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Study on modified differential transform method for free vibration analysis of uniform Euler-Bernoulli beam

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Abstract. A simulation method called modified differential transform is studied to solve the free vibration problems of uniform Euler-Bernoulli beam. First of all, the modified differential transform method is derived. Secondly, the modified differential transformation is applied to uniform Euler-Bernoulli beam free-free vibration. And then a set of differential equations are established. Through algebraic operations on these equations, we can get any natural frequency and normalized mode shape. Thirdly, the FEM is applied to obtain the numerical solutions. Finally, mode experimental method (MEM) is conducted to obtain experimental data for analysis by signal processing with LMS Test.lab Vibration testing and analysis system. Experimental data and simulation results are illustrated to be in comparison with the analytical solutions. The results show that the modified differential transform method can achieve good results in predicting the solution of such problems.

Keywords: modified differential transform method; mode experimental method; uniform Euler–Bernoulli beam; FEM; free vibration

1. Introduction

In recent years, many researchers have worked on engineering problems related to static and dynamic analysis of Euler–Bernoulli beam. The vibration problems of uniform Euler–Bernoulli beams can be solved by analytical or approximate approaches, Andrew Dimarogonas (1996) William T. Thomson (1981). The closed-form solutions for free vibration under various boundary conditions have been reported. Yeih (1999) obtained the natural frequencies and natural modes of an Euler–Bernoulli beam by a dual multiple reciprocity method (MRM) and the singular value decomposition method, which was able to avoid the spurious eigenvalue problems and modes. Taleb and Suppiger (1961) derived the frequency equation of a simply supported stepped beam. Register (1994) built a general expression for the modal frequencies and investigated the eigenvalue for a beam with symmetric spring boundary conditions. Naguleswaran (2002a, b) investigated natural frequencies, sensitivity and mode shape details of a Euler–Bernoulli beam. Rao and Misra (1989) studied the vibration of uniform beam with elastic supports at its ends. Yankelevesky and Eisenberger (1986) performed an exact analytical solution for a finite element

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beam-column resting on Winkler foundation leading to derivation of exact static stiffness matrix. Wang (1996) studied the dynamic analysis of generally supported beam using Fourier series. Sairigul and Aksu (1986) used the finite difference formulation to study a Timoshenko beam and they included the frequencies of Euler–Bernoulli beam with two step changes in cross-section and with clamped ends. An innovative method of solving these problems was presented by Lai (2008). With this method, the Adomian decomposition method (ADM) was applied to solve the Euler-Bernoulli beam vibration problem. Kim (2001) studied the vibration of uniform beams with generally restrained boundary conditions using Fourier series. Naguleswaran (2004) obtained an approximate solution to the transverse vibration of uniform Euler–Bernoulli beam under linearly varying axial force. He's variational iteration method was developed as a modification of a general Lagrange multiplier method, He (2000). Bayat (2013) used Hamiltonian Approach (HA) to analysis the nonlinear free vibration of Simply-Supported (S-S) and for the Clamped-Clamped (C-C) Euler-Bernoulli beams fixed at one end subjected to the axial loads.

Integral transform methods such as the Laplace and the Fourier transform methods are widely used in engineering problems. However, integral transform methods are very complicated and difficult to solve nonlinear problems. The concept of the differential transform method (DTM) was the first introduced in solving differential equations in the analysis of circuits by Zhou (1986). Being simple and widely applicable, DTM has been used in mechanical problems concerning the system of differential equations. The differential transform method is based on the Taylor's series expansion and provides a straightforward means of solving linear and non-linear differential equations. Furthermore, the method may be employed for the solution of both ordinary and partial differential equations. Jang (2001) applied a two-dimensional differential transform method to the solution of partial differential equations. Hassan (2002a, b) adopted the differential transformation method to solve some eigenvalue problems. Chen (1996) applied DTM to solve the free and forced vibration problems of non-uniform beams on a non-homogenous elastic foundation, and then, proposed a method to solve eigenvalue problem, Chen (1999). Ozgumus and Kaya (2008) applied DTM for free vibration analysis of double-tapered rotating Timoshenko beams. Balkaya (2009) used DTM to obtain natural frequencies of prismatic Euler–Bernoulli and Timoshenko beams resting on Winkler or Pasternak elastic foundations. Catal (2012) applied DTM for forced vibration differential equations of motion of Euler-Bernoulli beams with different boundary conditions and dynamic loads. Malik (1998) conducted the solution of continuous system by differential transformation.

