the maximum circumferential stress approach put forward in (Erdogan and Sih 1963) for the crack case. It is based on the condition

$$\sigma_{\theta\theta}(\Delta_{PS}, \theta) \geq \sigma_u \quad (10a)$$

with

$$\Delta_{PS} = \frac{1}{2\pi} l_{ch} \quad (10b)$$

The latter (Seweryn *et al.* 1997), requiring that the average stress ahead of the notch tip must reach a critical value, is expressed by the first equation of (1)

$$\int_0^{\Delta_{AS}} \sigma_{\theta\theta}(r, \theta) dr \geq \sigma_u \Delta_{AS} \quad (11a)$$

where

$$\Delta_{AS} = \frac{2}{\pi} l_{ch} \quad (11b)$$

Notice that according to both criteria:

1. the crack advance results a material property (Eqs. (10b)-(11b)) and not a structural one as for the FFM case (where it depends on ω , Eq. (9));
2. the critical crack propagation angle θ_c is the same (values are reported in Table 2), obtained by imposing the condition

$$\frac{d}{d\theta} (f_{\theta\theta}^{\text{II}}) = 0 \quad (12)$$

3. through some analytical manipulations, the expressions for the critical generalized SIF and

Table 2 Point stress (PS) and average stress (AS) criterion parameters (crack propagation angles $|\theta_c|$ and functions g) dependent on the notch amplitude ω

ω (deg)	$ \theta_c $ (deg)	g_K (PS)	g_Δ (PS)	g_K (AS)	g_Δ (AS)
0	70.5	0.8660	0.1591	0.8660	0.6366
20	66.9	0.8749	0.1591	0.9024	0.6366
30	65.1	0.8787	0.1591	0.9175	0.6366
40	63.2	0.8820	0.1591	0.9295	0.6366
50	61.4	0.8848	0.1591	0.9377	0.6366
60	59.5	0.8872	0.1591	0.9416	0.6366
70	57.6	0.8891	0.1591	0.9403	0.6366
80	55.6	0.8904	0.1591	0.9331	0.6366
90	53.7	0.8914	0.1591	0.9193	0.6366
100	51.8	0.8918	0.1591	0.8984	0.6366

crack advance (constant in these cases) can be recast in the same forms of Eqs. (8) and (9). Functions g have been reported in Table 2.

With respect to FFM, PS and AS criteria both provide lower critical angles and higher failure loads. Predictions differ more for what concerns lower notch opening angles, while they approach each other for ω tending to 100° .

In the next section, only the results related to the AS criterion will be considered (and compared with FFM ones), since it has been generally found to be more precise than those provided by the PS approach.

Eventually, let us remind that a coupled criterion similar to FFM, but based on a point stress condition, was also proposed (Leguillon 2002) and applied to mixed mode (Yosibash *et al.* 2006).

4. Experimental validation

In order to prove the soundness of the present approach, a comparison with experimental results is performed. As already stated, since for $\omega = 0^\circ$ the criterion coincides with Linear Elastic Fracture Mechanics (LEFM), whose reliability has been widely proved in the past, the attention is focused on the case $\omega > 0^\circ$. Many data related to pure mode II loading are available in the Literature (Seweryn *et al.* 1997, Seweryn and Lukaszewicz 2002, Ayatollahi and Torabi 2011, Ayatollahi *et al.* 2011a, b): in all the present cases the effect of the notch root radius will be neglected, consistently with the sharpness measured during experiments. Indeed, for a detailed discussion on the root radius effects on notched elements subjected to mode II (and more in general to mixed mode) loading conditions, the reader can refer to Lazzarin and Filippi (2006), Priel *et al.* (2008).

Let us start by considering the Arcan tests carried out on double V-notched PMMA samples (Seweryn *et al.* 1997): four notch amplitudes were tested corresponding to $\omega = 20^\circ, 40^\circ, 60^\circ$ and 80° . The material properties provided in (Seweryn *et al.* 1997) were: $K_{Ic} = 0.37 \text{ MPa}\sqrt{m}$, $\sigma_u = 115 \text{ MPa}$. On the other hand, in a subsequent work (Seweryn and Lukaszewicz 2002), they were interestingly revalued by one of the Authors, who proposed: $K_{Ic} = 120 \text{ MPa}\sqrt{m}$, $\sigma_u = 102.8 \text{ MPa}$.

These values are here implemented ($l_{ch} \sim 0.1363$ mm), supposing (and hoping) that the more recent data should be more accurate: it is important to underline, however, that the material fracture toughness K_{Ic} was derived from a best fit procedure on the AS criterion.

As regards the critical generalized SIF (i.e., the failure load), results are presented in Fig. 2: a satisfactory agreement is generally found, also for $\omega=20^\circ$, which revealed to be the most problematic situation in the mixed mode analysis (Sapora *et al.* 2013). In this case, the percent deviation according to FFM is approximately -16%. On the other hand, the error grows up to +20% as concerns the AS criterion for $\omega=60^\circ$. Fig. 3 shows the comparison on crack propagation angles: also in this case, FFM predictions result accurate. The most significant deviation (+5° compared to the mean value) is found again for $\omega=20^\circ$, where an important experimental scattering is indeed observed. Estimations by the AS approach reveal to be more precise, also because based on an optimal K_{Ic} - value.

