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Elastic flexural and torsional buckling behavior of pre-twisted bar under axial load

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Abstract. According to deformation features of pre-twisted bar, its elastic bending and torsion buckling equation is developed in the paper. The equation indicates that the bending buckling deformations in two main bending directions are coupled with each other, bending and twist buckling deformations are coupled with each other as well. However, for pre-twisted bar with dual-axis symmetry cross-section, bending buckling deformations are independent to the twist buckling deformation. The research indicates that the elastic torsion buckling load is not related to the pre-twisted angle, and equals to the torsion buckling load of the straight bar. Finite element analysis to pre-twisted bar with different pre-twisted angle is performed , the prediction shows that the assumption of a plane elastic bending buckling deformation curve proposed in previous literature (Shadnam and Abbasnia 2002) may not be accurate, and the curve deviates more from a plane with increasing of the pre-twisting angle. Finally, the parameters analysis is carried out to obtain the relationships between elastic bending buckling critical capacity, the effect of different pre-twisted angles and bending rigidity ratios are studied. The numerical results show that the existence of the pre-twisted angle leads to "resistance" effect of the stronger axis on buckling deformation, and enhances the elastic bending buckling critical capacity. It is noted that the "resistance" is getting stronger and the elastic buckling capacity is higher as the cross section bending rigidity ratio increases.

Keywords: : pre-twisted bar; flexural buckling; torsional buckling; coupling; finite element

1. Introduction

The pre-twisted beam, also known as naturally twisted beam, presents initially twisted shape in the natural state. Some researchers called it Naturally Twisted Beam (Zelenina and Zubov 2006). Naturally twisted beams have been widely applied in aviation and mechanical engineering, such as rotating helicopter blades, gears, turbine blades, and so on. As the prevailing research needs, the studies focused on material strength and vibration performance (Leung 2010, Banerjee 2004, Yu and Liao 2005, Hsu 2009, Leung 2010, Yu *et al.* 2011). In recent years, a number of new building structures appear, such as network shell, cable structure and tension-membrane structure system. Accordingly, it is necessary to adopt irregular shape components, for example, the design used a

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large number of initial bending and torsion members in the “bird nest” of Beijing, China. In 1999, the meeting on bridge at the United States, JF Barnett presented an exciting idea - “A Bridge for the Bridge” (Barnett 1999). Followed by Carlo H Sequin from University of California at Berkeley, a professor at the basis of its “Moebius Bridge” concept (Sequin 2000), the initially state is naturally twisted state.

In the past decade, research scholars put forward using the naturally twisted beam in frames and support members of wall structures (Shadnam and Abbasnia 2002, Leung 2010). However, because of the existence of the naturally twisted angle ω, bending buckling displacements coupled with each other, the buckling displacements will have no different from flexible shaft under direct pressure. As a result, research on the pre-twisted axis bar buckling performance has important theoretical and practical meaning for the engineers.

2. The elastic torsion buckling behavior

2.1 The elastic torsion buckling bearing capacity

In the global coordinate system *O_XYZ*, the stress behavior of the elastic pre-twisted cross-shaped bar under pined-pined condition is studied. The bar length is l and the axial force is P. In any position $Z = z$, the local coordinate system $G\_\xi\eta z$, Gz axis and OZ axis are coincidence, $G\xi$ and $G\eta$ are the main bending axis of the section.

In current work, it is assumed that the rotation angle of $G\_\xi\eta z$ relative to the *O_XYZ* is $\psi+kz$ in position $Z=z$, which the kz is the pre-twisted angle. In the adjacent position $Z=z+dz$, the rotation angle is $(\psi+kz)+(d\psi+kdz)$. When any section fiber DE is to $D'E''$, the rotation angle ϕ appears. The parameter ρ is the distance from the point E'' to section shear center S. Because the fiber is tilt after being twisted, the force acting on point E'' is $\sigma dA=P/A\cdot dA$, which is $\sigma' dA$ at the horizontal direction, the formed torque is $\sigma' dA\rho$ around the shear center S, as shown in Fig. 1.