In this paper, a modified differential transformation (MDTM) is introduced to solve the free vibration problem of a free-free end uniform Euler–Bernoulli beam. Using the MDTM, the governing differential equation becomes an algebraic equation and boundary conditions become simple algebraic frequency equations which are suitable for computation. Based on the method, *i*-th natural frequency and the closed form series solution of *i*-th mode shape can be obtained. Moreover, finite element method and mode experimental method are applied to getting natural frequency. Finally, results of the three methods are compared for the uniform Euler Bernoulli beam to verify the accuracy and efficiency of the present method.

2. Modified Differential Transform Method (MDTM)

2.1 Differential Transform Method (DTM)

DTM is an iterative method for obtaining the solution of differential equations in the form of Taylor series. It is different from high order Taylor series expansions which require the computation of derivatives of the data functions. DTM constructs an analytical solution in the form of polynomials and involves less computational effort in comparison with Taylor series solution in solving higher order problems.

Let *y(x)* be analytic in a domain D and let $x=x_i$ represent any point in D. The Taylor series expansion function of *y(x)* is of the form

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_i)^k\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_i} \tag{1}$$

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}(x)^k\left[\frac{d^k y(x)}{dx^k}\right]_{x_i=0} \quad \forall x\subset D \tag{2}$$

From Eq. (1) and Eq. (2), a *n*-th order differential transform of Taylor series expansion function of *y(x)* is defined about a point $x=x_i$ as

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_i} \tag{3}$$

Where it may be noted that upper case symbol *Y(k)* is used to denote the differential transform of a function represented by a corresponding lower case symbol *y(x)*. At x_i=0 the function *y(x)* may be expressed in terms of the differential transforms *Y(k)* as

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}x^k Y(k) \tag{4}$$

In real applications, the function *y(x)* is expressed by a finite series and Eq. (4) can be written as

$$y(x)=\sum_{k=0}^{n}\frac{1}{k!}x^k Y(k) \tag{5}$$

Here, *n* is series size and the value depends on the convergence of the natural frequency.

2.2 Modified differential transform method

Although the DTM is used to provide approximate solutions for many nonlinear problems in terms of convergent series, the method has some drawbacks. The series solution always converges in a very small region, and it has slow convergent rate or completely divergent in the wider region. So as to overcome the shortcomings, we develop modified differential transform method for the numerical solution of differential equations.

When applying Eq. (5) to solve engineering problems, the function *y(x)* can be represented by a finite-term Taylor series plus a modified as shown below

$$y(x)=\sum_{k=0}^{n}x^k Y(k)+R_{n+1}(x) \tag{6}$$

In order to speed up the convergent rate and to improve the accuracy of solutions, the entire domain D ($\forall x \subset D$) is usually split into multiple sub-intervals. At first, the differential transform method is used to solve the original equation in the first sub-interval. After that the final values of the first sub-interval are adopted as the initial values of the second interval and the original equation is solved under these new initial values. The same procedure is repeated in all the later sub-intervals until the solution of all domain D is achieved.

Reference to Multi-step DTM, Mohammad MehdiRashidi (2011), assuming that the interval [0, x] is divided into equi-length M numbered sub-intervals [x_{m-1}, x_m], m=1,2, …, M. Where $n=K\bullet M$. In fact, the MDTM assumes the following solution.

$$R_{n+1}(x) = \begin{cases} R_1(x),\ x \in [0,\ x_1] \\ R_2(x),\ x \in [x_1,\ x_2] \\ \quad n \\ R_{n+1}(x),\ x \in [x_{m-1},\ x_m] \end{cases} \tag{7}$$

The new algorithm MDTM, is simple for computational performance for all values. It is easily observed that if the step size is x, the MDTM reduces to the classical DTM. The main advantage of the new algorithm is that the obtained series solution converges for wide time regions and can approximate vibration solutions.

