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# Size-dependent thermal behaviors of axially traveling nanobeams based on a strain gradient theory

Cheng Li*

*School of Urban Rail Transportation, Soochow University, Suzhou 215006, China*

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**Abstract.** This work is concerned with transverse vibrations of axially traveling nanobeams including strain gradient and thermal effects. The strain gradient elasticity theory and the temperature field are taken into consideration. A new higher-order differential equation of motion is derived from the variational principle and the corresponding higher-order non-classical boundary conditions including simple, clamped, cantilevered supports and their higher-order "offspring" are established. Effects of strain gradient nanoscale parameter, temperature change, shape parameter and axial traction on the natural frequencies are presented and discussed through some numerical examples. It is concluded that the factors mentioned above significantly influence the dynamic behaviors of an axially traveling nanobeam. In particular, the strain gradient effect tends to induce higher vibration frequencies as compared to an axially traveling macro beams based on the classical vibration theory without strain gradient effect.

**Keywords:** critical speed; natural frequency; size dependent; strain gradient; thermal effect

## 1. Introduction

Nanotechnology has been a subject of intensive technological interest today due to its great potential applications beyond the ability of micro-technology, i.e., excessive strain, excessive stiffness, gigahertz oscillator and so on. Due to the requirement of enormous computing effort in molecular dynamic simulation even for an average molecular structure with limited molecule counts, it would appear that a continuum molecular model, such as that of nonlocal elasticity theory, is essential and will potentially play a critical part in the analyses related to nanotechnology applications, e.g., as subminiature belts, nano-machinery, nano-electromechanical system (NEMS). These are all typical and key components in engineering devices at nanometer scale. Taking the subminiature belt as an example, it involves transverse vibrations of axially traveling nanobeams.

There exists aplenty literature on the study of axially traveling structures at macro scales (Öz and Pakdemirli 1999, Hatami *et al*. 2007, Tang *et al*. 2009). Although analysis on axially traveling classical beams is well established, there is virtually no research work at present on similar problems at nanoscale, i.e. axially traveling nanobeams either using a continuum model or a numerical approach, such as the nonlocal elasticity or molecular dynamics simulation,

*Corresponding author, Associate Professor, E-mail: licheng@suda.edu.cn


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respectively. Because the length scales associated with nanotechnology are at molecular level, the applicability of classical continuum models should be revisited. For example, phonon scattering experiment indicates elastic wave with high frequency is dispersive and its wave speed is associated with frequency. On the contrary, the speed of elastic wave is a constant in classical continuum theory. In fracture mechanics, the classical model suggests that even a very small load causes infinite stress at a crack tip and the crack growth occurs absolutely. However, every material has its own fatigue limit and there is already theory established which confirms that no stress singularity at a flaw tip exists when the small size effects are considered (Eringen *et al*. 1977).

While classical continuum mechanics assumes that the stress at a given point is dependent uniquely on the strain state at that same point, namely, it does not admit intrinsic size dependence in the elastic solutions of inclusions and inhomogeneities, nonlocal theory regards the stress at a given point as a function of the strain stares of all points in the body and so size effects often become prominent. The theory of nonlocal continuum mechanics was pioneered by Eringen (1972, 1983). This new continuum theory differs greatly from the classical continuum theory, because the latter ignores the long range intermolecular force or it is inherently size-independent, while the former exhibits essential size-dependent characteristics. Nonlocal theory contains information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter. The original form of Eringen's nonlocal theory contains an integral type constitutive relation. However, by introducing higher-order derivatives into such constitutive equations, the integral type of nonlocal theory becomes differential type or strain gradient type, and the latter assumes the stress is a function of the strain and the gradients of strain at the same point (Mindlin 1965). In this paper, the strain gradient type of nonlocal theory is employed to investigate the axially traveling nanobeams.

The strain gradient type nonlocal theory has been widely used in nanomechanics including dislocation, fracture mechanics, fluid surface tension and wave propagation in composites, etc. Currently, research investigations are focused on mechanical characteristics of carbon nanotubes or nanobeam-like structures based on the strain gradient theory (Peddieson *et al*. 2003, Zhang *et al*. 2004, Lu *et al*. 2006, Wang *et al*. 2006, Lee and Chang 2009, Lim *et al*. 2009, Lim 2010, Lim *et al*. 2010, Li *et al*. 2011a,b,c, Lim and Yang 2011, Pradhan and Kumar 2011, Wang 2011, Aksencer and Aydogdu 2012, Lim *et al*. 2012, Yu and Lim 2013). The strain gradient elasticity model has been one of the most popular approaches in nanostructures. However, there are two kinds of strain gradient elasticity model and both are reported extensively. The first strain gradient model (see e.g., Peddieson *et al*. 2003) combined the classical equilibrium with strain gradient constitution, and obtained a finite-order governing equation for nanostructures. It was reported that an increase in strain gradient nanoscale parameter results in decreasing nanostructural stiffness. Therefore, strain gradient effects are associated with decreased natural frequency, decreased wave velocity, decreased critical buckling load and increased bending deflection. Hence, the nanostructures are softened accounting for strain gradient elasticity effects in the first model. The second strain gradient model, or called a new effective strain gradient model was firstly presented by Lim *et al*. (2009), Lim (2010). Unlike the first strain gradient model, this effective strain gradient model was derived according to the energy variation principle and a higher-order governing equation was obtained. Some effective strain gradient types of nonlocal quantity (i.e., effective nonlocal stress, or effective nonlocal bending moment, etc.) are defined and they are expressed in infinite series of the corresponding common strain gradient nonlocal quantities. During the past years, such effective strain gradient model was employed for tensile, buckling, bending and vibration analyses of nanobeams or CNTs extensively (see e.g., Lim *et al*. 2012, Yu and Lim 2013). These analyses reported new solutions that nanostructural stiffness may be enhanced with stronger size effects, i.e., lower deflection, higher buckling load or natural frequency under the effective strain gradient effects. People are puzzled about these two strain gradient models because they declared some opposite conclusions. In order to verify the two strain gradient models, Li *et al*. (2011d) developed a semi-continuum model with discrete atomic layers in the thickness direction to investigate the bending behaviors of an ultra-thin beam with nanoscale thickness. The relaxation effect was considered in one atomic layer on each of the upper and lower surfaces and a comparison research with the two strain gradient models indicates inconclusive due to different material parameters. The two strain gradient models were proved to be both acceptable. The long range attractive or repulsive interaction, or the looser or tighter atomic lattice near surface than the bulk results in the two different strain gradient models. Therefore, the different physical phenomena (i.e., the stiffness of nanostructures is lower or higher than that predicted by classical mechanics theory) observed in two strain gradient models do exist simultaneously and they are related to the types of surface effects in nanostructures.

There is virtually no paper which considers strain gradient thermal effects for an axially traveling nanobeam available currently, especially that using the new effective strain gradient model. Therefore, this paper tried to develop a new strain gradient type nonlocal approach for axially traveling nanobeams based on the effective strain gradient model. Because thermal effect is essential to the mechanical behavior of nanostructures, recently, Lim and Yang (2011) presented a static strain gradient thermal-elasticity model for nanobeams and they obtained some exact strain gradient solutions for nanobeam deformation due to thermal loads. The work (Lim and Yang 2011) verifies again that stiffness enhancement effects in the effective strain gradient theory. Wang (2011) investigated the vibration and stability of nanotubes conveying fluid using the effective strain gradient stress model, and the effects of strain gradient nanoscale parameter on natural frequency and critical flow velocity is presented in detail. In the present work, thermal effect is taken into account and effective strain gradient effects on the higher-order governing equation and non-classical boundary conditions of an axially traveling nanobeam are proposed. Subsequently the transverse vibration of such a nanobeam under the influences of various geometric and material parameters is discussed. The conclusion is obvious that due to size-dependent nanoscale effects, the vibration of a traveling nanobeam is significantly different from its behavior according to the classical vibration theory. In particular, strain gradient effect, axial velocity, temperature change and density are concluded to play significant roles in the dynamic behavior of an axially traveling nanobeam.