Because the angle ϕ is small, the following equation is given by

$$\sin\phi \approx \phi = \frac{E'E''}{\mathrm{d}z} = \frac{\rho(\mathrm{d}\psi + k\mathrm{d}z)}{\mathrm{d}z} \tag{1}$$

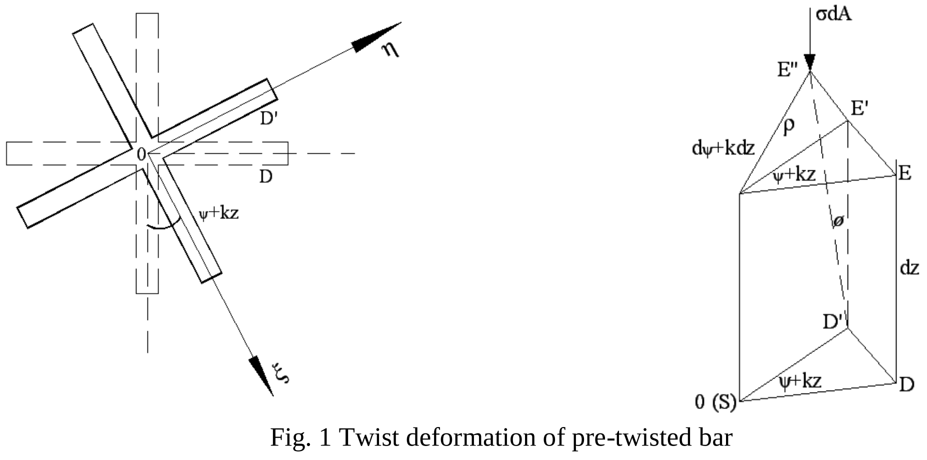


Fig. 1 Twist deformation of pre-twisted bar

Thus, in the fiber point *E''*, the horizontal component is

$$\sigma' \mathrm{d}A = \sigma \mathrm{d}A \sin\psi = \sigma \mathrm{d}A \rho(\psi' + k) \tag{2}$$

The non-uniform torque in the total cross-section is

$$M_z = \int_A \sigma \mathrm{d}A \rho^2 (\psi' + k) = \frac{P}{A}\int_A \rho^2 \mathrm{d}A(\psi' + k) \tag{3}$$

It can be seen that the Wagner effect factor is equal to that of the straight bar from the Eq. (3), But the Wagner torque adds one: $\frac{P}{A}\int_A \rho^2 k dA$. Introduction the section characteristic, $\int_A \rho^2 dA = I_\xi + I_\eta = i_0^2 A$, the parameter i_0 is gyration radius, The Eq. (3) can be expressed

$$M_z = P i_0^2 (\psi' + k) \tag{4}$$

According to the torque equal equation $M_z = M_k + M_w$ (M_k and M_w are the free torque and warping torque), The elastic buckling equation of the pre-twisted bar can be obtained as following

$$EI_w \psi''' + (P i_0^2 - GI_k)\psi' + P i_0^2 k = 0 \tag{5}$$

Assume: $\lambda_1^2 = \frac{P i_0^2}{EI_w}$, $\lambda_2^2 = \frac{GI_k}{EI_w}$, $\lambda^2 = \lambda_1^2 - \lambda_2^2$, Eq. (5) can be expressed as

$$\psi''' + (\lambda_1^2 - \lambda_2^2)\psi' + \lambda_1^2 k = 0 \tag{6}$$

The general result of differential Eq. (6) can be obtained using MATLAB

$$\psi = C_1 \sin(\lambda z) + C_2 \cos(\lambda z) + C_3 - \lambda_1^2 kz / \lambda^2 \tag{7}$$

By introducing the boundary conditions, we can get the following equations

$$\psi(0) = \psi(l) = 0, \ \ \psi''(0) = \psi''(l) = 0 \tag{8}$$

Thus

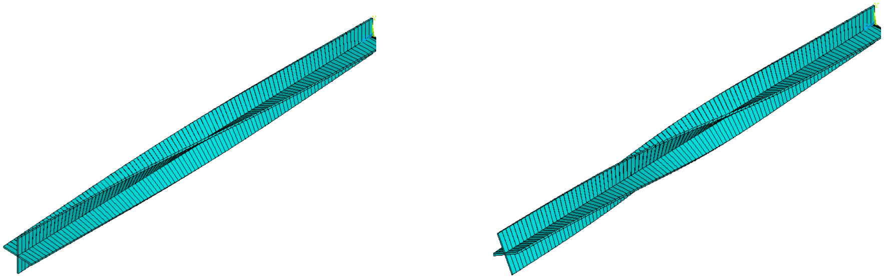
$$-C_1 \lambda^2 \sin \lambda l = 0 \ \ C_2 = 0 \ \ C_3 = 0 \tag{9}$$

At the same time, as the factor $C_1 \neq 0$, the equation $\lambda_{\min} = \pi / l$ is given.

