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The receding contact problem of two elastic layers supported by two elastic quarter planes

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Abstract. The receding contact problem for two elastic layers whose elastic constants and heights are different supported by two elastic quarter planes is considered. The lower layer is supported by two elastic quarter planes and the upper elastic layer is subjected to symmetrical distributed load whose heights are 2a on its top surface. It is assumed that the contact between all surfaces is frictionless and the effect of gravity force is neglected. The problem is formulated and solved by using Theory of Elasticity and Integral Transform Technique. The problem is reduced to a system of singular integral equations in which contact pressures are the unknown functions by using integral transform technique and boundary conditions of the problem. Stresses and displacements are expressed depending on the contact pressures using Fourier and Mellin formula technique. The singular integral equation is solved numerically by using Gauss-Jacobi integration formulation. Numerical results are obtained for various dimensionless quantities for the contact pressures and the contact areas are presented in graphics and tables.

Keywords: contact mechanics; theory of elasticity; quarter plane; integral transform technique; elastic layer

1. Introduction

In engineering mechanics, the contact problems have different applications to a variety of structures of practical interest. For example; foundations, roads, railways, airfield pavements, rolling mills, ball and roller bearings are some application areas of the contact problem. Although developments in the contact problems did not appear in the literature until the beginning of this century, the studies have been accelerated recently due to the improvements in computer technology.

The frictionless contact problem of the layer which locates into elastic half-plane, quarter plane, rigid plane or another plane has been studied by many researchers so far. The first study about contact problems was made by Hertz therefore contact problems are known as “Hertz Contact Problem” in the literature Hertz (1895). The solution methods of contact problems by elasticity were given by Galin (1961) and the application methods of integral transformations in

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solutions were given by Uffliand (1961). Layer was loaded by rigid or elastic punch and located into the half-plane is solved by Ratwani (1973), Erdogan (1974). Contact problem, when the half-plane and the layer is anisotropic, was discussed by Kahya *et al*. (2007). Keer *et al*. (1972) solved while the elastic layer was loaded by distributed load. El-Borgi *et al*. (2006) considered a receding contact plane problem between a functionally graded layer and a homogeneous substrate. Comez *et al*. (2004) studied contact problem of the layer which suppressed by curvilinear punch and rest on elastic layer which is tied totally from the bottom. Contact problems about the quarter-plane and the development of the numerical methods about this problem were discussed by Gerber (1968). Solution methods of the key type problems and quarter-plane and stresses which come out in contact distances were discussed by Dundurs and Lee (1972), Bakioglu (1976). By discussing the contact problem of the distributed and singular loaded layer which sits into quarter-planes, stress and strain in the form of different loading and material conditions were studied by Aksogan *et al*. (1996), (1997), (1999), Akavci (1999). Examination of quarter-planes of boundary value problems were discussed by Aghili (1999). Ke and Wang (2005) investigated two-dimensional contact mechanics of functionally graded materials with arbitrary spatial variations of material properties. Fracture dynamic problems for elastic cracked solids with allowance for crack faces contact interaction is solved by Oleksandr *et al*. (2007). Two-dimensional frictionless contact problem of a coating structure consisting of a surface coating, a functionally graded layer and a substrate under a rigid cylindrical punch is investigated by Yang and Ke (2008). Contact mechanics of thin films bonded to graded coatings is investigated both analytically and numerically by Guler *et al*. (2012). Carbone and Mangialardi (2008) developed a numerical procedure to analyze the adhesive contact between a soft elastic layer and a rough rigid substrate. Contact mechanics for randomly rough surfaces is solved by Persson (2006). The contact pressure distribution beneath a square block sliding along an elastically similar half-plane, in the presence of frictions are found by Karuppanan *et al*. (2008). Plane contact problem of an infinite long cylinder pressured into an elastic laminated semi-space is considered by Perkowski *et al*. (2007). Mahmoud *et al*. are studied incremental finite element model for the solution of the two dimensional quasi-static frictionless nonlinear viscoelastic contact problems with large deformations. Long and Wang (2012) investigated effects of surface tension on axisymmetric Hertzian contact problem.

In this study, the receding contact problem for two elastic layers whose elastic constants and heights are different has been considered. The layers and quarter planes are homogeneous and isotropic. The lower layer is supported by two elastic quarter planes and the upper elastic layer is subjected to symmetrical distributed load whose lengths are $2a$ on its top surface. It is assumed that the contact between all surfaces is frictionless and the effect of gravity force is neglected. The problem is formulated and solved by using Theory of Elasticity and Integral Transform Technique. The problem is reduced to a system of singular integral equations in which contact pressure are the unknown functions by using integral transform technique and boundary conditions of the problem. Stresses and displacements are expressed depending on the contact pressure using Fourier and Mellin formula technique. The singular integral equation is solved numerically by using Gauss-Jacobi integration. Finally, numerical results are analyzed and conclusions are plotted.