3. The uniform Euler-Bernoulli beam vibration problem analysis by MDTM

3.1 The beam governing differential equation

The governing differential equation of a rectangle uniform Euler-Bernoulli beam whose length is L undergoing free harmonic vibration is

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{8}$$

Here, A is the cross section area of the beam. E is Young modulus. I is the inertia of the beam. ρ is mass per unit volume. To any vibration mode, the beam flapwise displacement $y(x,t)$ is

$$y(x,t) = Y(x)T(t) \tag{9}$$

Here, $Y(x)$ is the beam mode deflection. $T(t)$ is the harmonic function of time, t. If ω is used to express the angular frequency of $T(t)$, then

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 Y(x)T(t) \tag{10}$$

Substituting Eq. (10) into Eq. (8) yields

$$EI\frac{\partial^4 Y(x)}{\partial x^4} - \rho A\omega^2 Y(x) = 0 \tag{11}$$

Introducing the following dimensionless quantities. ξ is dimensionless distance to the left end of the beam. $y(\xi)$ is dimensionless transverse mode.

Using

$$\xi = \frac{x}{L}, \quad y(\xi) = \frac{Y(x)}{L}, \quad y(\xi) = \frac{Y(x)}{L}, \quad \lambda^2 = \frac{\rho A w^2 L^4}{EI} \tag{12}$$

Eq. (11) becomes a dimensionless form

$$\frac{\partial^4 Y(\xi)}{\partial \xi^4} - \lambda^2 y(\xi) = 0 \tag{13}$$

The corresponding boundary conditions are at x=0

$$y(x,t) = \frac{\partial y(x,t)}{\partial x} = 0 \tag{14}$$

At x=L

$$y(x,t) = \frac{\partial y(x,t)}{\partial x} = 0 \tag{15}$$

The corresponding initial conditions is at t=0

$$y(x,t) = \frac{\partial y(x,t)}{\partial x} = 0 \tag{16}$$

Using Eq. (12), boundary conditions are

$$\begin{aligned} y(\xi) = \frac{\partial y(\xi)}{\partial \xi} = 0 \quad & at \quad \xi = 0 \\ y(\xi) = \frac{\partial y(\xi)}{\partial \xi} = 0 \quad & at \quad \xi = L/L = 1 \end{aligned} \tag{17}$$

3.2 MDTM application

We analyze the beam equation and solve the frequency with MDTM and Mode Superposition Method. Taking differential transform of (13) and (6), we obtain

$$Y(k+4) = \frac{k!\lambda^2 Y(k)}{(k+4)!} \tag{18}$$

At ξ=0, Y(0)=0, Y(1)=0, ξ=1, the boundary conditions change to

$$\sum_{k=0}^{\infty} Y(k) = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} k(k-1)Y(k) = 0 \tag{19}$$

Assigning

$$Y(2) = c \quad \text{and} \quad Y(3) = d \tag{20}$$

Table 1 The result of *Y* parameters

$Y(k)$	Result	$Y(k)$	Result
$Y(4)$	0	$Y(16)$	0
$Y(5)$	0	$Y(17)$	0
$Y(6)$	$\frac{\lambda^2 c}{360}$	$Y(18)$	$\frac{\lambda^8 c}{3201186852\ 864000}$
$Y(7)$	$\frac{\lambda^2 d}{840}$	$Y(19)$	$\frac{\lambda^8 d}{2027418340\ 1472000}$
$Y(8)$	0	$Y(20)$	0
$Y(9)$	0	$Y(21)$	0
$Y(10)$	$\frac{\lambda^4 c}{1814400}$	$Y(22)$	$\frac{\lambda^{10} c}{5620003638\ 8880384000\ 0}$
$Y(11)$	$\frac{\lambda^4 d}{6652800}$	$Y(23)$	$\frac{\lambda^{10} d}{4308669456\ 4808294400\ 00}$
$Y(12)$	0	$Y(24)$	0
$Y(13)$	0	$Y(25)$	0
$Y(14)$	$\frac{\lambda^6 c}{4358914560\ 0}$	$Y(26)$	$\frac{\lambda^{12} c}{2016457305\ 6330281779\ 2000000}$
$Y(15)$	$\frac{\lambda^6 d}{2179457280\ 00}$	$Y(27)$	$\frac{\lambda^{12} d}{1814811570\ 6972536012\ 8000000}$