Thus, the elastic torsion buckling Load can be calculated

$$P_w' = \frac{1}{i_0^2}\left(GI_k + \frac{\pi^2 EI_w}{l^2}\right) \tag{10}$$

We can get that the elastic torsion buckling load is not related to the pre-twisted angle, and is equal to the torsion buckling load of the straight bar.



(a) $\omega=\pi/2$ (b) $\omega=3\pi/4$

Fig. 2 Finite element model in different pre-twisted angle

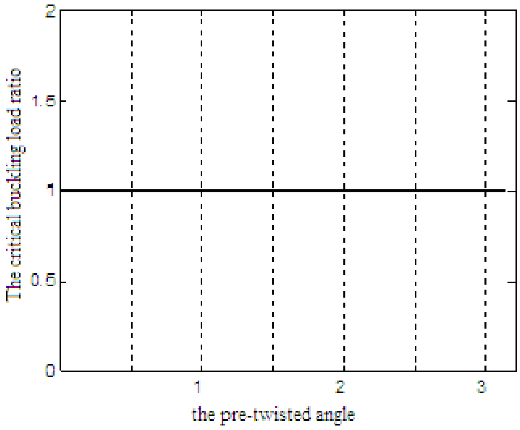


Fig. 3 Relation of buckling critical load ratio and pre-twisted angle

2.2 Verification using finite element analysis

The buckling behavior of the elastic pre-twisted cross-shaped bar is investigated using finite element software ANSYS. The bar length l is 6000mm, width of the cross-shaped section and plate thickness are 240mm and 10mm, respectively; the pre-twisted angle is ω, the steel material modulus E is 2.1×10^5 Mpa. Based on ANSYS "direction point" characteristics (ANSYS Inc. 2007), a 3D finite element model with different pre-twisted angle is developed as shown in Fig. 2.

According to Eq. (10), the elastic torsion buckling capacity is 2.824×10^6 N, which is equal to the finite element prediction. Based on the numerical result, the elastic torsion buckling capacity is not related to the pre-twisted angle ω. The critical buckling load ratio $\alpha=P'_w/P_w$ is 1 (P'_w is the elastic torsion buckling load of the pre-twisted bar, P_w is for the straight bar), as shown in Fig. 3.

3. The elastic flexural-torsion buckling behavior

3.1 The coupled elastic bending displacement

In the global coordinate system O_XYZ, the elastic flexural buckling behavior of the pre-twisted bar under pined-pined condition is studied, the bar length is l and the axial force is P. At any position $Z = z$, we introduce local coordinate system $G\_\xi\eta z$, Gz axis and OZ axis are coincidence, $G\xi$ and $G\eta$ are the main bending axis of the section, the twisted angle of $G\_\xi\eta z$ relative to the O_XYZ is $\omega = kz$. The linear displacements are u and v along the axis $G\xi$ and $G\eta$, the rotations displacements are θ and φ. So at position $Z = z + dz$, the linear displacements are $u+u'dz$ and $v+v'dz$ along the axis $\overline{G\xi}$ and $\overline{G\eta}$, as shown in Fig. 4.

The incremental displacements of adjacent section in the local coordinate system are given by

$$\Delta u = (u + u'\mathrm{d}z)\cos \mathrm{d}\omega - (v + v'\mathrm{d}z)\sin \mathrm{d}\omega - u \tag{11}$$

$$\Delta v = (v + v'\mathrm{d}z)\cos \mathrm{d}\omega + (u + u'\mathrm{d}z)\sin \mathrm{d}\omega - v \tag{12}$$

To the first order, $\cos d\omega = 1$, $\sin d\omega = d\omega$ and $\omega = kdz$, these are

$$\Delta u = (u' - kv)\mathrm{d}z \tag{13}$$

$$\Delta v = (v' + ku)\mathrm{d}z \tag{14}$$

From the definition of shear strain (Fig. 5), the following equations can be obtained

$$\gamma_{z\xi} = -\varphi + u' - kv,\ \gamma_{z\eta} = \theta + v' + ku \tag{15}$$