2. Formula of the problem

Consider the symmetric contact problem for the quarter planes and two elastic layers with

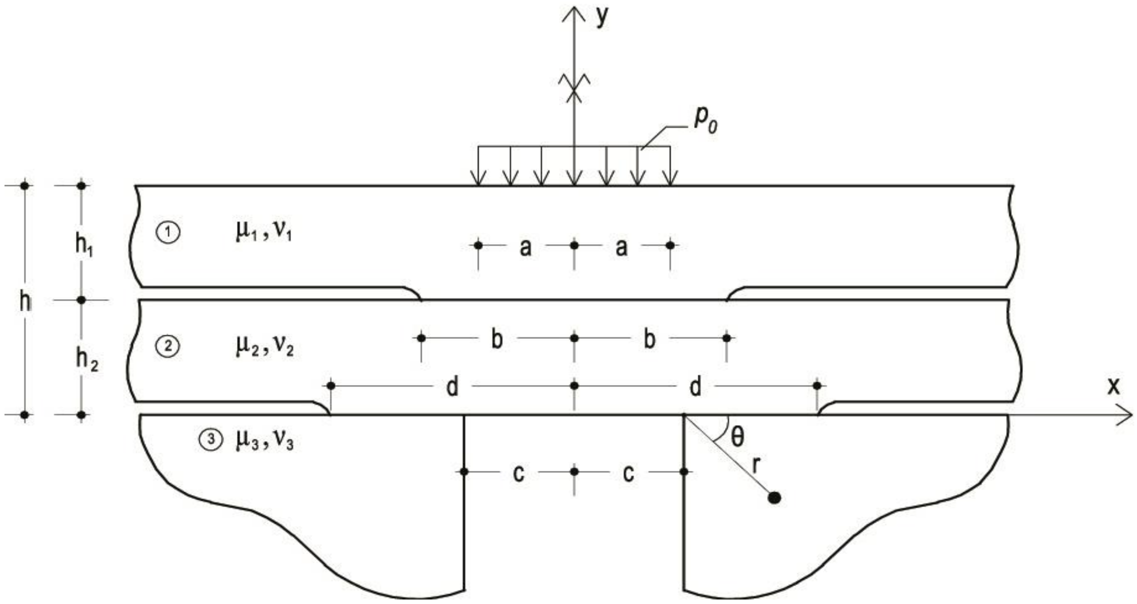


Fig. 1 The receding contact problem of two elastic layers which sets on two elastic quarter planes In the absence of body forces, two-dimensional Navier equations can be written as given in (1a) and (1b)

different elastic constants and heights shown Fig. 1. While upper layer and lower layer are in contact over the interval $(-b, b)$, the lower layer and quarter planes are in contact over the interval (c, d). The thickness of the upper layer and lower layer are h_1 and h_2, respectively. μ_i and ν_i $(i = 1,2,3)$ are elastic constants of the layers and quarter planes. The subscript $i(i = 1,2,3)$ refers to the layers and quarter planes respectively. Thickness in z-direction is taken to be unit.

$$\mu \nabla^2 u + (\lambda + \mu) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{1a}$$

$$\mu \nabla^2 v + (\lambda + \mu) \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{1b}$$

In these expressions, λ and μ are the Lame constant and the shear modulus, u and v are the displacement components in x and y-directions, respectively. The problem is symmetrical according to the y- axis and the following conditions must also be satisfied.

$$u_i(x, y) = -u_i(-x, y) \tag{2a}$$

$$v_i(x, y) = v_i(-x, y) \tag{2b}$$

Due to the symmetry, it is enough to consider the problem in the region of $0 \le x < \infty$. Displacement of each layer may be expressed as the Fourier sine and Fourier cosine transform of the unknown functions $\phi_i(x, y)$ and $\psi_i(x, y)$ as

$$u_i\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \phi_i(x,y)\sin\left(\alpha x\right)d\alpha,\quad (i=1,2) \tag{3a}$$

$$v_i\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \psi_i(x,y)\cos\left(\alpha x\right)d\alpha,\quad (i=1,2) \tag{3b}$$

Substituting (3a) and (3b) into (1a) and (1b) and solving the resulting ordinary differential equation system, one may obtain the unknown functions $\phi_i(x,y)$ and $\psi_i(x,y)$. Using these solutions into (3a) and (3b), the displacements u_i and v_i for each layer can be determined as

$$u_i\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \left\{\left[A_i+B_iy\right]e^{-\alpha y}+\left[C_i+D_iy\right]e^{\alpha y}\right\}\sin\left(\alpha x\right)d\alpha \tag{4a}$$

$$v_i\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \left\{\left[A_i+B_i\left(\tfrac{\kappa_i}{\alpha}+y\right)\right]e^{-\alpha y}+\left[-C_i+D_i\left(\tfrac{\kappa_i}{\alpha}-y\right)\right]e^{\alpha y}\right\}\cos\left(\alpha x\right)d\alpha \tag{4b}$$

Where $\kappa_i=(3-4\nu_i)$ for plane strain and ν_i is the Poisson's ratio. A_i, B_i, C_i and D_i $(i=1,2)$ are the unknown functions for the layers which will be determined from boundary conditions of the problem. The subscripts 1 and 2 to the upper and lower elastic layers, respectively. Using Hooke's law and Eqs. (4a) and (4b), the stress components may be expressed as follows

$$\frac{1}{2\mu_i}\sigma_{x_i}\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \left\{\left[\alpha\left(A_i+B_iy\right)-\left(\frac{3-\kappa_i}{2}\right)B_i\right]e^{-\alpha y}+\left[\alpha\left(C_i+D_iy\right)+\left(\frac{3-\kappa_i}{2}\right)D_i\right]e^{\alpha y}\right\}\cos\left(\alpha x\right)d\alpha \tag{5a}$$