## 2. Strain gradient equation of motion and boundary conditions

A uniform elastic nanobeam of mass density $\rho$, length $L$ and axial velocity $u$ is considered. The $x$-axis is along the axial direction and $y$-axis lies on the transverse direction. The beam is subjected to uniformly distributed load $p_0$ along its axial direction. Transverse vibration in the presence of an axial velocity is studied, considering different end boundary conditions which are to be specified in various cases of study. The variational principle is applied to derive the higher-order equation of motion and the corresponding higher-order boundary conditions that governs the transverse vibration.

For free vibrations of an axially traveling nanobeam, the strain energy density $e$ at an arbitrary point of a deformed nanobeam is given by (Lim 2010, Lim and Yang 2011)

$$e=\frac{1}{2}E\varepsilon_{xx}^{2}+\frac{1}{2}E\sum_{n=1}^{\infty}(-1)^{n+1}\tau^{2n}\left(\frac{d^{n}\varepsilon_{xx}}{d\overline{x}^{n}}\right)^{2}+E\sum_{n=1}^{\infty}\left\{\tau^{2(n+1)}\sum_{m=1}^{n}\left[(-1)^{m+1}\frac{d^{m}\varepsilon_{xx}}{d\overline{x}^{m}}\frac{d^{(2(n+1)-m)}\varepsilon_{xx}}{d\overline{x}^{(2(n+1)-m)}}\right]\right\} \tag{1}$$

where $\overline{x}=x/L$ is the dimensionless axial coordinate, $\tau=e_0 a/L$ the dimensionless strain gradient nanoscale parameter, $E$ Young's modulus, $\varepsilon_{xx}$ the normal axial strain, $e_0$ a constant dependent on material, and $a$ an internal characteristic length (e.g., lattice parameter, C-C bond length, granular distance, etc. (Eringen 1983)).

The total strain energy in the deformed body is

$$E_e=\int_V e\,dV \tag{2}$$

Following the variational principle, the first variation of strain energy can be expressed as (Lim 2010, Lim and Yang 2011)

$$\begin{aligned}\delta E_e=&\frac{EI}{L}\int_0^1\left[-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial^2\overline{w}}{\partial\overline{x}^2}\right)^{\langle 2n\rangle}\right]\delta\overline{w}\,d\overline{x}\\
&+\frac{EI}{L}\left[\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}\delta\overline{w}-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n-1\rangle}\frac{\partial\delta\overline{w}}{\partial\overline{x}}\right.\\
&+\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}\frac{\partial^2\delta\overline{w}}{\partial\overline{x}^2}-\sum_{n=1}^{\infty}2n\tau^{2(n+1)}\left(\frac{\partial^2\overline{w}}{\partial\overline{x}^2}\right)^{\langle 2n\rangle}\frac{\partial^3\delta\overline{w}}{\partial\overline{x}^3}+\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}\frac{\partial^4\delta\overline{w}}{\partial\overline{x}^4}\\
&\left.-\sum_{n=1}^{\infty}2n\tau^{2(n+2)}\left(\frac{\partial^2\overline{w}}{\partial\overline{x}^2}\right)^{\langle 2n\rangle}\frac{\partial^5\delta\overline{w}}{\partial\overline{x}^5}+\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}\frac{\partial^6\delta\overline{w}}{\partial\overline{x}^6}+\cdots\right]_0^1\end{aligned} \tag{3}$$

where $\overline{w}=w/L$ and $w$ are the dimensionless and dimensional transverse displacements, respectively, $I=\iint_A y^2 dA$ is the second moment of cross-sectional area $A$ and $\langle n\rangle$ represents the $n$th-order derivative with respect to $\overline{x}$.

Assuming the nanobeam is homogenous and subject to a uniform thermal load over its entire length, as a result of thermal expansion, an additional axial force is given by (Avsec and Oblak 2007)

$$N=\alpha\theta EA \tag{4}$$

where $\theta$ is the temperature difference between the actual and initial or reference temperature and $\alpha$ is the linear thermal expansion coefficient. For room or low temperature $\alpha=-1.6\times10^{-6}\,\mathrm{K}^{-1}$ while for high temperature $\alpha=1.1\times10^{-6}\,\mathrm{K}^{-1}$ (Yao and Han 2007). The work done by the axial force is

$$E_\alpha=\frac{N}{2}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2dx=\frac{\alpha\theta EAL}{2}\int_0^1\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^2d\overline{x} \tag{5}$$

The first variation of the work (virtual work) above is given by

$$\delta E_\alpha = \alpha\theta EAL\int_0^1 \frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \delta\bar{w}}{\partial \bar{x}}d\bar{x} = \alpha\theta EAL\left[\frac{\partial \bar{w}}{\partial \bar{x}}\delta\bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\delta\bar{w}d\bar{x}\right] \tag{6}$$

The work done by the uniformly distributed load is

$$E_p = \int_0^L p_0 w dx = p_0 L^2 \int_0^1 \bar{w}d\bar{x} \tag{7}$$

and the corresponding virtual work of Eq. (7) is obtained by its first variation as

$$\delta E_p = p_0 L^2 \int_0^1 \delta\bar{w}d\bar{x} \tag{8}$$

The kinetic energy due to axial velocity and transverse motion is

$$\begin{aligned} E_k &= \frac{\rho A}{2}\int_0^L \left[\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x}\right)^2 + u^2\right]dx \\ &= \frac{\rho AL}{2}\int_0^1 \left[\frac{L^2}{T^2}\left(\frac{\partial \bar{w}}{\partial \bar{t}}\right)^2 + \frac{\bar{u}^2 EI}{\rho AL^2}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 + \frac{2\bar{u}L}{T}\sqrt{\frac{EI}{\rho AL^2}}\frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \bar{w}}{\partial \bar{t}} + \frac{\bar{u}^2 EI}{\rho AL^2}\right]d\bar{x} \end{aligned} \tag{9}$$

where $\bar{t} = t/T$ is the dimensionless time in which $t$ is the temporal coordinate, $T$ is a characteristic time, and $\bar{u} = u\sqrt{\rho AL^2/EI}$ is the dimensionless axial velocity. Hence variation of the kinetic energy gives

$$\begin{aligned} \delta E_k &= \frac{\rho AL^3}{T^2}\int_0^1 \frac{\partial \bar{w}}{\partial \bar{t}}\frac{\partial \delta\bar{w}}{\partial \bar{t}}d\bar{x} + \frac{\bar{u}^2 EI}{L}\int_0^1 \frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \delta\bar{w}}{\partial \bar{x}}d\bar{x} + \frac{\rho A\bar{u}L^2}{T}\sqrt{\frac{EI}{\rho AL^2}}\int_0^1\left(\frac{\partial \bar{w}}{\partial \bar{x}}\frac{\partial \delta\bar{w}}{\partial \bar{t}} + \frac{\partial \bar{w}}{\partial \bar{t}}\frac{\partial \delta\bar{w}}{\partial \bar{x}}\right)d\bar{x} \\ &= \frac{\rho AL^3}{T^2}\left[\frac{\partial \bar{w}}{\partial \bar{t}}\delta\bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2}\delta\bar{w}d\bar{x}\right] + \frac{\bar{u}^2 EI}{L}\left[\frac{\partial \bar{w}}{\partial \bar{x}}\delta\bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}\delta\bar{w}d\bar{x}\right] \\ &+ \frac{\rho A\bar{u}L^2}{T}\sqrt{\frac{EI}{\rho AL^2}}\left[\frac{\partial \bar{w}}{\partial \bar{x}}\delta\bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{x}\partial \bar{t}}\delta\bar{w}d\bar{x} + \frac{\partial \bar{w}}{\partial \bar{t}}\delta\bar{w}\Big|_0^1 - \int_0^1 \frac{\partial^2 \bar{w}}{\partial \bar{x}\partial \bar{t}}\delta\bar{w}d\bar{x}\right] \end{aligned} \tag{10}$$