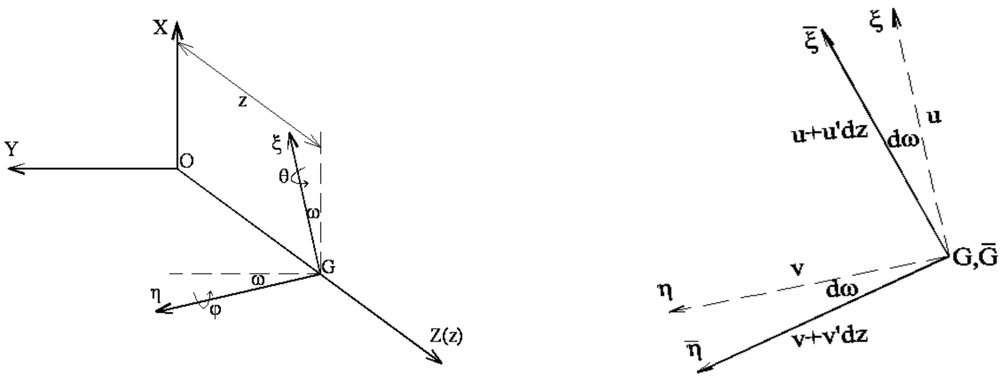


Fig. 4 Global and local coordinate of pre-twisted bar

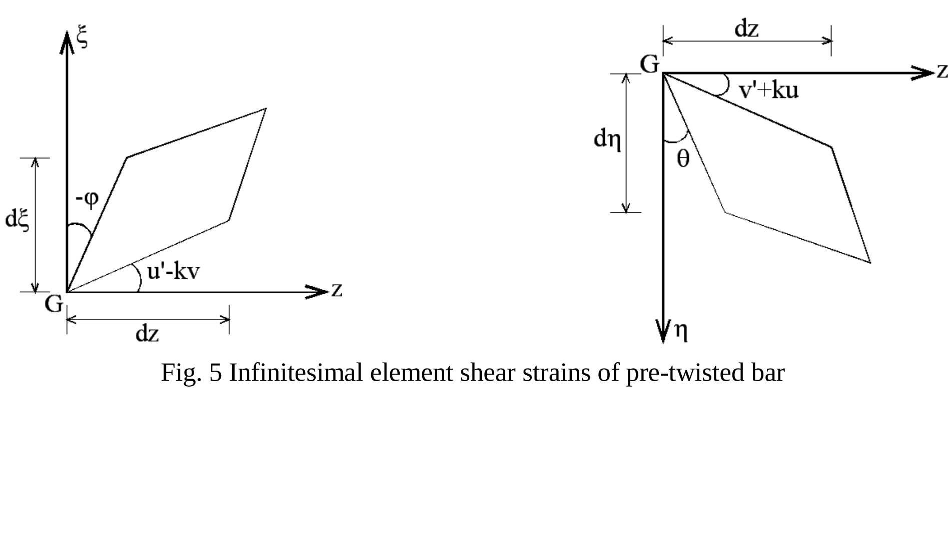


Fig. 5 Infinitesimal element shear strains of pre-twisted bar

Without considering shear deformation of the Euler beam

$$\theta = -v' - ku\,,\ \varphi = u' - kv \tag{16}$$

The axial displacement *W* of any point in $G\xi\eta$ plane along the *GZ* axis is given by

$$W = \eta\theta - \xi\varphi \tag{17}$$

By introducing the global and local coordinate system conversion relation

$$\xi = X\cos\omega + Y\sin\omega,\ \ \eta = -X\sin\omega + Y\cos\omega \tag{18}$$

Substituting Eqs. (16)-(18) into Eq. (17), we can get

$$W = (-X\sin\omega + Y\cos\omega)(-v' - ku) - (X\cos\omega + Y\sin\omega)(u' - kv) \tag{19}$$

The normal strain is

$$\begin{aligned}\varepsilon_z = \frac{\partial W}{\partial z} &= (-X\sin\omega + Y\cos\omega)(-v'' - ku') - (X\cos\omega + Y\sin\omega)(u'' - kv') \\ &(-X\cos\omega - Y\sin\omega)k(-v' - ku) - (-X\sin\omega + Y\cos\omega)k(u' - kv)\end{aligned} \tag{20}$$