$$\frac{1}{2\mu_i}\sigma_{y_i}\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha\left(A_i+B_iy\right)+\left(\frac{1+\kappa_i}{2}\right)B_i\right]e^{-\alpha y}+\left[-\alpha\left(C_i+D_iy\right)+\left(\frac{1+\kappa_i}{2}\right)D_i\right]e^{\alpha y}\right\}\cos\left(\alpha x\right)d\alpha \tag{5b}$$

$$\frac{1}{2\mu_i}\tau_{xy_i}\left(x,y\right)=\frac{2}{\pi}\int_0^\infty \left\{-\left[\alpha\left(A_i+B_iy\right)+\left(\frac{\kappa_i-1}{2}\right)B_i\right]e^{-\alpha y}+\left[\alpha\left(C_i+D_iy\right)-\left(\frac{\kappa_i-1}{2}\right)D_i\right]e^{\alpha y}\right\}\sin\left(\alpha x\right)d\alpha \tag{5c}$$

The Airy stress function method is applied to the case in polar coordinates; the stresses are expressed as follows

$$\sigma_r=\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r} \tag{6a}$$

$$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2} \tag{6b}$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{6c}$$

Taking the Mellin transform of the latter equation, the solution in the transform domain is found to be.

$$2\mu\left(\frac{\partial u_\theta}{\partial r}r^2\right)^M = s\left(s+1\right)i\left[F_1e^{is\theta}-F_2e^{-is\theta}\right]+\left[(s+1)(s+2)+(1-\nu)(-4s-4)\right]\left[G_1e^{i(s+2)\theta}-G_2e^{-i(s+2)\theta}\right] \tag{7a}$$

$$\left(r^2\sigma_r\right)^M = \left(\frac{\partial^2}{\partial\theta^2}-s\right)\left[\phi^M\right] \tag{7b}$$

$$\left(r^2\tau_{r\theta}\right)^M = (s+1)\frac{\partial}{\partial\theta}\left[\phi^M\right] \tag{7d}$$

$$\phi^M(s,\theta) = F_1e^{is\theta}+F_2e^{-is\theta}+G_1e^{i(s+2)\theta}+G_2e^{-i(s+2)\theta} \tag{7e}$$

Where u_θ is the displacement, the stress compounds for the quarter planes are $\sigma_r(r,\theta), \sigma_\theta(r,\theta), \tau_{r\theta}(r,\theta)$ and ϕ^M Airy stress function which will be used for the quarter planes in the solution of the problem is carried out by using Mellin transform technique.

3. The boundary conditions and the system of integral equations

The boundary conditions of the receding contact problem for the elastic layers can be written as

$$\tau_{xy}^{(1)}(x,h)=0 \qquad (0\le x<\infty) \tag{8a}$$

$$\sigma_y^{(1)}(x,h)=\begin{cases}-p_0 & (0\le x<a)\\ 0 \quad ; & (a\le x<\infty)\end{cases} \tag{8b}$$

$$\tau_{xy}^{(1)}(x,h_2)=0 \qquad (0\le x<\infty) \tag{8c}$$

$$\tau_{xy}^{(2)}(x,h_2)=0 \qquad (0\le x<\infty) \tag{8d}$$

$$\sigma_y^{(1)}(x,h_2)=\begin{cases}-p_1(x) & (0\le x<b)\\ 0 \quad ; & (b\le x<\infty)\end{cases} \tag{8e}$$

$$\sigma_y^{(1)}(x,h_2)=\sigma_y^{(2)}(x,h_2) \qquad (0\le x<\infty) \tag{8f}$$

$$\sigma_y^{(2)}(x,0)=\begin{cases}-p_2(x) & (c<x<d)\\ 0 \quad ; & \left(0\le x<c\ ,\ \ d\le x<\infty\right)\end{cases} \tag{8g}$$

$$\tau_{xy}^{(2)}(x,0)=0 \qquad (0\le x<\infty) \tag{8h}$$

$$\frac{\partial v_2(x,0)}{\partial x}=\frac{\partial u_\theta}{\partial r} \qquad (c<x<d) \tag{9}$$

$$\frac{\partial}{\partial x}\left[\mathrm{v}_1(x,h_2)-v_2(x,h_2)\right]=0 \qquad (0\leq x<b) \tag{10}$$

For the quarter planes, the boundary conditions in polar coordinates are

$$\sigma_\theta(r,\theta)=-p_2(r) \qquad (c<r<d) \qquad (\theta=0) \tag{11a}$$

$$\tau_{r\theta}(r,\theta)=0 \qquad (d<r<\infty) \qquad (\theta=0) \tag{11b}$$

$$\sigma_\theta(r,\theta)=0 \qquad (0<r<\infty) \qquad (\theta=\pi/2) \tag{11c}$$

$$\tau_{r\theta}(r,\theta)=0 \qquad (0<r<\infty) \qquad (\theta=\pi/2) \tag{11d}$$

Additional conditions are needed for a complete solution of the problem. The contact pressures must satisfy the equilibrium conditions, which can be expressed as

$$\int_{-b}^{b} p_1(x_1)dx_1=2ap_0 \tag{12a}$$

$$\int_{c}^{d} p_2(x_2)dx_2=ap_0 \tag{12b}$$

$2a$ is length of distributed load. b is the half-width of the contact area between two elastic layers. $(c-d)$ is the half-width of the contact area between lower layer and the quarter planes. p_0 is a known distributed load, $p_1(x)$ and $p_2(x)$ are the unknown contact pressures on the contact areas (b) and $(c-d)$, respectively. In which $p_2(r)$ is the stress between the upper layer and the quarter plane.