The second data set taken into account is that related to V-notched Brazilian Disk (VBD) tests carried out on graphite samples (Ayatollahi and Torabi 2011, Ayatollahi *et al.* 2011a). Specimens were machined with $\omega=30^\circ$, 60° and 90° . The graphite properties measured are: $K_{Ic} = 1.00 \text{ MPa}\sqrt{\text{m}}$, $\sigma_u = 27.5 \text{ Mpa}$, corresponding to $l_{ch} \sim 1.322$ mm. The material thus reveals to be less brittle than the previous one (PMMA). Results are presented in Figs. 2 and 3, for what concerns the critical loads and propagation angles, respectively. Experimental values were deduced from the graphics presented in (Ayatollahi and Torabi 2011).

As regards K_{IIc}^* (Fig. 2), theoretical FFM predictions are very accurate, the mean square error being always below 5%. The same occurs for estimations on θ_c : from Fig. 3 the good matching between experimental and FFM values can be evinced, generally better than that found by implementing the AS criterion.

Let us now consider the VBD samples tested in (Ayatollahi *et al.* 2011b) for $\omega=30^\circ$, 60° and

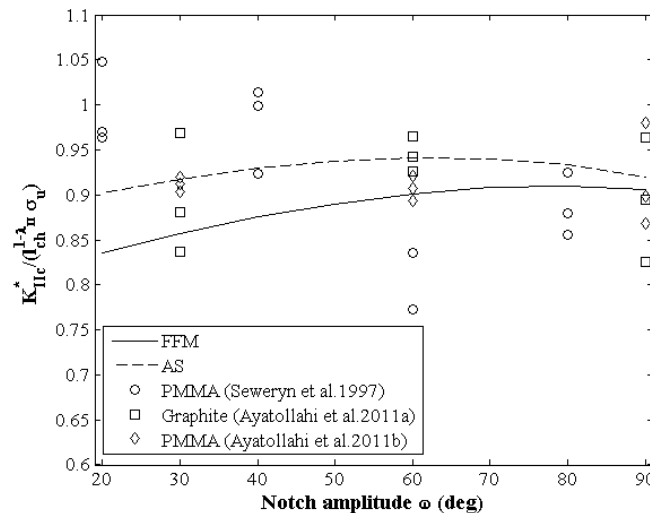


Fig. 2 FFM (continuous line) and AS criterion (dashed line) predictions on the dimensionless critical generalized SIF K_{IIc}^* for all the experiments considered in the present analysis

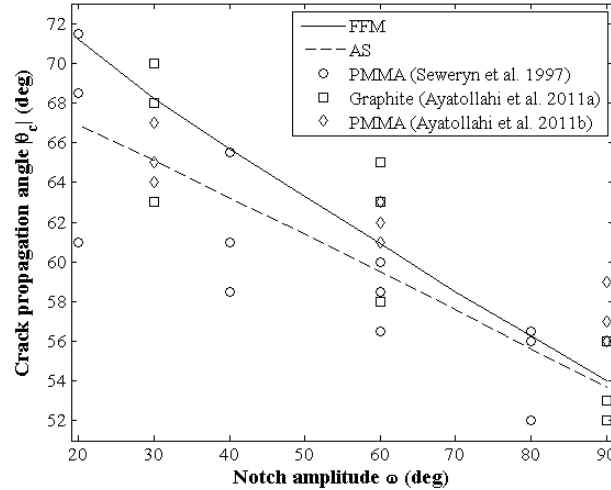


Fig. 3 FFM (continuous line) and AS criterion (dashed line) predictions on the crack propagation angle θ_c for all the experiments considered in the present analysis

90°. The considered material was PMMA, with the following properties: $K_{Ic} = 1.96 \text{ MPa}\sqrt{\text{m}}$, $\sigma_u = 70.5 \text{ MPa}$, corresponding to $l_{ch} \sim 0.7729 \text{ mm}$. Both FFM predictions on K_{IIc}^* (Fig. 2) and θ_c (Fig. 3) are found to be very accurate: for what concerns the first case the maximum deviation is observed for $\omega = 30^\circ$ (+5.5%), while as regards the second case the greatest error corresponds to $\omega = 90^\circ$ (-3.5%). Although good, AS estimations on θ_c are generally less precise.

Before concluding, it is worth observing that other experimental results can be found in the Literature. Dunn *et al.* (1997), for instance, tested PMMA samples under specific mode II loading conditions. On the other hand, they considered just one angle ($\omega = 90^\circ$) and they did not provide the value for the fracture toughness K_{Ic} in the paper. Thus, we have not considered this data, although the measured crack propagation angle ($\theta_c = 58^\circ$) approaches the theoretical one (Tables 1 and 2). Eventually, some tests on single V-notched PPMA samples were carried out in (Kim and Cho 2008) for different notch amplitudes. Anyway, the Authors faced some drawbacks during experiments (several specimens broke at the ends instead that from the notch tip, especially under mode II loading), and we have not included them in the analysis.

5. Conclusions

We applied the FFM criterion to determine the critical load in V-notched structures under Mode II loading. The problem is more complex than for simple Mode I loading, since, beyond the failure load, also the crack propagation angle at the re-entrant corner tip is unknown. Indeed, this parameter is found to depend only on the notch amplitude and not on the material. A comparison with different experimental data (for both the failure load and the crack orientation) has been performed, proving once more the soundness of the present approach. A good matching with the average stress criterion estimations has also been proved: the best criterion varies from case to case.

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