At k=0, substituting Eqs. (19) and (20) into (18), we have

$$Y(4) = 0 \tag{21}$$

Following the same process, we can get the result corresponding to k=12.
From Table 1 and Eqs. (18)-(21), we obtain the following simplified equations

$$\begin{aligned} &Y(4k) = 0 && k = 0, 1, 2, \cdots, \\ &Y(4k+1) = 0 && k = 0, 1, 2, \cdots, \\ &Y(4k+2) = \frac{2!\lambda^{2k} c}{(4k+2)!} && k = 1, 2, \cdots, \\ &Y(4k+3) = \frac{3!\lambda^{2k} c}{(4k+3)!} && k = 1, 2, \cdots, \end{aligned} \tag{22}$$

Substituting Eq. (22) into (19) wields

$$\begin{cases} \sum_{k=0}^{\infty} Y(k) = \sum_{k=0}^{\infty} \frac{2!\lambda^{2k} c}{(4k+2)!} + \sum_{k=0}^{\infty} \frac{3!\lambda^{2k} d}{(4k+3)!} = 0 \\ \sum_{k=0}^{\infty} k(k-1)Y(k) = \sum_{k=0}^{\infty} \frac{2!\lambda^{2k} c}{(4k)!} + \sum_{k=0}^{\infty} \frac{3!\lambda^{2k} d}{(4k+1)!} = 0 \end{cases} \tag{23}$$

From Eq. (23), we can get

$$\begin{cases} c/d = -\sum_{k=0}^{\infty} \frac{3!\lambda^{2k}}{(4k+3)!} / \sum_{k=0}^{\infty} \frac{2!\lambda^{2k}}{(4k+3)!} = 0 \\ c/d = -\sum_{k=0}^{\infty} \frac{3!\lambda^{2k}}{(4k+1)!} / \sum_{k=0}^{\infty} \frac{2!\lambda^{2k}}{(4k)!} = 0 \end{cases} \tag{24}$$

λ is obtained by the determinant of Eq. (23), $f^k(\lambda)$ is polynomial of λ corresponding to k (k is the variable in the summation, from 0 to infinite). We get frequency equation

$$f^{(k)}(\lambda) = \sum_{k=0}^{\infty} \frac{3!\lambda^{2k}}{(4k+3)!} \sum_{k=0}^{\infty} \frac{2!\lambda^{2k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{2!\lambda^{2k}}{(4k+2)!} \sum_{k=0}^{\infty} \frac{3!\lambda^{2k}}{(4k+1)!} = 0 \tag{25}$$

Solving Eq. (25), we get $\lambda = \lambda_i^k, i \in [1, \infty)$. It is the i-th estimated dimensionless natural frequency corresponding to k, which is decided by the following equation, where ε is a small value set by us.

$$\left| \lambda_i^{(k)} - \lambda_i^{(k-1)} \right| \le \varepsilon \tag{26}$$

If Eq. (26) is satisfied, then λ_i^k is the eigenvalue λ_i. The eigenvalue or mode function describing the instantaneous deflected shape of the beam for a given λ may then be obtained by using Eqs. (20)-(22) in (6). We obtain the eigenfuntions *y(x)* as Eq. (27). Following an identical procedure, one can obtain the frequency equations and mode functions for other types of beams as well.