Using the boundary conditions given by (8a-8g), the unknown constants A_i, B_i, C_i and D_i $(i=1,2)$, appearing in the stress and the displacement expressions for the layers, can be obtained in terms of the unknown contact pressures $p_1(x)$ and $p_2(x)$. Thus, the stresses and displacements can be expressed depending on the unknown contact pressures $p_1(x)$ and $p_2(x)$ which have not yet determined. Substituting Eqs. (7a)-(7e) into boundary conditions for the quarter planes (11a-11d), is obtained boundary condition given by (9) by using Mellin transform technique.

The solution (2) and (5) with A_i, B_i, C_i and D_i $(i=1,2)$ satisfies all the boundary conditions stated by Eqs. (8a)-(8g) expect the mixed conditions (9) and (10). The unknown functions $p_1(x)$ and $p_2(x)$ are determined from the mixed conditions which have not yet been satisfied. These conditions give the following system of integral equations, after some routine manipulations and using the symmetry consideration

$p_1(x)=p_1(-x)$ and $p_2(x)=p_2(-x)$:

$$\frac{1}{\pi}\int_{-b}^{b}\left[\frac{1}{t-x}-M_{11}(x,t)-R_{11}(x,t)\right]p_1(t_1)dt_1+\frac{1}{\pi}\int_{c}^{d}R_{12}(x,t)p_2(t_2)dt_2=\frac{1}{\pi}p_0M(x) \tag{13a}$$

$$\frac{1}{\pi}\int_{-b}^{b} M_{21}(x,t)p_1(t_1)dt_1 + \frac{1}{\pi}\int_{c}^{d}\left[\frac{1}{t+x} - \frac{1}{t-x} + M_{22}(x,t) - \frac{\mu_2}{\mu_3}\frac{1+\kappa_3}{1+\kappa_2}k_{22}(x,t)\right]p_2(t_2)dt_2 = 0 \quad (13b)$$

Where $M_{11}(x_1,t_1), R_{11}(x_1,t_1), R_{12}(x_1,t_2), M_{21}(x_2,t_1), M_{22}(x_2,t_2), M(x)$ and $k_{22}(x_2,t_2)$ are given by (A1-A7) in Appendix.

4. The solution of the system of the integral equations

The numerical solutions of the integral equations will be achieved by Gauss-Jacobi Integration Formulation which was investigated by Krenk (1975), Erdogan *et al*. (1973).

To simplify the numerical analysis of the integral equation, the following dimensionless quantities can be introduced.

$$\alpha = \frac{z}{h_2} \quad (14a)$$

$$t_1 = br_1 \quad (14b)$$

$$t_2 = \frac{d-c}{2}r_2 + \frac{d+c}{2} \quad (14c)$$

$$x_1 = bs_1 \quad (14d)$$

$$x_2 = \frac{d-c}{2}s_2 + \frac{d+c}{2} \quad (14e)$$

$$g(r_1) = \frac{p_1(br_1)}{p_0} \quad (14f)$$

$$g(r_2) = \frac{p_2\left(\frac{d-c}{2}r_2 + \frac{d+c}{2}\right)}{p_0} \quad (14g)$$

Using these dimensionless quantities, the integral Eqs. (13a) and (13b) can be written as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{r_1-s_1} - \bar{M}_{11}(s,r) - \bar{R}_{11}(s,r)\right]g_1(r_1)dr_1 + \frac{1}{\pi}\int_{-1}^{1}\bar{R}_{12}(s,r)g_2(r_2)dr_2 = \frac{1}{\pi}\bar{M}(s_1) \quad (15a)$$

$$\frac{1}{\pi}\int_{-1}^{1}\bar{M}_{21}(s,r)g(r_1)dr_1 + \frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{(r_2+s_2)+2(\frac{d+c}{d-c})} - \frac{1}{(r_2-s_2)} + \bar{M}_{22}(s,r) - \frac{\mu_2}{\mu_3}\frac{1+\kappa_3}{1+\kappa_2}k_{22}(s,r)\right]g(r_2)dr_2 = 0 \quad (15b)$$

Similarly, the equilibrium conditions become

$$\frac{b}{2a}\int_{-1}^{1} g_1(r_1)dr_1 = 1 \tag{16a}$$

$$\frac{d-c}{2a}\int_{-1}^{1} g_2(r_2)dr_2 = 1 \tag{16b}$$

The solution of the integral equations can be expressed as

$$g_j(r_j) = G_j(r_j)w_j(r_j) \tag{17}$$

One may notice that because of the smooth contact at the end point b, the unknown function $p_1(x)$ is zero at the ends, thereby the index of integral Eq. (15a) is -1.