For static equilibrium, the variational principle requires that

$$\delta\int_0^T \left(E_e + E_\alpha + E_p - E_k\right)dt = 0 \tag{11}$$

which yields

$$\frac{EIT}{L}\int_0^1\int_0^1\left[-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^{\langle 2n\rangle}-\frac{\alpha\theta AL^2}{I}\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{p_0L^3}{EI}\right.$$
$$\left.+\frac{\rho AL^4}{EIT^2}\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+\bar{u}^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\frac{2\rho A\bar{u}L^3}{EIT}\sqrt{\frac{EI}{\rho AL^2}}\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{t}}\right]\delta\bar{w}\,d\bar{x}d\bar{t}$$
$$+\frac{EIT}{L}\int_0^1\left[\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^{\langle 2n\rangle}\delta\bar{w}-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^{\langle 2n-1\rangle}\frac{\partial\delta\bar{w}}{\partial\bar{x}}\right.$$
$$+\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^{\langle 2n\rangle}\frac{\partial^2\delta\bar{w}}{\partial\bar{x}^2}-\sum_{n=1}^{\infty}2n\,\tau^{2(n+1)}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^{\langle 2n\rangle}\frac{\partial^3\delta\bar{w}}{\partial\bar{x}^3}$$
$$+\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^{\langle 2n\rangle}\frac{\partial^4\delta\bar{w}}{\partial\bar{x}^4}-\sum_{n=1}^{\infty}2n\tau^{2(n+2)}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^{\langle 2n\rangle}\frac{\partial^5\delta\bar{w}}{\partial\bar{x}^5}$$
$$+\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\left(\frac{\partial\bar{w}}{\partial\bar{x}}\right)^{\langle 2n\rangle}\frac{\partial^6\delta\bar{w}}{\partial\bar{x}^6}+\cdots$$
$$\left.+\frac{\alpha\theta AL^2}{I}\frac{\partial\bar{w}}{\partial\bar{x}}\delta\bar{w}-\bar{u}^2\frac{\partial\bar{w}}{\partial\bar{x}}\delta\bar{w}-\frac{\rho A\bar{u}L^3}{EIT}\sqrt{\frac{EI}{\rho AL^2}}\frac{\partial\bar{w}}{\partial\bar{x}}\delta\bar{w}\right]_0^1 d\bar{t}=0 \tag{12}$$

where

$$\int_0^1\frac{\partial\bar{w}}{\partial\bar{t}}\delta\bar{w}\Big|_{\bar{t}=0}^{\bar{t}=1}d\bar{x}=0 \tag{13}$$

is assumed because the motion is periodic and at $\bar{t}=0$ and $\bar{t}=1$, the quantity within the integration have the same values. Because $\delta\bar{w}$ in Eq. (12) cannot vanish, hence the higher-order governing equation of motion is

$$-\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2}\right)^{\langle 2n\rangle}+\left(\bar{u}^2-\alpha\theta\kappa\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\bar{\rho}\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+2\bar{u}\sqrt{\bar{\rho}}\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{t}}+\bar{p}_0=0 \tag{14}$$

where $\kappa=AL^2/I$ is a dimensionless shape parameter, $\bar{\rho}=\rho AL^4/EIT^2$ and $\bar{p}_0=p_0L^3/EI$ are the dimensionless density and load, respectively. The basic and higher-order strain gradient boundary conditions are also obtained from Eq. (12) and they are listed in Appendix.

To investigate the effective strain gradient and thermal effects, the first few strain gradient terms in the series of Eq. (14) are retained. These are the most significant strain gradient components which exhibit the obvious effective strain gradient effects. As a result, a truncated governing equation of motion with strain gradient influence is obtained as

$$-\tau^2\frac{\partial^6\bar{w}}{\partial\bar{x}^6}+\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+\left(\bar{u}^2-\alpha\theta\kappa\right)\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\bar{\rho}\frac{\partial^2\bar{w}}{\partial\bar{t}^2}+2\bar{u}\sqrt{\bar{\rho}}\frac{\partial^2\bar{w}}{\partial\bar{x}\partial\bar{t}}+\bar{p}_0=0 \tag{15}$$

For linear free vibration of a nanobeam, the vibration modes are harmonic in time. Hence the time-dependent transverse deformation of the nanobeam can be represented by

$$\bar{w}(\bar{x},\bar{t})=\varphi(\bar{x})e^{i\omega_n\bar{t}} \tag{16}$$

where $\varphi(\bar{x})$ is the dimensionless vibration amplitude shape function, $n$ = 1, 2, 3, K denotes the vibration mode number, and $\omega_n$ is the dimensionless natural frequency. The substitution of Eq. (16) into Eq. (15) yields a characteristic equation as

$$-\tau^2 \frac{d^6\varphi}{d\bar{x}^6} + \frac{d^4\varphi}{d\bar{x}^4} + \left(\bar{u}^2 - \alpha\theta\kappa\right)\frac{d^2\varphi}{d\bar{x}^2} + 2i\bar{u}\,\omega_n\sqrt{\bar{\rho}}\,\frac{d\varphi}{d\bar{x}} - \bar{\rho}\omega_n^2\varphi + \bar{p}_0 = 0 \tag{17}$$

The solution to the governing equation Eq. (17) can be expressed as

$$\varphi(\bar{x}) = C_1 e^{k_1\bar{x}} + C_2 e^{k_2\bar{x}} + C_3 e^{k_3\bar{x}} + C_4 e^{k_4\bar{x}} + C_5 e^{k_5\bar{x}} + C_6 e^{k_6\bar{x}} \tag{18}$$

where $k_j$ ($j$=1,2,…6) are the six roots of the characteristic equation (17) and $C_j$ ($j$=1,2,…6) are the corresponding coefficients which can be determined from the six higher-order strain gradient boundary conditions in Eq. (A3) in Appendix.

## 3. Numerical examples and discussion

In order to illustrate the effects of strain gradient nanoscale parameter $\tau$, dimensionless velocity $\bar{u}$, temperature $\theta$ and density $\bar{\rho}$ on the transverse vibration of an axially traveling nanobeam, two examples with typical boundary conditions are presented and discussed in detail.

### *3.1 Simply supported nanobeams*

The boundary conditions for a simply supported nanobeam are shown in Eq. (A4) or (A5) in Appendix. The substitution of Eq. (18) into Eq. (A5) yields a system of homogeneous equations in a matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} & e^{k_5} & e^{k_6} \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 & k_5^2 & k_6^2 \\ k_1^2 e^{k_1} & k_2^2 e^{k_2} & k_3^2 e^{k_3} & k_4^2 e^{k_4} & k_5^2 e^{k_5} & k_6^2 e^{k_6} \\ k_1^4 & k_2^4 & k_3^4 & k_4^4 & k_5^4 & k_6^4 \\ k_1^4 e^{k_1} & k_2^4 e^{k_2} & k_3^4 e^{k_3} & k_4^4 e^{k_4} & k_5^4 e^{k_5} & k_6^4 e^{k_6} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = 0 \tag{19}$$

For nontrivial solutions, the determinant of the matrix in Eq. (19) must be zero, or

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} & e^{k_5} & e^{k_6} \\ k_1^2 & k_2^2 & k_3^2 & k_4^2 & k_5^2 & k_6^2 \\ k_1^2 e^{k_1} & k_2^2 e^{k_2} & k_3^2 e^{k_3} & k_4^2 e^{k_4} & k_5^2 e^{k_5} & k_6^2 e^{k_6} \\ k_1^4 & k_2^4 & k_3^4 & k_4^4 & k_5^4 & k_6^4 \\ k_1^4 e^{k_1} & k_2^4 e^{k_2} & k_3^4 e^{k_3} & k_4^4 e^{k_4} & k_5^4 e^{k_5} & k_6^4 e^{k_6} \end{vmatrix} = 0 \tag{20}$$

Combining Eq. (20) and the characteristic equation (17), the relation between natural frequency and strain gradient nanoscale parameter is obtained. The numerical solution for varying velocity