Substituting Eq. (18) into Eq. (20) gives

$$\varepsilon_z = -\xi(u'' - 2kv' - k^2u) - \eta(v'' + 2ku' - k^2v) \tag{21}$$

The moments in any position are then given by

$$M_\xi = \iint_A \eta E\varepsilon_z \mathrm{d}\xi\mathrm{d}\eta,\ M_\eta = \iint_A -\xi E\varepsilon_z \mathrm{d}\xi\mathrm{d}\eta \tag{22}$$

Substituting Eq. (21) into Eq. (22), considering the equation $I_{\xi\eta} = \int_A \xi\eta\mathrm{d}\xi\mathrm{d}\eta = 0$, Eqs. (23)-(24) can be derived

$$M_\xi = -EI_\xi(v'' + 2ku' - k^2v) \tag{23}$$

$$M_\eta = EI_\eta(u'' - 2kv' - k^2u) \tag{24}$$

Where the beam's moment of inertia about the $G\xi$ and $G\eta$ axes are equations $I_\xi = \int_A \eta^2\mathrm{d}\xi\mathrm{d}\eta$ and $I_\eta = \int_A \xi^2\mathrm{d}\xi\mathrm{d}\eta$.

3.2 The coupled elastic flexural-torsion buckling equation

The following section establishes the balance buckling equations of the single-axis symmetric *I*-shaped pre-twisted bar, in the small bending and torsion deformation state, as shown in Fig. 6. The section shear center's displacement is *u*, the twisted angle is ψ+kz. Based on Eqs. (23)-(24) and the moment balance conditions, we can get

$$EI_\eta(u'' - 2kv' - k^2u) + P(u + e\psi + ekz) = 0 \tag{25}$$

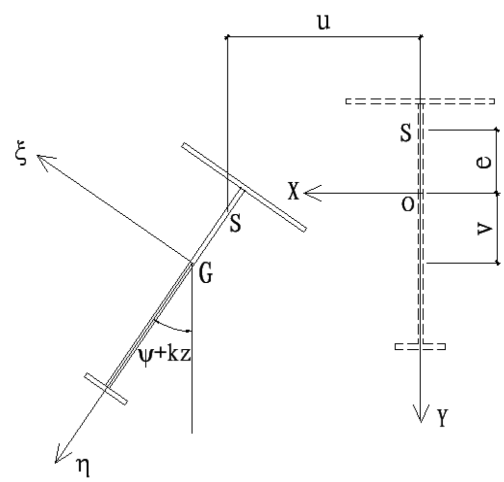


Fig. 6 Pre-twisted bar deformation of bending and torsion

$$EI_{\xi}(v'' + 2ku' - k^2 v) + Pv = 0 \tag{26}$$

It should be noted that the longitudinal axis will appear tilt when the pre-twisted bar happen to bending deformations around the axis of symmetry. Here assumes that the rotation angle around the longitudinal axis is θ_1 when the bending deformations happened around symmetry axis *OY*, the rotation angle around the longitudinal axis is θ_2 when the bending deformations happened around non-symmetry axis *OX*. By employing Eq. (16), the shear of cross-section shape center is given by

$$P\sin\theta_1 \approx P\,\mathrm{t}\,g\theta_1 = P(u' - kv),\ P\sin\theta_2 \approx P\,\mathrm{t}\,g\theta_2 = P(v' + ku) \tag{27}$$

Thus, the torque from the shear around the shear center *S* is given

$$P(u' - kv)e \times \cos(\psi + kz) \tag{28}$$

$$P(v' + ku)e \times \sin(\psi + kz) \tag{29}$$

The distance between the shear center *S* and any point is $\rho = \sqrt{x^2 + (y+e)^2}$, so the Wagner torque can be defined by

$$M_z = \int_A \sigma \mathrm{d}A \rho^2 (\psi' + k) = \frac{P}{A}\int_A [x^2 + (y+e)^2)]\mathrm{d}A(\psi' + k) = \frac{P}{A}(I_x + I_y + Ae^2)(\psi' + k) = Pi_0^2(\psi' + k) \tag{30}$$

Where: $i_0^2 = (I_\xi + I_\eta)/A + e^2$. The non-uniform torque can be further expressed as

$$M_z = Pi_0^2(\psi' + k) + P(u' - kv)e \times \cos(\psi + kz) - P(v' + ku)e \times \sin(\psi + kz) \tag{31}$$