Where $w_j(r_j)$ is the weight function of $g_j(r_j)$. Using the Gauss-Jacobi integration formulas, the integral Eq. (15a) and equilibrium conditions (16a) become

$$\sum_{i=1}^{N} W_{1i}^{N}\left[\frac{1}{r_{1i}-s_{1k}} + k_{11}(s_{1k},r_{1i})\right]G_1(r_{1i}) + \sum_{i=1}^{N} W_{2i}^{N}\bar{R}_{12}(s_{1k},r_{2i})G_2(r_{2i}) = \frac{1}{\pi}\bar{M} \qquad (k=1,2,...,N+1) \tag{18a}$$

$$\frac{b}{2a}\sum_{i=1}^{N} W_{1i}^{N}G_1(r_{1i}) - \frac{1}{\pi} = 0 \tag{18b}$$

where

$$\begin{aligned} &w_j(r_j) = (1-r_j)^{\alpha_j}(1+r_j)^{\beta_j}, \qquad j=1,2 \\ &\alpha_1 = \beta_1 = 0.5 \end{aligned} \tag{19}$$

$r_{1i}$and s_{1k} are the roots of the related Jacobi polynomials and W_{1i}^{N} is the weighting constant

$$r_{1i} = \cos\left(\frac{i\pi}{N+1}\right) \qquad (i=1,2,...,N) \tag{20a}$$

$$s_{1k} = \cos\left(\frac{\pi}{2}\frac{2k-1}{N+1}\right) \qquad (k=1,2,...,N+1) \tag{20b}$$

$$\mathrm{W}_{1i}^{N} = \frac{1-\mathrm{r}_{1i}^{2}}{N+1} \qquad (i=1,2,...,N) \tag{20c}$$

One may notice that because of the smooth contact at the end point d, unknown functions $p_2(x)$ is zero at the ends. Unknown function $p_2(x)$ is infinite in the point c which in the interior edge of the quarter plane, thereby the index of integral Eq. (15b) is 0.

Where $w_j(r_j)$ is the weight function of $g_j(r_j)$. Using the Gauss-Jacobi integration formulas the integral Eq. (15b) and equilibrium conditions (16b) become

$$\sum_{i=1}^{N} W_{1i}^{N}\bar{\bar{M}}_{21}(s_{2k},r_{1i})G_1(r_{1i}) + \sum_{i=1}^{N} W_{2i}^{N}\left[-\frac{1}{r_{2i}-s_{2k}} + K_{22}(s_{2k},r_{2i})\right]G_2(r_{2i}) = 0 \qquad (k=1,2,...,N+1) \tag{21a}$$

$$\frac{d-c}{2a}\sum_{i=1}^{N} W_{2i}^{N} G_2(r_{2i}) - \frac{1}{\pi} = 0 \tag{21b}$$

where

$$\begin{aligned} & w_j(r_j) = (1-r_j)^{\alpha_j}(1+r_j)^{\beta_j}, \qquad j = 1,2 \\ & \alpha_2 = 0.5 \\ & \frac{\mu_2}{\mu_3}\frac{1+\kappa_3}{1+\kappa_2}\left(2\lambda^2 - 1 + \cos\pi\lambda\right)\cos\pi\lambda - \sin^2\pi\lambda = 0 \\ & \beta_2 = \lambda_1 - 1 \end{aligned} \tag{22}$$

$r_{2i}$and s_{2k} are the roots of the related Jacobi polynomials and W_{2i}^{N} is the weighting constant:

$$P_N^{(\alpha_2,\beta_2)}(r_{2i}) = 0 \qquad (i = 1,2,...,N) \tag{23a}$$

$$P_N^{(\alpha_2-1,\beta_2-1)}(s_{2k}) = 0 \qquad (k = 1,2,...,N) \tag{23b}$$

$$\mathrm{W}_{2i}^{N} = -\frac{1}{\pi}\frac{2N+\alpha_2+\beta_2+2}{(N+1)!(N+\alpha_2+\beta_2+1)}\frac{\Gamma(N+\alpha_2+1)\Gamma(N+\beta_2+1)}{\Gamma(N+\alpha_2+\beta_2+1)}\frac{2^{\alpha_2+\beta_2}}{P_N^{(\alpha_2+\beta_2)}(r_{2i})P_{N+1}^{(\alpha_2+\beta_2)}(r_{2i})} \tag{23c}$$

It can be seen that the extra equations in (18a) and (21a) correspond to the consistency condition of the original integral Eqs. (15a) and (15b). It may be also shown that the $(N/2+1)$-th equations in (18a) and (21a) are automatically satisfied. Thus, Eqs. (18a), (18b), (21a) and (21b) give $2N+2$ algebraic equations to determine the $2N+2$ unknowns $G_1(r_{1i})$ and $G_2(r_{2i})$, b and $(c-d)$. The system of equations are linear $G_1(r_{1i})$ and $G_2(r_{2i})$, but highly nonlinear in b and $(c-d)$. Therefore, an interpolation and iteration scheme had to be used to obtain these two unknowns.

5. Numerical results

The resulting values are contact areas and contact pressures between two elastic layers and between the quarter planes and the lower layer for various dimensionless quantities, such as (a/h_2), (c/h_2), (h_1/h_2), (κ_2,κ_3) and (μ_3/μ_2).

In Table 1, the contact areas between two elastic layers (b/h_2) and between the lower layer and quarter planes $((d-c)/h_2)$ are analyzed for quantities of the materials (κ_2,κ_3) by depending on the various of value distance between the two quarter planes (c/h_2). With increasing distance between the two quarter planes, the contact area between the elastic layers increases. On the contrary, contact area between the lower layer and the two quarter planes decreases.