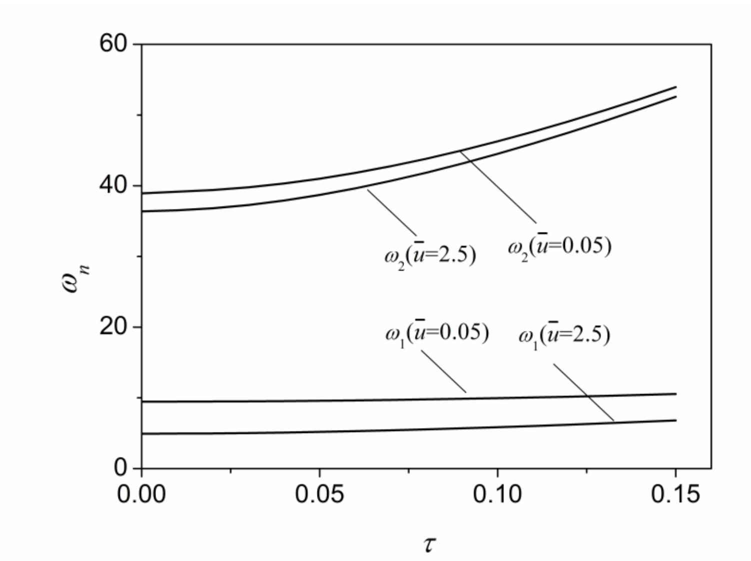


Fig. 1 The effect of strain gradient nanoscale parameter on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$, $\theta=50K$ and $\kappa=10^4$

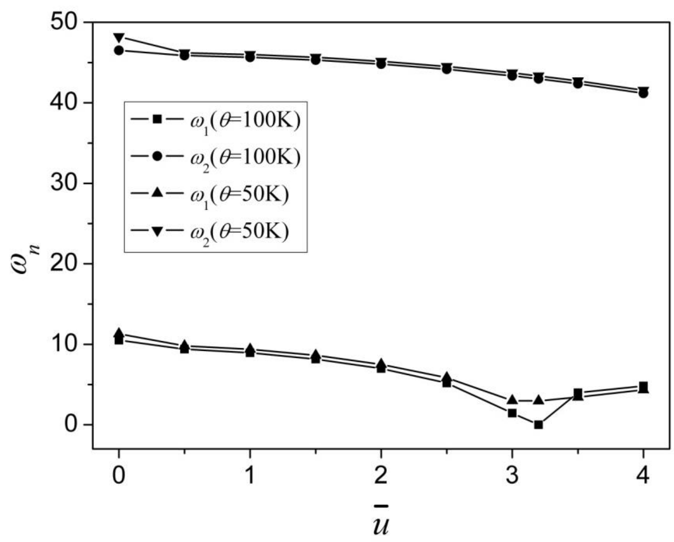


Fig. 2 The effect of axial velocity on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$, τ=0.1 and $\kappa=10^4$

shown is illustrated in Fig. 1 for $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$ (Yao and Han 2007), $\theta=50K$ and $\kappa=10^4$.

In Fig. 1, it is noticed that the natural frequency increases with increasing $\tau$, i.e. stiffness is strengthened with the presence of stronger strain gradient effect. The second mode frequency increases faster than the first mode. For example, when $\tau$ increases from 0.01 to 0.15, $\omega_n$ increases

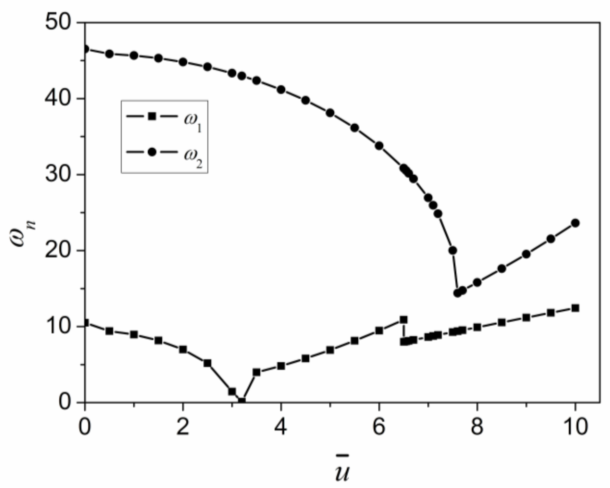


Fig. 3 The effect of axial velocity on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha= -1.6\times10^{-6}K^{-1}$, $\tau = 0.1$, $\kappa=10^4$ and $\theta = 100K$

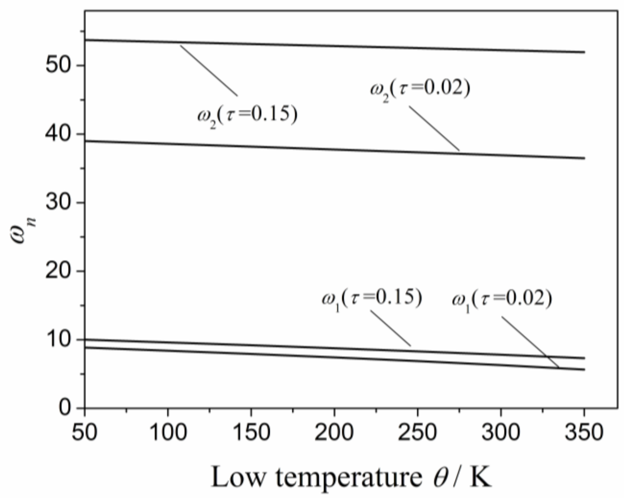


Fig. 4 The effect of low temperature on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha= -1.6\times10^{-6}K^{-1}$, $\bar{u}=1$ and $\kappa=10^4$

approximately 37.63% and 43.96% for $n$=1 and $n$=2, respectively with $\bar{u}=2.5$. In addition, a higher dimensionless axial velocity causes lower frequency.

The relation between $\omega_n$ and $\bar{u}$ is shown in Fig. 2. It is observed that a higher temperature induces a lower frequency but the thermal effect is not as significant as $\tau$. Although $\omega_n$ decreases with increasing $\bar{u}$, it is noted that first mode frequency $\omega_1$ approaches zero when $\bar{u}\approx3.2$. This value ($\bar{u}_{\text{cri}}\approx3.2$) is called the first critical velocity for $\omega_1$. Afterwards, $\omega_1$ starts to increases beyond this critical speed.

The velocity which makes $\omega_n$ to be a minimum locally within its immediate vicinity, including

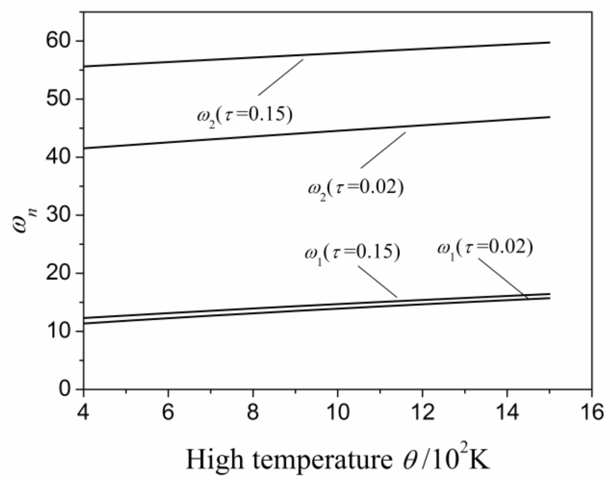


Fig. 5 The effect of high temperature on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha$= $1.1\times10^{-6}K^{-1}$, $\bar{u}=1$ and $\kappa$=$10^4$