$$y_i(x) = \sum_{k=0}^{\infty} \frac{\lambda_i^{2k}}{(4k+2)!} x^{(4k+2)} - \frac{\sum_{k=0}^{\infty} \frac{\lambda_i^{2k}}{(4k+2)!}}{\sum_{k=0}^{\infty} \frac{\lambda_i^{2k}}{(4k+3)!}} \sum_{k=0}^{\infty} \frac{\lambda_i^{2k}}{(4k+3)!} x^{(4k+3)} \tag{27}$$

4. Example

Let's take a look at free vibration of an approximate rectangle uniform Euler-Bernoulli beam (free at the end). The basic parameters are shown in Table 2 (In the case, we consider the cross-section of beam as approximately rectangle).

Table 2 The basic parameters of beam

Length (m)	Width (m)	Thick (m)	Young modulus E(Gpa)	Density ρ (Kg/m3)	Material	Poisson's ratio
12	2.12	2.06	174	7200	QT600	0.275

4.1 MDTM solution

Solving Eq. (25) using C++ program, and neglecting imaginary roots, we have the two real roots

$$\lambda_1^6 = 54.586438 \quad \text{and} \quad \lambda_2^6 = 61.596035 \tag{28}$$

When k=5, by the same method we obtain

$$\lambda_1^5 = 54.586375 \tag{29}$$

From Eqs. (25), (28) and (29) we have

$$\left|\lambda_1^6 - \lambda_1^5\right| = 0.000063 \le 0.0001 \tag{30}$$

Here we take ε is 0.0001. So we have $\lambda_1^6 = 54.586438$ with λ_1 being the first-order natural frequency. Substituting λ_1 into *Y(k)*, we obtain the series solution of the normalized mode shape

$$\begin{aligned} y_1(x) = 120.314(&0.5x^2 - 0.503x^3 + 0.354x^6 - 0.186x^7 + 0.017x^{10} - 0.005x^{11} \\ &+ 1.545\times10^{-4}x^{14} - 2.577\times10^{-5}x^{15} + 4.889\times10^{-6}x^{20} - 1.413\times10^{-7}x^{21} + \cdots) \end{aligned} \tag{31}$$

Following the same procedure as shown above, the second-order and third-order natural frequencies and mode shapes can be calculated. The convergence of natural frequencies λ_1 to λ_4 are shown in Fig. 1. They are 54.586438, 61.596035, 69.075648, 100.246996, respectively. The normalized mode shapes are shown in Fig. 2.

4.2 FEM solution

According to the design drawings, we builted the 3D model with UG software (NX 6.0, Siemens company, Germany), and divided grid cell using tetrahedral grid. Grid size is 40 mm. 3D

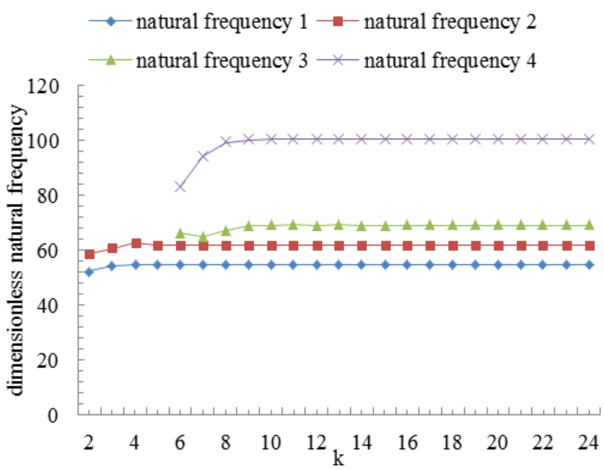

Fig. 1 The convergence of dimensionless natural frequencies of a free-free approximate rectangle uniform beam as functions of *k*