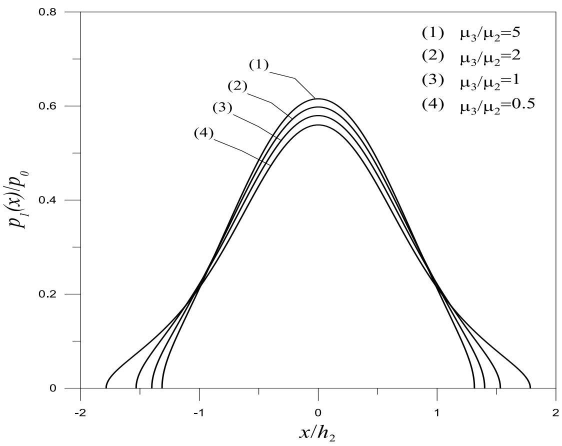
Table 2 shows the variation of size of the contact areas (b/h_2) and $((d-c)/h_2)$ with (c/h_2) for various values of length of distributed load (a/h_2). With increasing distance between the two quarter planes, the contact area between the elastic layers increases, but the contact area between the lower layer and the two quarter planes decreases.

Table 1 Variation of the contact areas (b/h_2) and ($(d-c)/h_2$) with quantities of the materials (κ_2/κ_3) depending on the distance between the two quarter planes (c/h_2), (a/h_2=0.5, a/h_2=0.5, h_1/h_2=1, κ_2=2, μ_2/μ_1=2)

c/h_2	$\kappa_2=\kappa_3=1.25$		$\kappa_2=\kappa_3=2$		$\kappa_2=\kappa_3=2.5$	
	b/h_2	$(d-c)/h_2$	b/h_2	$(d-c)/h_2$	b/h_2	$(d-c)/h_2$
0.01	1.224843	1.530731	1.28842	1.561148	1.328485	1.581076
0.05	1.228639	1.485792	1.293031	1.516519	1.333523	1.536648
0.1	1.234141	1.427584	1.299625	1.458815	1.340663	1.479264
0.2	1.248	1.305697	1.315922	1.338231	1.358092	1.359496
0.4	1.291107	1.052237	1.365116	1.08763	1.409706	1.110678
0.5	1.323315	0.928894	1.401032	0.965228	1.446864	0.988867
0.6	1.364811	0.813174	1.44656	0.84979	1.493538	0.873611
0.75	1.447688	0.659903	1.535358	0.695553	1.583466	0.718774
1	1.647764	0.46526	1.739802	0.496484	1.786203	0.516879

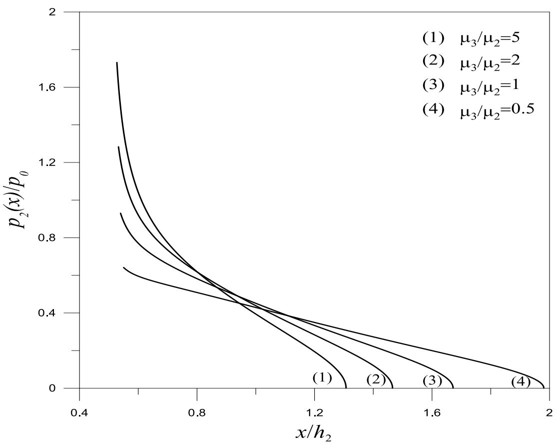
Table 2 Variation of the contact areas (b/h_2) and ($(d-c)/h_2$) with length of distributed load (a/h_2) depending on the distance between the two quarter planes (c/h_2), (h_1/h_2=1, $\kappa_1=\kappa_2=\kappa_3$=2, μ_2/μ_1=2, μ_3/μ_2=2)

c/h_2	$a/h_2=0.1$		$a/h_2=0.5$		$a/h_2=1$	
	b/h_2	$(d-c)/h_2$	b/h_2	$(d-c)/h_2$	b/h_2	$(d-c)/h_2$
0.01	1.171683	1.485043	1.28842	1.561148	1.608344	1.800536
0.05	1.177525	1.44125	1.293031	1.516519	1.609762	1.753812
0.1	1.186085	1.384639	1.299625	1.458815	1.611512	1.693408
0.2	1.20774	1.266621	1.315922	1.33823	1.615275	1.566558
0.4	1.273267	1.024201	1.365116	1.08763	1.62786	1.295324
0.5	1.319969	0.907778	1.401032	0.965228	1.639846	1.15673
0.6	1.37774	0.799268	1.44656	0.849789	1.658122	1.021359
0.75	1.48621	0.656007	1.535358	0.695553	1.701206	0.83324
1	1.723954	0.427744	1.739802	0.496484	1.824652	0.581203



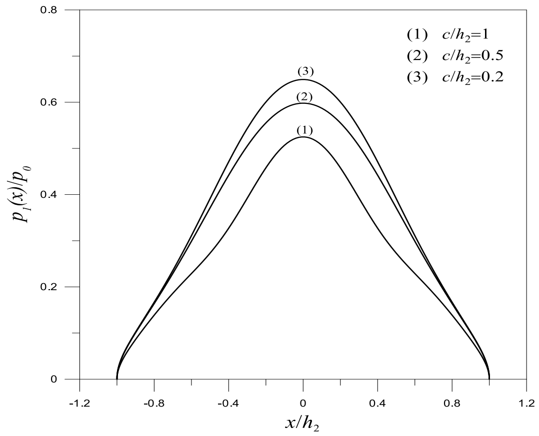
a/h_2=0.5, h_1/h_2=1, c/h_2=0.5, μ_2/μ_1=2, $\kappa_{1,2,3}$=2