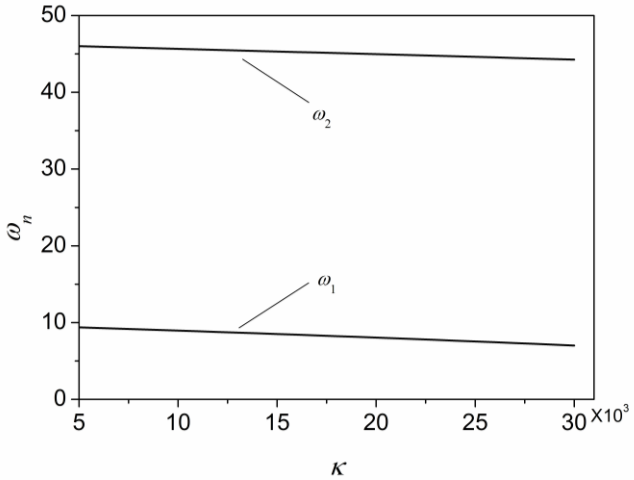


Fig. 6 The effect of shape parameter on the first two mode frequencies for simply supported nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha$= $-1.6\times10^{-6}K^{-1}$, $\bar{u}=1$, $\tau$ = 1 and $\theta$ = 100$K$

the possibility of zero frequency, is called the critical velocity $\bar{u}_{\mathrm{cri}}$. There are infinitely many critical velocities when $\bar{u}$ increases. Therefore, it is expected that $\omega_n$ continues to rise and fall as that for $\omega_1$ as shown in Fig. 3. For instance, the first critical velocity is approximately $\bar{u}_{\mathrm{cri}}\approx3.2$ for $\omega_1$ and $\bar{u}_{\mathrm{cri}}\approx7.7$ for $\omega_2$. The second critical velocity for $\omega_1$ is also observed as approximately 6.5. The strain gradient solutions of the existence of repeated critical velocities is a new physical phenomenon observed only for traveling strain gradient nanobeams because the classical vibration theory for traveling beams (Hatami 2007, Pakdemirli and Ö z 2008) only one critical velocity for each mode of frequency without the occurrence of rise and fall for $\omega_n$ as that observed in Fig. 3.

As stated above, the thermal effect is less significant as compared with the strain gradient effect and it is relation with $\omega_n$ is shown in Figs. 4 and 5, where $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$ (for low temperature) or $\alpha=1.1\times10^{-6}K^{-1}$ (for high temperature) (Yao and Han 2007), $\bar{u}=1$, and $\kappa=10^4$. The natural frequency decreases with increasing the temperature change at low temperature while it increases with increasing temperature change at high temperature. For instance, when $\theta$ increases from 50K to 350K, $\omega_n$ decreases approximately 36% and 6.5% for $n$=1 and $n$=2, respectively, and with $\tau$=0.02. Similarly, when $\theta$ increases from 400K to 1500K, $\omega_n$ increases approximately 38% and 13% with $\tau$=0.02 for $n$=1 and $n$=2, respectively.

Fig. 6 shows the correlation between $\omega_n$ and the shape parameter $\kappa$ defined in Eq. (14). It is seen that shape parameter causes $\omega_n$ to decrease.

### 3.2 Clamped nanobeams

Next we consider a fully clamped nanobeam, and the boundary conditions for such axially traveling nanostructures are expressed in Eq. (A6) in Appendix. Using Eq. (18) and Eq. (A6), a system of homogeneous equations expressed in a matrix form is obtained as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} & e^{k_5} & e^{k_6} \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ k_1e^{k_1} & k_2e^{k_2} & k_3e^{k_3} & k_4e^{k_4} & k_5e^{k_5} & k_6e^{k_6} \\ K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{11}e^{k_1} & K_{12}e^{k_2} & K_{13}e^{k_3} & K_{14}e^{k_4} & K_{15}e^{k_5} & K_{16}e^{k_6} \end{pmatrix} \begin{pmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \\ C_{5n} \\ C_{6n} \end{pmatrix} = 0 \tag{21}$$

where $K_{1i} = k_i^3 + 3\tau^2 k_i^5$ $(i=1,2,...6)$. For nontrivial solutions, the determinant of matrix in Eq. (21) must vanish which results in

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} & e^{k_5} & e^{k_6} \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ k_1e^{k_1} & k_2e^{k_2} & k_3e^{k_3} & k_4e^{k_4} & k_5e^{k_5} & k_6e^{k_6} \\ K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{11}e^{k_1} & K_{12}e^{k_2} & K_{13}e^{k_3} & K_{14}e^{k_4} & K_{15}e^{k_5} & K_{16}e^{k_6} \end{vmatrix} = 0 \tag{22}$$

Following the similar analytical and numerical procedure as described in Sec. 3.1, the behaviors of $\omega_n$ with respect to changes in various parameters are presented in Figs. 7-10. The responses observed earlier for simply supported nanobeams in Sec. 3.1 are valid here except that $\omega_n$ for a clamped nanobeam is higher than the corresponding frequency for simply supported ones. Again, the strain gradient effect is more significant than thermal effect. For example, $\omega_1$ and $\omega_2$ increase approximately 14% and 36% when $\tau$ increases from 0.01 to 0.15, respectively. On the other hand, they decrease approximately 3.6% and 10% when $\theta$ increases from 500K to 10000K for $\kappa$=1000 while for $\kappa$=10 the corresponding values are 0.094% and 0.035%, respectively. It is observed that the higher shape parameter $\kappa$ causes $\omega_n$ to decrease in high temperature field. Particularly, $e_0a$ is the principal small scale parameter in strain gradient theory, and Fig. 10 shows

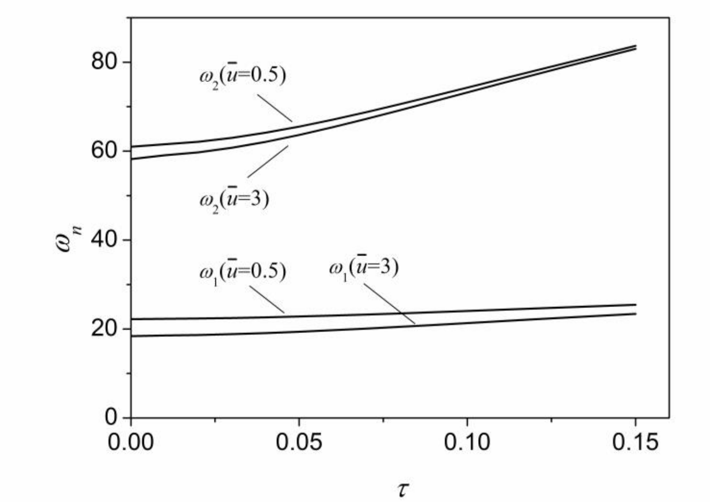


Fig. 7 The effect of strain gradient nanoscale parameter on the first two mode frequencies for clamped nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$, $\theta=50K$ and $\kappa$=10

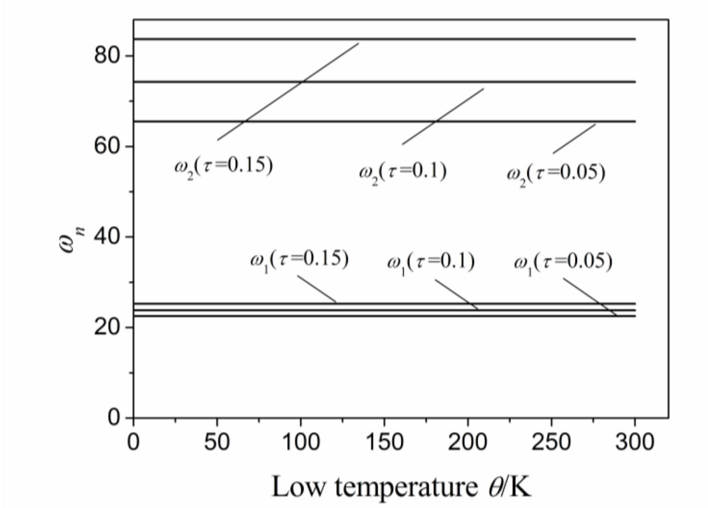


Fig. 8 The effect of low temperature on the first two mode frequencies for clamped nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha=-1.6\times10^{-6}K^{-1}$, $\bar{u}=1$ and $\kappa$=10

the correlation between the internal variable $e_0a$ and natural frequencies where the length of the axially traveling nanobeam is assumed to be 5nm.