Because the twist angle ψ is small, cosψ=1, sinψ=0, the Eq. (31) can be further simplified to

$$M_z = Pi_0^2(\psi' + k) + P(u' - kv)e \times \cos kz - P(v' + ku)e \times \sin kz \tag{32}$$

Thus, the twist balance equation is given

$$EI_w\psi'''+(Pi_0^2-GI_k)\psi'+Pi_0^2k+P(u'-kv)e\times\cos kz-P(v'+ku)e\times\sin kz=0 \tag{33}$$

Eqs. (25)-(26) and Eq. (33) compose the coupling bending and torsion buckling differential equations of the pre-twisted bar. Here, the bending buckling displacement coupled with each other around bending principle axis, the buckling displacement coupled with each other between the bending and twist buckling displacements as well. If the pre-twisted bar have dual-axis symmetric section, namely, e is equal to zero, and the above bending and twist buckling coupled equation can be expressed as

$$\begin{aligned} &EI_\eta(u''-2kv'-k^2u)+Pu=0\\ &EI_\xi(v''+2ku'-k^2v)+Pv=0\\ &EI_w\psi'''+(Pi_0^2-GI_k)\psi'+Pi_0^2k=0 \end{aligned} \tag{34}$$

From Eq. (34), it is noted that the bending buckling displacement coupled with each other; on the other hand, the bending buckling displacement does not couple with the twisted buckling displacement.

3.3 Verification using finite element analysis

For the pre-twisted bar of dual-axis symmetric section, previous work (Shadnam and Abbasnia 2002) assumed that the two principle bending direction of buckling displacement u and v obey linear relations, namely, $u=e^{mz}$, $v=\zeta e^{mz}$ (ζ is constant coefficient), which changed the second order coupled differential equations into fourth-order algebraic equations, and got the upper limit value of elastic bending buckling capacity. However, the equation for solving is complex, the general or semi-general analysis methods of second-order coupling differential equations still needs further research.

3.3.1 Whether the buckling displacement is a plane curve

The buckling behavior of the elastic pre-twisted rectangular section bar is studied numerically. The representative bar length l is 6000mm, the cross-shaped section's width and height are 200mm and 400mm, the pre-twisted angle is ω, the steel material modulus E is 2.1×10^5 Mpa. Based on "direction point" characteristics (ANSYS Inc. 2007), different finite element models with different pre-twisted angles are developed.

From the prediction of finite element analysis, the elastic bending displacement curve in the OXY plane projection with different pre-twisted angle is obtained. If the curve is a plane curve, it should be a straight line projection. As shown in Fig. 7, the curve is spatial. When the pre-twisted angle is larger, the deviated degree of the curve on the plane is more pronounced.

Fig. 8 denotes the elastic buckling curve normalized ratio of displacement in the X and Y direction with different pre-twisted angle. It is noted that if the pre-twisted angle ω is larger, the spatial curves deviation degree is greater, and the ratio of displacement was changed nonlinearly.