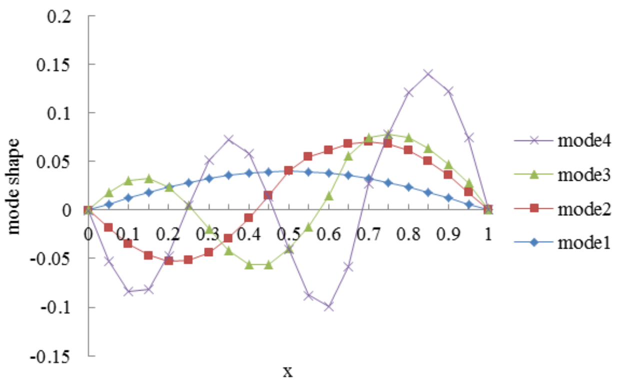

Fig. 2 The first four normalized mode shape

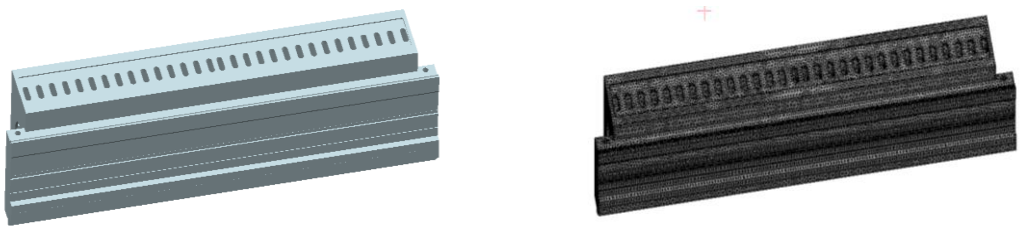

Fig. 3 3D model and FEM model of beam

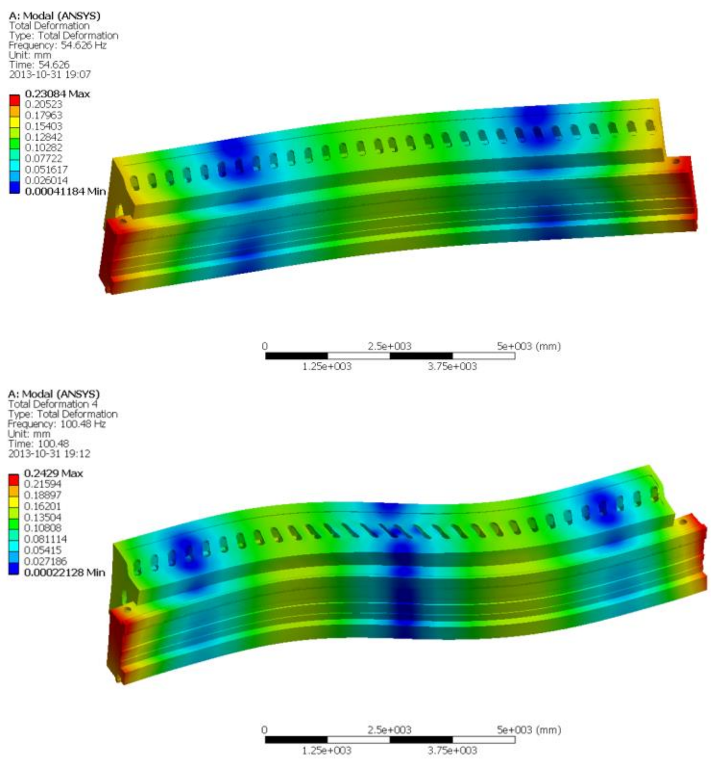

Fig. 4 The first and fourth mode of beam

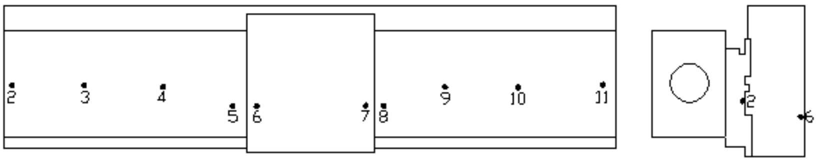

Fig. 5 Schematic of test points arrangement

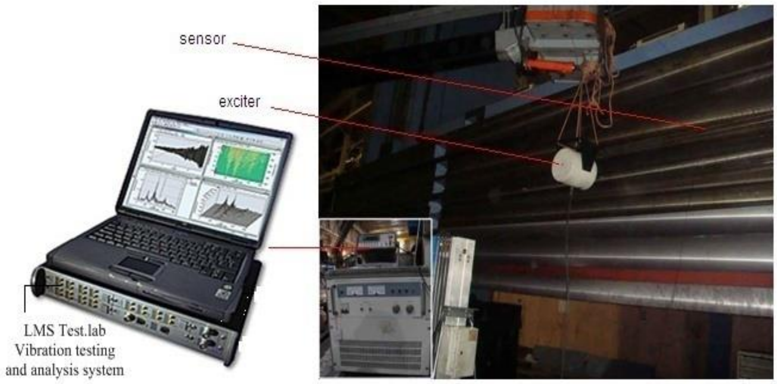

Fig. 6 Picture of test setup up

model and FE model are shown in Fig. 3.

Using Ansys 12.1 (Ansys company, USA), we analyzed the mode of beam and obtained the mode image, shown in Fig. 4. The first-order to fourth-order frequencies are 54.626, 61.339, 69.299, 100.48, respectively. They are very close to our results.

4.3 Experiment solution

We performed an experiment of free vibration at Beijing No.1 Machine Tool Plant. We hung up the beam with overhead crane and steel wire rope. Beam was in free state. We conducted the mode shape and frequency experiment.

Exciter was installed at the center of the beam, which was the No.1 test point. The arrangement of test points are shown in Fig. 5. We completed data acquisition and signal processing analysis with LMS Test.lab Vibration testing and analysis system (LMS company, Belgium), shown in Fig. 6. We scanned the beam at a dynamic sweep mode shape. Excitation frequency range was 0Hz to 150Hz, exciting force was about 100KN, and scanning speed was 0.01Hz/s.

We found five peaks of the frequency response functions of all test points. They were around 12.03Hz, 54.10Hz, 61.11Hz, 68.44Hz, and 99.89Hz, respectively. The analysis of the mode shapes of frequencies range indicates that the beam occurs rigid body resonance at 12.03Hz. Fine sweeping was performed around all four frequencies. For example, a frequency range of 53.5Hz to55.5Hz with a scanning speed of 0.004Hz/ s, and scanning time of 500s, were set to sweep the 54.1Hz zone. A frequency range of 97Hz to 103Hz, scanning speed of 0.01Hz/s and scanning time of 600s were set to sweep 99.89Hz zone. The frequency functions of the No.3 test point was