### *3.3 Cantilever nanobeams*

The last case is for a cantilever nanobeam and the boundary conditions are shown in Appendix. Combining Eqs. (18) and (A7) in Appendix yields a system of homogeneous equations which can be expressed in a matrix form as

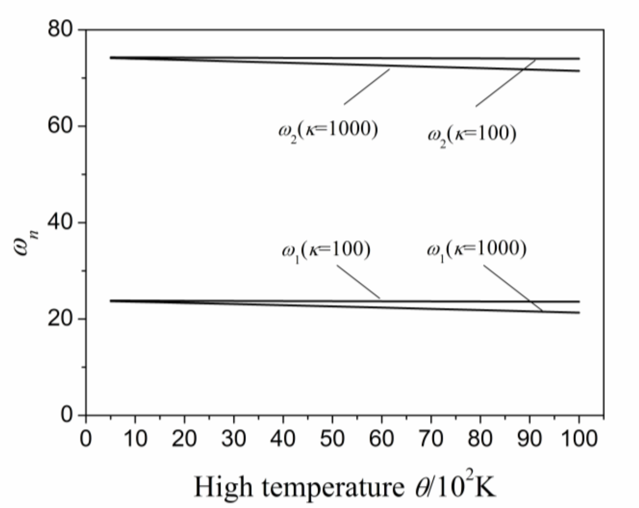


Fig. 9 The effect of high temperature on the first two mode frequencies for clamped nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha$= $1.1\times10^{-6}K^{-1}$, $\bar{u}=1$ and $\tau$=0.1

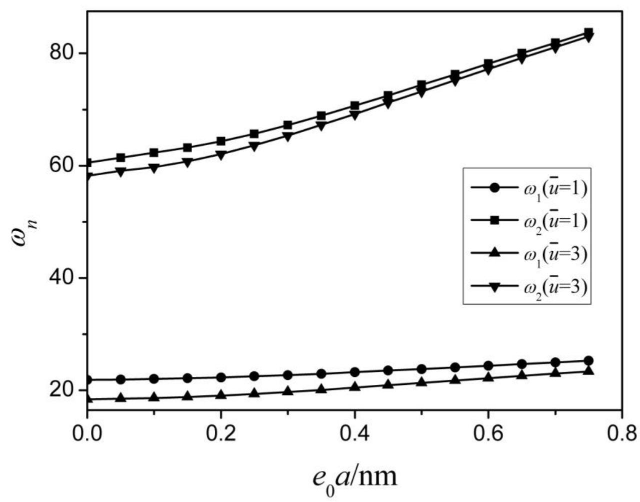


Fig. 10 The effect of internal variable $e_0a$ on the first two mode frequencies for clamped nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha$= $-1.6\times10^{-6}K^{-1}$, $\theta$ = 50$K$, $\kappa$=10 and $L$ = 5nm

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31}e^{k_1} & K_{32}e^{k_2} & K_{33}e^{k_3} & K_{34}e^{k_4} & K_{35}e^{k_5} & K_{36}e^{k_6} \\ k_1^4e^{k_1} & k_2^4e^{k_2} & k_3^4e^{k_3} & k_4^4e^{k_4} & k_5^4e^{k_5} & k_6^4e^{k_6} \\ k_1^2e^{k_1} & k_2^2e^{k_2} & k_3^2e^{k_3} & k_4^2e^{k_4} & k_5^2e^{k_5} & k_6^2e^{k_6} \end{pmatrix} \begin{pmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \\ C_{5n} \\ C_{6n} \end{pmatrix} = 0 \qquad (23)$$

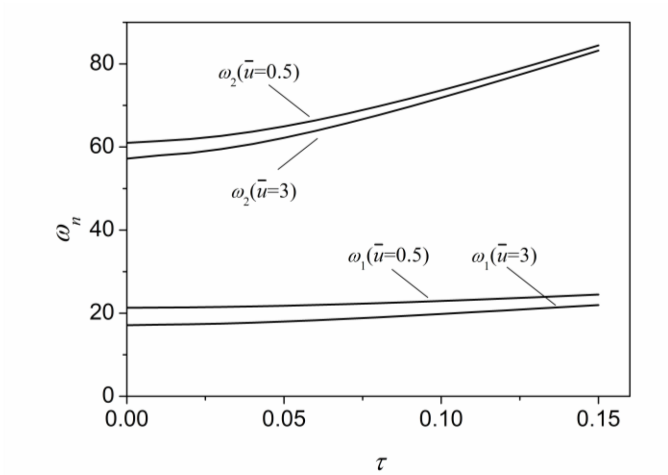


Fig. 11 The effect of strain gradient nanoscale parameter on the first two mode frequencies for cantilever nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha = -1.6\times10^{-6}K^{-1}$, $\theta$ = 50$K$, $\kappa$=10

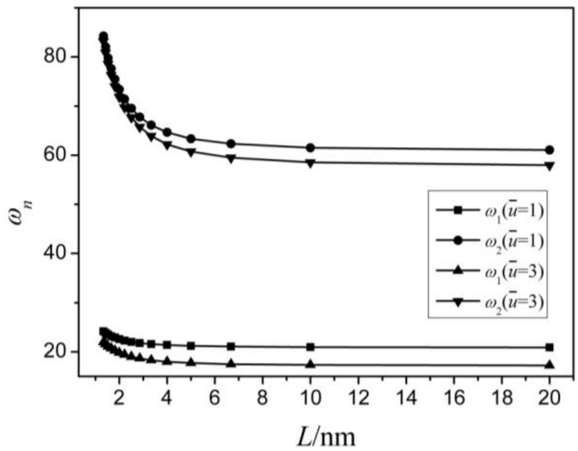


Fig. 12 Size-dependence of the axially traveling cantilever nanobeams $\bar{\rho}=1$, $p_0=1$, $\alpha = -1.6\times10^{-6}K^{-1}$, $\theta$ = 50$K$, $\kappa$=10 and $e_0a$=0.2nm

where $K_{2i}=\tau^2k_i^3+3\tau^4k_i^5$, $K_{3i}=-k_i^3+\tau^2k_i^5-(\alpha\theta\kappa+\bar{u}^2+\bar{u}\sqrt{\bar{\rho}})k_i$ $(i=1,2,\ldots 6)$. For nontrivial solution, the determinant of the matrix must vanish, or

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31}e^{k_1} & K_{32}e^{k_2} & K_{33}e^{k_3} & K_{34}e^{k_4} & K_{35}e^{k_5} & K_{36}e^{k_6} \\ k_1^4e^{k_1} & k_2^4e^{k_2} & k_3^4e^{k_3} & k_4^4e^{k_4} & k_5^4e^{k_5} & k_6^4e^{k_6} \\ k_1^2e^{k_1} & k_2^2e^{k_2} & k_3^2e^{k_3} & k_4^2e^{k_4} & k_5^2e^{k_5} & k_6^2e^{k_6} \end{vmatrix}=0 \tag{24}$$

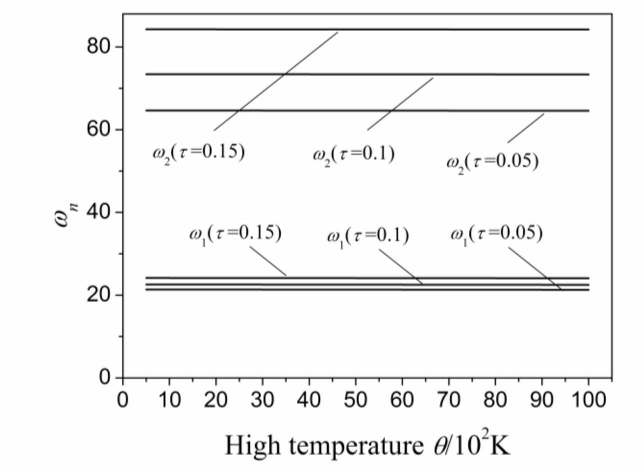


Fig. 13 The effect of high temperature on the first two mode frequencies for cantilever nanobeams with $\bar{\rho}=1$, $p_0=1$, $\alpha$= $1.1\times10^{-6}K^{-1}$, $\bar{u}=1$ and $\kappa$=10

Following the same procedure described in the previous sections, Figs. 11-13 are presented to show the effects of strain gradient nanoscale parameter, the length of nanobeam and temperature change on the dimensionless vibration frequency. The observation and discussion on the responses of $\omega_n$ for variations in various parameters are again valid in this example. Moreover, size dependence is obviously seen in Fig. 12, where the first two mode frequencies decrease with increasing length of the axially traveling nanobeam, especially when the length is very small.