3.3.2 The buckling capacity predicted from finite element analysis

The critical buckling load ratio can be defined as

$$\alpha=P_{cr}'/P_{cr} \tag{35}$$

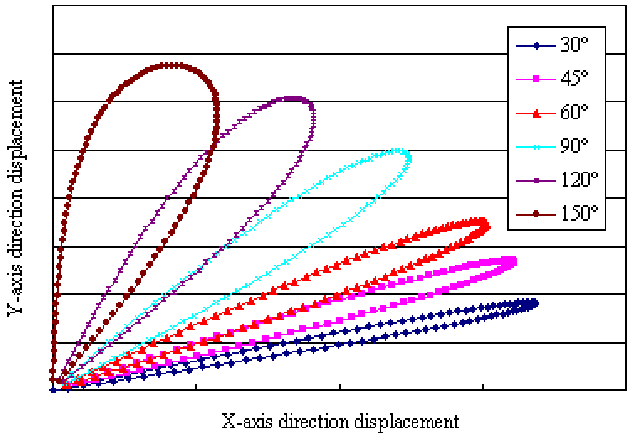


Fig. 7 Buckling displacement in different pre-twisted angle

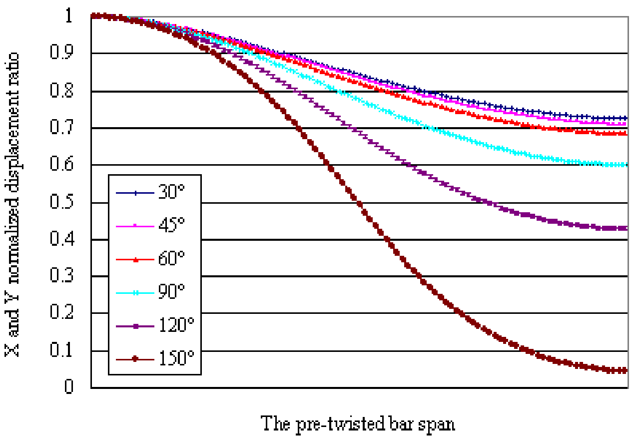


Fig. 8 Normalized displacement ratio in different pre-twisted angle between *X* and *Y* direction

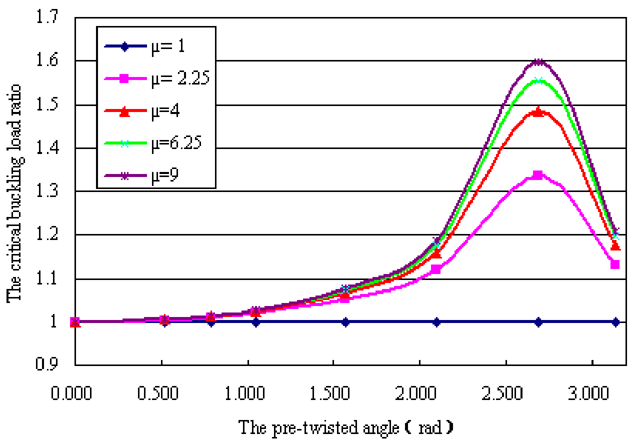


Fig. 9 Relations between critical bending buckling load ratio α and pre-twisted angle ω

The parameter P'_{cr} denotes the critical buckling load of the pre-twisted bar, and P_{cr} is the critical buckling load of the non-twisted bar.

The definition of bending rigidity parameters is given by

$$\mu = (I_{\xi} / I_{\eta}) \geq 1 \tag{36}$$

According to the developed model and numerical results, the relation between the flexural buckling bearing load and bending rigidity ratio μ is studied, which is changed by the section height for 200mm, 300mm, 400mm, 500mm, 600mm, respectively. From Fig. 9, the following statements can be concluded:

(1) When bending rigidity ratio μ is equal to 1, the flexible buckling capacity of pre-twisted bar is the same as the non- twisted straight bar.

(2) When bending rigidity ratio μ is greater than 1 and $\omega \in (0, \pi / 2)$, the critical buckling load slowly increases with the increasing of pre-twisted angle. When $\omega \in (\pi / 2, 5\pi / 6)$, the critical buckling load increases sharply. When $\omega \in (5\pi / 6, \pi)$, the critical buckling load shows downward trend.

(3) When the bending rigidity ratio μ is larger, the critical buckling load ratio is greater. The existence of the pre-twisted angle leads to “resistance” effect of the stronger axis on buckling deformation in the direction of other axis, which enhances the elastic bending buckling critical capacity. The “resistance” is getting stronger and the elastic buckling capacity is higher with the increasing of the cross section bending rigidity ratio.

4. Conclusions

(1) The elastic torsion buckling capacity is not related to the pre-twisted angle ω.

(2) For dual-axis symmetry pre-twisted bar, the bending buckling displacement does coupled with each other; however, the bending buckling displacement does not couple with the twisted buckling displacement. In current work, the elastic bending and torsion buckling problem can be simplified to a separate elastic bending buckling and torsion buckling problem. But for the pre-twisted bar of single axis symmetry, the flexural and torsion buckling displacements are coupled with each other.

(3) The flexural buckling curve is spatial because of the pre-twisted angle. When the pre-twisted angle is larger, the deviated degree of the curve on the plane is more pronounced. The assumption of plane deformation curve method will not be applicable to the pre-twisted bar with large pre-twisted angle.

(4) The paper adopts the straight beam element “infinite approximation” to pre-twisted element; this method doesn’t consider geometric discontinuities, more accurate pre-twisted element (Petrov and Geradin 1998, Yu *et al.* 2009, Zupan and Saje 2004) will be studied in future work.

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