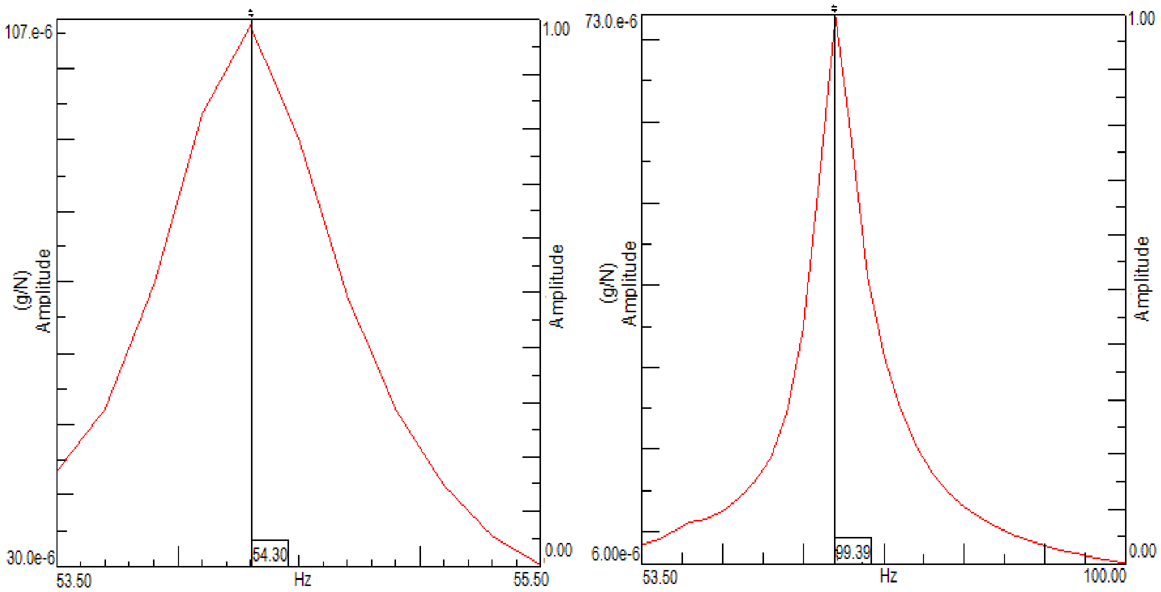

Fig. 7 Frequency functions of 54.30Hz and 99.39Hz

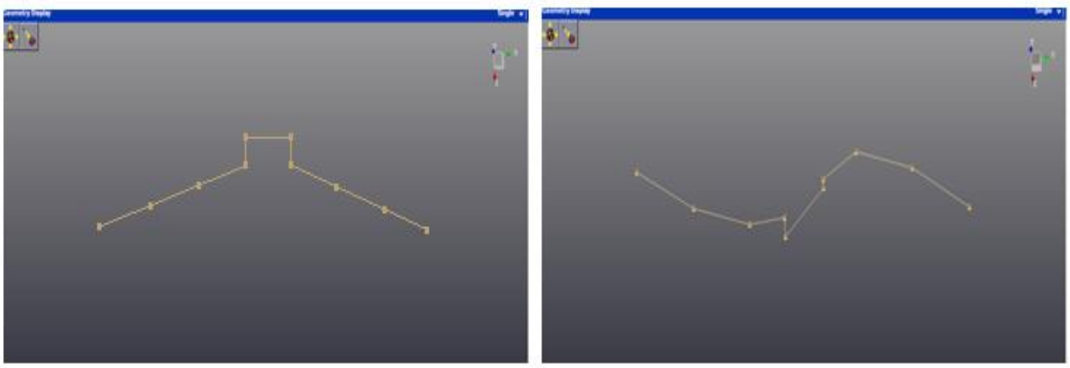

Fig. 8 Mode shapes of 54.30Hz and 99.39Hz

analyzed as shown in Fig. 7. With LSCE operation of LMS Analysis Software, we got the natural frequencies of beam, 54.30Hz, 61.19Hz, 68.57Hz, and 99.39Hz. Using LMS Analysis Software, we also obtained the mode shapes as shown in Fig. 8. Comparing Fig. 2 and Fig. 8, it was found that the first and the second mode shapes given by the MDTM methods and experiments are very similar.

4.4 Results

The results of MDTM, FEM and MEM (mode experimental method) are compared in Table 3. Considering the cross-section of experimental beam is approximate rectangle in MDTM, the effect of steel wire rope and experimental conditions in MEM, boundary condition and numerical calculate in FEM, the MDTM gives accurate results.

Table 3 Comparison of results of three methods

Natural Frequency	MDTM	FEM	MEN
λ_1	54.586438	54.626	54.30
Λ_2	61.596035	61.339	61.19
Λ_3	69.075648	69.299	68.57
Λ_4	100.246996	100.48	99.39

5. Conclusions

In this paper, the modified differential transform method is employed to solve the free vibration problems of uniform Euler-Bernoulli beam. With the method, the closed form solution of the problems can be obtained including some natural frequencies (eigenvalues) and mode shapes (eigenfunctions) for the fourth-order differential equation. The problem is successfully transformed into algebraic equations, any ***i***-th natural frequency and normalized mode shape could be obtained by solving these equations. As an evaluation, the 3D model of uniform Euler-Bernoulli beam was created by using UG Software and finite element model. Their numerical solutions of natural frequencies were obtained. Besides a vibration experiment was performed by applying the LMS Test.lab Vibration testing and analysis system. The three results are highly compatible to prove the accuracy and efficiency of the presented method.

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