## 5. Conclusions

Strain gradient and thermal effects on the transverse vibration frequency and critical axial velocities are considered in this article for an axially traveling nanobeam. Using a variational principle, an effective strain gradient higher-order equation of motion and the corresponding non-classical, higher-order boundary conditions are first derived. Some practical numerical examples are solved and discussed in detail. It is concluded that many geometric, material and environment parameters affects the frequency response and critical velocity, including temperature change and in particular the strain gradient nanoscale parameter which has significant effect on the nanoscaled vibration behaviors. The strain gradient nanoscale parameter significantly strengthens the stiffness and thus increases the natural frequency of vibration for an axially traveling nanobeam. The thermal effect is weaker and it is concluded that in low temperature field, it causes the frequency to increase while in high temperature field, it makes the frequency decrease. In addition, higher nanobeam shape parameter induces lower frequency. Meanwhile, the natural frequencies drop drastically with increasing, but small, axial velocity. If the axial velocity is rather high, the frequencies rise and fall with increasing axial velocity and there exists the critical velocities at local minima to the immediate vicinity. This observation differs significantly from the conclusion according to the classical vibration theory. It is obvious that effective strain gradient plays an essential role in size dependence of a traveling nanobeam vibration and the thermal effect should not be ignored neither.

## Acknowledgments

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## References

Aksencer, T. and Aydogdu, M. (2012), "Forced transverse vibration of nanoplates using nonlocal elasticity", *Phys. E*, **44**, 1752-1759.

Avsec, J. and Oblak, M. (2007), "Thermal vibrational analysis for simply supported beam and clamped beam", *J. Sound Vib.*, **308**, 514-525.

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**, 1-16.

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**, 4703-4710.

Eringen, A.C., Speziale, C.G. and Kim, B.S. (1977), "Crack tip problem in non-local elasticity", *J. Mech. Phys. Solids*, **25**, 339-355.

Hatami, S., Azhari, M. and Saadatpour, M.M. (2007), "Free vibration of moving laminated composite plates", *Compos. Struct.*, **80**, 609-620.

Lee, H.L. and Chang, W.J. (2009), "Vibration analysis of a viscous-fluid-conveying single-walled carbon nanotube embedded in an elastic medium", *Phys. E*, **41**, 529-532.

Li, C., Lim, C.W. and Yu, J.L. (2011a), "Dynamics and stability of transverse vibrations of nonlocal nanobeams with a variable axial load", *Smart Mater. Struct.*, **20**, 015023.

Li, C., Lim, C.W., Yu, J.L. and Zeng, Q.C. (2011b), "Analytical solutions for vibration of simply supported nonlocal nanobeams with an axial force", *Int. J. Struct. Stab. Dyn.*, **11**, 257-271.

Li, C., Lim, C.W., Yu, J.L. and Zeng, Q.C. (2011c), "Transverse vibration of pre-tensioned nonlocal nanobeams with precise internal axial loads", *Sci. Chin. Tech. Sci.*, **54**, 2007-2013.

Li, C., Zheng, Z.J., Yu, J.L. and Lim, C.W. (2011d), "Static analysis of ultra-thin beams based on a semi-continuum model", *Acta Mech. Sin.*, **27**, 713-719.

Lim, C.W. (2010), "On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection", *Appl. Math. Mech.*, **31**, 37-54.

Lim, C.W., Li, C. and Yu, J.L. (2009), "The effects of stiffness strengthening nonlocal stress and axial tension on free vibration of cantilever nanobeams", *Interaction Multiscale Mech.*, **2**, 223-233.

Lim, C.W., Li, C. and Yu, J.L. (2012), "Free torsional vibration of nanotubes based on nonlocal stress theory", *J. Sound Vib.*, **331**, 2798-2808.

Lim, C.W., Niu, J.C. and Yu, Y.M. (2010), "Nonlocal stress theory for buckling instability of nanotubes: new predictions on stiffness strengthening effects of nanoscales", *J. Comput. Theore. Nanosci.*, **7**, 2104-2111.

Lim, C.W. and Yang, Q. (2011), "Nonlocal thermal-elasticity for nanobeam deformation: exact solutions with stiffness enhancement effects", *J. Appl. Phys.*, **110**, 013514.

Lu, P., Lee, H.P., Lu, C. and Zhang, P.Q. (2006), "Dynamic properties of flexural beams using a nonlocal elasticity model", *J. Appl. Phys.*, **99**, 073510.

Mindlin, R.D. (1965), "Second gradient of strain and surface tension in linear elasticity", *Int. J. Solids Struct.*, **1**, 417-438.

Ö z, H.R. and Pakdemirli, M. (1999), "Vibrations of an axially moving beam with time-dependent velocity", *J. Sound Vib.*, **227**, 239-257.

Pakdemirli, M. and Ö z, H.R. (2008), "Infinite mode analysis and truncation to resonant modes of axially accelerating beam vibrations", *J. Sound Vib.*, **311**, 1052-1074.

Peddieson, J., Buchanan, G.G. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci.*, **41**, 305-312.

Pradhan, S.C. and Kumar, A. (2011), "Vibration analysis of orthotropic graphene sheets using nonlocal

elasticity theory and differential quadrature method", *Compos. Struct.*, **93**, 774-779.
Tang, Y.Q., Chen, L.Q. and Yang, X.D. (2009), "Nonlinear vibrations of axially moving Timoshenko beams under weak and strong external excitations", *J. Sound Vib.*, **320**, 1078-1099.
Wang, L. (2011), "A modified nonlocal beam model for vibration and stability of nanotubes conveying fluid", *Phys. E*, **44**, 25-28.
Wang, Q., Zhou, G.Y. and Lin, K.C. (2006), "Scale effect on wave propagation of double-walled carbon nanotubes", *Int. J. Solids Struct.*, **43**, 6071-6084.
Yao, X.H. and Han, Q. (2007), "Investigation of axially compressed buckling of a multi-walled carbon nanotube under temperature field", *Compos. Sci. Tech.*, **67**, 125-134.
Yu, Y.M. and Lim, C.W. (2013), "Nonlinear constitutive model for axisymmetric bending of annular graphene-like nanoplate with gradient elasticity enhancement effects", *J. Eng. Mech.-ASCE*, **139**, 1025-1035.
Zhang, Y.Q., Liu, G.R. and Wang, J.S. (2004), "Small-scale effects on buckling of multiwalled carbon nanotubes under axial compression", *Phys. Rev. B*, **70**, 205430.

## Appendix

The non-classical boundary conditions derived from Eq. (12) are obtained as

$$\left.\begin{aligned}
\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}+\left(\alpha\theta\kappa-\overline{u}^{2}-\overline{u}\sqrt{\overline{\rho}}\right)\frac{\partial\overline{w}}{\partial\overline{x}}=0 \quad &\text{or} \quad \overline{w}=0\\
\sum_{n=1}^{\infty}(2n-3)\tau^{2(n-1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n-1\rangle}=0 \quad &\text{or} \quad \frac{\partial\overline{w}}{\partial\overline{x}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}=0 \quad &\text{or} \quad \frac{\partial^{2}\overline{w}}{\partial\overline{x}^{2}}=0\\
\sum_{n=1}^{\infty}2n\tau^{2(n+1)}\left(\frac{\partial^{2}\overline{w}}{\partial\overline{x}^{2}}\right)^{\langle 2n\rangle}=0 \quad &\text{or} \quad \frac{\partial^{3}\overline{w}}{\partial\overline{x}^{3}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+1)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}=0 \quad &\text{or} \quad \frac{\partial^{4}\overline{w}}{\partial\overline{x}^{4}}=0\\
\sum_{n=1}^{\infty}2n\tau^{2(n+2)}\left(\frac{\partial^{2}\overline{w}}{\partial\overline{x}^{2}}\right)^{\langle 2n\rangle}=0 \quad &\text{or} \quad \frac{\partial^{5}\overline{w}}{\partial\overline{x}^{5}}=0\\
\sum_{n=1}^{\infty}(2n-1)\tau^{2(n+2)}\left(\frac{\partial\overline{w}}{\partial\overline{x}}\right)^{\langle 2n\rangle}=0 \quad &\text{or} \quad \frac{\partial^{6}\overline{w}}{\partial\overline{x}^{6}}=0\\
\vdots \quad &\text{or} \quad \vdots
\end{aligned}\right\}_{\overline{x}=0,1} \tag{A1}$$