Fig. 2 The Contact pressure distribution between two layers with variation of elastic constants (μ_3/μ_2)



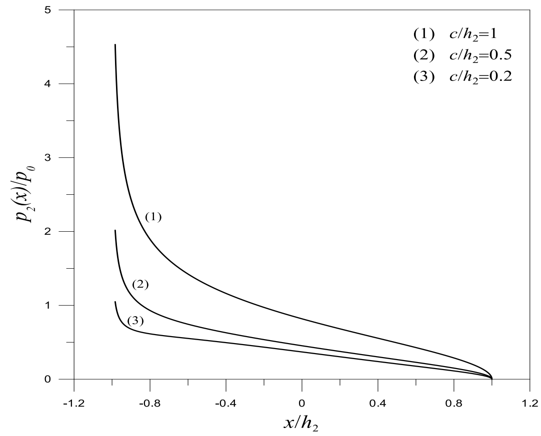
a/h_2=0.5, h_1/h_2=1, c/h_2=0.5, μ_2/μ_1=2, $\kappa_{1,2,3}$=2

Fig. 3 The Contact pressure distribution surfaces between the lower layer and quarter plane with variation elastic constants (μ_3/μ_2)



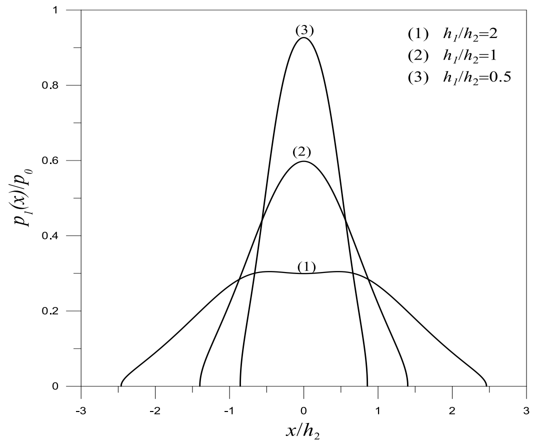
a/h_2=0.5, h_1/h_2=1, μ_2/μ_1=2, μ_3/μ_2=2, $\kappa_{1,2,3}$=2

Fig. 4 The Contact pressure distribution between two layers with variation of the distance between two quarter planes (c/h_2)



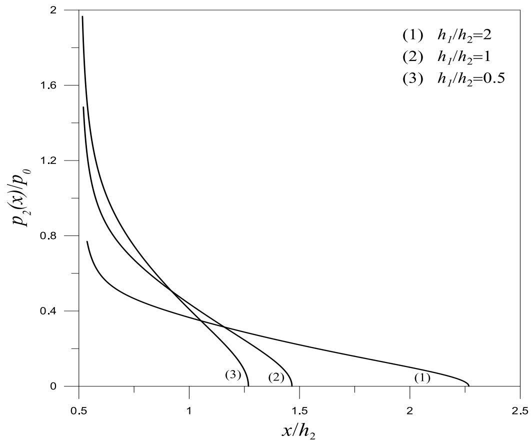
a/h_2=0.5, h_1/h_2=1, μ_2/μ_1=2, μ_3/μ_2=2, $\kappa_{1,2,3}$=2

Fig. 5 The Contact pressure distribution surfaces between the lower layer and quarter plane with variation the distance between two quarter planes (c/h_2)



a/h_2=0.5, c/h_2=0.5, μ_2/μ_1=2, μ_3/μ_2=2, $\kappa_{1,2,3}$=2

Fig. 6 Contact pressure distribution between two elastic layers with variation of (h_1/h_2)



a/h_2=0.5, c/h_2=0.5, μ_2/μ_1=2, μ_3/μ_2=2, $\kappa_{1,2,3}$=2

Fig. 7 Contact pressure distribution between the the lower layer and quarter plane with variation (h_1/h_2)

Figs. 2 and 3 show $p_1(x)/p_0$ and $p_2(x)/p_0$ the dimensionless contact pressure distributions. In the event of increase lower layer and the quarter plane ratio shear modules, it is indicated that the contact pressure distributions at the contact surfaces between two elastic layers and between the lower layer and quarter plane increase.

The contact pressure distributions for various values of (c/h_2) are shown in Figs. 4 and 5. With increasing distance between the two quarter planes, the contact pressure $p_1(x)/p_0$ decreases. On the contrary, the contact pressure $p_2(x)/p_0$ increases.

As seen in Fig. 6, the contact pressure $p_1(x)/p_0$ decreases with increasing of h_1/h_2 and in Fig. 7, the contact pressure $p_2(x)/p_0$ increases with increasing of h_1/h_2.

6. Conclusions

In this paper, the receding contact problem for two elastic layers whose elastic constants and heights are different and supported by two elastic quarter planes is considered. Dimensionless pressures distribution between two elastic layers $p_1(x)/p_0$ and between the quarter planes and the lower layer $p_2(x)/p_0$ and contact areas (b/h_2) and $((d-c)/h_2)$ are investigated for various dimensionless quantities, such as a/h_2, c/h_2, (κ_2, κ_3), h_1/h_2 and μ_3/μ_2 .