which can be grouped into the natural boundary conditions on the left and the geometric boundary conditions on the right. In each line, either the natural condition or the geometric condition vanishes but not both at the same time (Lim 2010). Subsequently, considering the most important strain gradient terms which reveal the significant strain gradient effects, we simplify the boundary conditions in Eq. (A1) as follows

$$\left.\begin{aligned}
-\frac{\partial^{3}\overline{w}}{\partial\overline{x}^{3}}+\tau^{2}\frac{\partial^{5}\overline{w}}{\partial\overline{x}^{5}}+\left(\alpha\theta\kappa-\overline{u}^{2}-\overline{u}\sqrt{\overline{\rho}}\right)\frac{\partial\overline{w}}{\partial\overline{x}}=0 \quad &\text{or} \quad \overline{w}=0\\
-\frac{\partial^{2}\overline{w}}{\partial\overline{x}^{2}}+\tau^{2}\frac{\partial^{4}\overline{w}}{\partial\overline{x}^{4}}=0 \quad &\text{or} \quad \frac{\partial\overline{w}}{\partial\overline{x}}=0\\
\tau^{2}\frac{\partial^{3}\overline{w}}{\partial\overline{x}^{3}}+3\tau^{4}\frac{\partial^{5}\overline{w}}{\partial\overline{x}^{5}}=0 \quad &\text{or} \quad \frac{\partial^{2}\overline{w}}{\partial\overline{x}^{2}}=0
\end{aligned}\right\}_{\overline{x}=0,1} \tag{A2}$$

Eq. (A2) is the corresponding boundary conditions with respect to the equation of motion (14). Substituting Eq. (16) into (A2) yields the following boundary conditions in amplitude shape field as

$$\left.\begin{array}{r} -\dfrac{d^3\varphi}{d\bar{x}^3}+\tau^2\dfrac{d^5\varphi}{d\bar{x}^5}+\left(\alpha\theta\kappa-\bar{u}^2-\bar{u}\sqrt{\bar{\rho}}\right)\dfrac{d\varphi}{d\bar{x}}=0 \quad \text{or} \qquad \varphi=0 \\ -\dfrac{d^2\varphi}{d\bar{x}^2}+\tau^2\dfrac{d^4\varphi}{d\bar{x}^4}=0 \quad \text{or} \qquad \dfrac{d\varphi}{d\bar{x}}=0 \\ \tau^2\dfrac{d^3\varphi}{d\bar{x}^3}+3\tau^4\dfrac{d^5\varphi}{d\bar{x}^5}=0 \quad \text{or} \qquad \dfrac{d^2\varphi}{d\bar{x}^2}=0 \end{array}\right\}_{\bar{x}=0,1} \tag{A3}$$

For a simply supported nanobeam, the boundary conditions consist of the classical and non-classical conditions. As the classical boundary conditions, the transverse deflection (or vibration amplitude shape) on the right of the first line in Eq. (A3), and its second-order derivative on the right of the third line at both ends should be zero. As the non-classical boundary conditions, the combined expression of the second-order derivative and the fourth-order derivative with a strain gradient nanoscale parameter on the left of the second line should vanish. The reason for the selection logic of the non-classical boundary conditions is the effective strain gradient nonlocal bending moment does not exist at both ends in the strain gradient theory. Please note that the dimensionless effective strain gradient nonlocal bending moment was defined as (Lim 2010)

$$\bar{M}_{\text{ef}}=-\frac{\partial^2\bar{w}}{\partial\bar{x}^2}+\tau^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4}+3\tau^4\frac{\partial^6\bar{w}}{\partial\bar{x}^6}+5\tau^6\frac{\partial^8\bar{w}}{\partial\bar{x}^8}+\cdots=\sum_{n=1}^{\infty}(2n-3)\tau^{2n-2}\frac{\partial^{2n}\bar{w}}{\partial\bar{x}^{2n}} \tag{A4}$$

where the boundary condition on the left of the second line in Eqs. (A2) or (A3) is just the simplified form of the effective strain gradient nonlocal bending moment. Therefore, the simply supported boundary conditions chosen from Eq. (A3) are given by

$$\begin{array}{lll} \varphi(0)=0 \;\; ; & \dfrac{d^2\varphi(0)}{d\bar{x}^2}=0 \;\; ; & -\dfrac{d^2\varphi(0)}{d\bar{x}^2}+\tau^2\dfrac{d^4\varphi(0)}{d\bar{x}^4}=0 \\ \varphi(1)=0 \;\; ; & \dfrac{d^2\varphi(1)}{d\bar{x}^2}=0 \;\; ; & -\dfrac{d^2\varphi(1)}{d\bar{x}^2}+\tau^2\dfrac{d^4\varphi(1)}{d\bar{x}^4}=0 \end{array} \tag{A5}$$

Eq. (A5) can also be expressed, after simplification as

$$\begin{array}{lll} \varphi(0)=0 \;\; ; & \dfrac{d^2\varphi(0)}{d\bar{x}^2}=0 \;\; ; & \dfrac{d^4\varphi(0)}{d\bar{x}^4}=0 \\ \varphi(1)=0 \;\; ; & \dfrac{d^2\varphi(1)}{d\bar{x}^2}=0 \;\; ; & \dfrac{d^4\varphi(1)}{d\bar{x}^4}=0 \end{array} \tag{A6}$$

Similarly, the boundary conditions of an axially traveling nanobeam with fully clamped ends from Eq. (A3) are

$$\begin{array}{lll} \varphi(0)=0 \;\; ; & \dfrac{d\varphi(0)}{d\bar{x}}=0 \;\; ; & \dfrac{d^3\varphi(0)}{d\bar{x}^3}+3\tau^2\dfrac{d^5\varphi(0)}{d\bar{x}^5}=0 \\ \varphi(1)=0 \;\; ; & \dfrac{d\varphi(1)}{d\bar{x}}=0 \;\; ; & \dfrac{d^3\varphi(1)}{d\bar{x}^3}+3\tau^2\dfrac{d^5\varphi(1)}{d\bar{x}^5}=0 \end{array} \tag{A7}$$

which consist of the classical boundary conditions that the deflection and its first-order derivative (rotating angle) at both ends are all zero, and the non-classical boundary conditions that the combination of the third-order derivative and the fifth-order derivative with a strain gradient nanoscale parameter on the left of the third line should vanish. In fact, it can be proved that such non-classical boundary conditions are part of the simplified form of the effective strain gradient rotating angle. Similarly, for axially traveling cantilever nanobeams, the boundary conditions are given by

$$\begin{aligned}&\varphi(0)=0 \quad ; \quad \tau^2\frac{d^3\varphi(0)}{d\overline{x}^3}+3\tau^4\frac{d^5\varphi(0)}{d\overline{x}^5}=0 \quad ; \quad -\frac{d^2\varphi(1)}{d\overline{x}^2}+\tau^2\frac{d^4\varphi(1)}{d\overline{x}^4}=0;\\ &\frac{d\varphi(0)}{d\overline{x}}=0 \quad ; \quad \frac{d^2\varphi(1)}{d\overline{x}^2}=0 \quad ; \quad -\frac{d^3\varphi(1)}{d\overline{x}^3}+\tau^2\frac{d^5\varphi(1)}{d\overline{x}^5}-\left(\alpha\theta\kappa+\overline{u}^2+\overline{u}\sqrt{\overline{\rho}}\right)\frac{d\varphi(1)}{d\overline{x}}=0\end{aligned} \tag{A8}$$

where the nanobeam fixed at $\overline{x}=0$ and its free end at $\overline{x}=1$.