• With increasing distance between the two quarter planes, the contact area between the elastic layers increases. On the contrary, the contact area between the lower layer and the two quarter planes decrease.

• The size of contact areas (b/h_2) and $((d-c)/h_2)$ decrease depending on increasing μ_3/μ_2.

• As it can be seen in the figures that increasing thickness of upper layer, contact pressure along to contact area between two elastic layers decreases, but contact pressure along to contact area the lower layer and quarter plane increases.

• The size of contact areas increase depending on increasing of the length of distributed load.

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Notations

The following symbols are used in this paper:

A_i, B_i, C_i, D_i : Coefficients,

a : Length of distributed load,

h_i : Thickness of layer $(h = h_1 + h_2)$

p_0 : Distributed load,

$p_1(x), p_2(x)$: Contact pressures,

u_i, v_i : Displacement components in cartesian coordinates,
u_r, u_θ : Displacement components in polar coordinates,
$\sigma_x, \sigma_y, \tau_{xy}$: Stress components in Cartesian coordinates,
$\sigma_r, \sigma_\theta, \tau_{r\theta}$: Stress components in polar coordinates,
x, y : Cartesian coordinates,
r, θ : Polar coordinates,
μ_i, ν_i : Elasticity constants,
κ_i : Material constants,
c : Distance between the quarter planes,
$b, (d-c)$: Contact areas.

Appendix A.

$$M_{11}(x_1,t_1)=\int_0^\infty\left[\frac{2m}{K}\right]\left[\frac{1}{\Delta_1}\left(-\frac{1}{2}e^{-4\alpha h_1}(-1+e^{4\alpha h_1}+4\alpha h_1 e^{2\alpha h_1})(1+\kappa_1)\right)+\left(\frac{1+\kappa_1}{2}\right)\right]\sin\alpha\left[t_1-x_1\right]d\alpha \tag{A1}$$

$$R_{11}(x_1,t_1)=\int_0^\infty\left[-\frac{2m}{K}\right]\left[\frac{1}{\Delta_2}\left(-\frac{1}{2}e^{-4\alpha h_2}(-1+e^{4\alpha h_2}+4\alpha h_2 e^{2\alpha h_2})(1+\kappa_1)\right)-\left(\frac{1+\kappa_2}{2m}\right)\right]\sin\alpha\left[t_1-x_1\right]d\alpha \tag{A2}$$

$$R_{12}(x_1,t_2)=\int_0^\infty\left[-\frac{2}{K}\right]\left[\frac{m}{\Delta_2}\left(\frac{1}{2}e^{-3\alpha h_2}\left(-1+\alpha h_2+e^{2\alpha h_2}(1+\alpha h_2)\right)\left(1+\kappa_2\right)\right)\right]\sin\left[\alpha x_2\right]\cos\left[\alpha t_2\right]d\alpha \tag{A3}$$

$$M(x)=\int_0^\infty\left[-\frac{4m}{K}\right]\left[\frac{1}{\Delta_1}\left(e^{-3\alpha h_1}(-1+\alpha h_1+e^{2\alpha h_1}(1+\alpha h_1)(1+\kappa_1)\right)\right]\sin\left[\alpha x\right]\frac{\sin\left[\alpha a\right]}{\alpha}d\alpha \tag{A4}$$

$$M_{21}(x_2,t_1)=\int_0^\infty\left[\frac{2m}{1+\kappa_2}\right]\left[\frac{1}{\Delta_2}\left(-e^{-3\alpha h_2}(-1+\alpha h_2+e^{2\alpha h_2}(1+\alpha h_2))\right)\left(1+\kappa_2\right)\right]\sin\alpha\left[t_1-x_2\right]d\alpha \tag{A5}$$

$$M_{22}(x_2,t_2)=\int_0^\infty\left[-\frac{2}{1+\kappa_2}\right]\left[\frac{m}{\Delta_2}\left(\frac{1}{2}e^{-4\alpha h_2}(-1+4\alpha h_2+4\alpha h_2 e^{2\alpha h_2})\left(1+\kappa_2\right)\right)+\left(\frac{1+\kappa_2}{2}\right)\right] \sin\left[\alpha x_2\right]\cos\left[\alpha t_2\right]dz \tag{A6}$$

$$k_{22}(x_2,t_2)=\frac{1}{t_2-x_2}+\int_0^\infty\left(\frac{sinh\pi y}{cosh\pi y-1-2y^2}-1\right)\frac{sin\left[\log\dfrac{t_2-c}{x_2-c}\right]y}{x_2-c}dy-\frac{\pi^2}{(x_2-c)\left(\pi^2-4\right)} \tag{A7}$$

The expressions for Δ_1, Δ_2 and K from Eqs. (A1)-(A7)

$$\Delta_1=\left[e^{-4\alpha h_1}+1-2e^{-2\alpha h_1}+\left(1+2\alpha^2 h_1^2\right)\right] \tag{A9}$$

$$\Delta_2=\left[-1-e^{-4\alpha h_2}+2e^{-2\alpha h_2}+\left(1+2\alpha^2 h_2^2\right)\right]m \tag{A10}$$

$$K=1+\kappa_2+m+m\kappa_1 \tag{A11}$$

$$m=\frac{\mu_2}{\mu_1} \tag{A12}$$