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Dynamic response of railway bridges traversed simultaneously by opposing moving trains

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Abstract. Bridges are vital components of the railroads. High speed of travel, the periodic and oscillatory nature of the loads and the comparable vehicle bridge weight ratio distinguish the railway bridges from the road bridges. The close proximity between estimations by some numerical methods and the measured data for the bridge-vehicle dynamic response under the moving load conditions has boosted the confidence in the numerical analyses. However, there is hardly any report regarding the responses of the railway bridges under the effect of the trains entering from the opposite directions while running at unequal speed and having dissimilar geometries. It is the purpose of this article to present an analytical method for the dynamic analysis of the railway bridges under the influence of two opposing series of moving loads. The bridge structural damping and many modes of vibrations are included. The concept of modal superposition is used to solve for the system motion equations. The method of solution is indeed a computer assisted analytical solution. It solves for the system motion equations and gives output in terms of the bridge deflection. Some case studies are also considered for the validation of the proposed method. Furthermore, the effects of varying some parameters such as the distance between the bogies, and the bogie wheelset distance are studied. Also, the conditions of resonance and cancellation in the dynamic response for a variety of vehicle-bridge specifications are investigated.

Keywords: railroad bridges; dynamic analysis; railroad vehicles; bridge oscillations; modal superposition; resonance and cancellation

1. Introduction

Expansion of railway industries based on its many advantages needed extended studies concerned with improving safety, profitability and speed of transporting goods and passengers. The higher speed of travel, safety and increased ride comfort and the lower running costs and the use of clean energy resources are some of the advantages that favor the widespread use of railway transportation.

Bridges are vital components of the railroads. On the bridge section, the track stiffness is lower and its composition is different, compared to the rest of the path. This can make the bridges as the bottle necks for the railway transportation. Generally, the use of the bridges in railroads is

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associated with shortening the routes and preventing increase in the cost of laying the tracks. They are therefore, very popular. In the past, the design of railway bridges included procedures that were almost the same as the design of road bridges. However, the specific particulates of the railway bridges and the rolling stocks in comparison with the road bridges and machineries warranted particular studies. Research results pointed to the increased attention on the details of the bridge and the railway rolling stocks. The high-speed trains travel at operational speed in the range of 250-350 km/hr that is much faster than the normal road vehicles. The railway bridges have higher vehicle to bridge weight ratio. There is regularity in the interactive loading between the rail vehicle and the bridge. Such regularity is caused by the similarity in the axle load and the wheelset distances in different wagons. All of the above noted factors are considered as the most important parameters that differentiate the railway bridges from the road bridges.

Variety of models has become available for the analysis of the dynamic behavior of the rolling stocks over the bridges. Some recent models by Yang *et al*. (2004), Xia and Zhang (2005), Liu *et al*. (2009), Garinei and Risitano (2008) include the moving load, the moving mass and the suspended mass models.

The moving load model was also used by Piccardo and Tubino (2012) for the analysis of the dynamic response of the Euler-Bernoullie beams. Fryba (1996) made general overview of the theories of the moving load, moving mass and the suspended moving mass models and presented the corresponding equations and general solutions. Many studies were also performed by other researchers that involved numerical solutions to the problem, (Dinh *et al*. 2009), (Majka and Hartnett (2009)). Study by Sridharan and Malik (1979) involved the numerical analysis of vibration of beams subjected to moving loads. Yau (2004) in a study on the vibration of simply supported beams due to moving loads added two trusses to the middle section of the beam as extra supports to the general structure and studied the effect of the ratio of the truss height to the beam span length. Garinei and Risitano (2008) assumed variable moving load and reported a solution for the high speed trains in their paper.

Generally, many reports in this field were based on the assumption of the moving load model. It provided simplified methods of solution with results within acceptable levels of accuracy. Therefore, such methods of solution are still in use. After studying the behavior of the bridge under single moving load, series of moving loads were also tested in order to improve the modeling techniques and also to increase the accuracy of the results. Fryba (1996) reported studying many cases of moving loads traversing bridges. However, he did not consider series of loads and modeled the case of a multi axle vehicle. Fryba (1996) then used numerical methods to solve the system motion equations. Ricciardelli and Briatico (2011) reported study of a supported beam loaded by a force with sinusoidal time variation moving at a constant speed. During this work, they challenged the accuracy of the approach used by many codes of practice for the evaluation of the transient response of footbridges to walkers. Wu and Dai (1987) proposed the idea of considering two moving loads traversing in similar directions and also in opposite directions. They initially solved the problem for a single span bridge. After comparing with the exact solutions, they analyzed the multi span bridges and used series of loads instead of a single load. The finite element method and the transfer matrices were used in order to solve the problem. They also added the effect of the speed of travel on the deflection of the midpoint of the bridge. It included loads moving in opposite directions at either constant or variable speed. Later on, in a study by Yang *et al*. (1997) a set of moving loads was considered and the phenomenon of resonance was investigated. The step functions due to the entry and the exit of the loads were modeled by using the equivalent series. As a matter of simplicity, the effects of damping were ignored. It was also assumed that the maximum deflection happens at the bridge midpoint at the time when the last vehicle passes by. Savin (2001) analytically calculated the response of the bridge midpoint under series of moving loads. He reached to the conclusion that under the effect of a set of moving loads the effect of the free vibrations cannot be neglected and it has to be considered in addition to the forced vibrations of the bridge. The effects of the different speed of travel were also investigated in practical as well as in theoretical studies. Amongst such studies, Wu *et al*. (2001) in two special cases studied the case of opposing traversing trains by using the FEM models.

Research in this area resorted to simplifications in order to avoid complexities and curb the huge number of calculations that would have otherwise been obligatory. This can be apparently altered by developing an exact analytical solution that reflects on the effect of damping in addition to the condition of opposing traversing vehicles.

It is the purpose of this research to develop an analytical solution for the dynamic response of the railway bridges under two series of moving loads traversing simultaneously in opposite directions. It is indeed a computer assisted analytical solution. It combines the accuracy of the closed form solutions and the speed of calculation that can be provided by computers. The outcome of the analysis is the displacement of the bridge in many selected points along the bridge span. The calculation time is small and the results are accurate. With 100 data points along the bridge and 1500 time intervals, the solution procedure takes only a few seconds. It also facilitates varying many vital design parameters and collecting the results in a matter of seconds. The method involves a step by step procedure in which movement of each load traversing the bridge is defined as an event. The time of the occurrence and the type of such events is clearly defined. This is then extended into the equations that govern the bridge vibrations under two opposing moving loads. Such a method results in the bridge dynamic model that is used to in order to reach to the analytical results. Verification of the proposed model is by comparing it with the results that are reported by other researchers. The model is also used to evaluate the percentage of error in using some of the existing simplifications. Such simplifications include the assumption of one moving load in place of each bogie (instead of two moving loads) and the assumption that the maximum deflection of the bridge happens at its mid plane. The conditions of resonance and cancellation for the case of the simultaneously travelling trains are also considered. This is performed while varying the distance between the bogies in a vehicle and the speed of travel of the two sets of the trains. Some suggestions are also offered in order to avoid the resonance phenomenon in the dynamic responses.

2. The Method of solution

2.1 The basic assumptions

The problem at hand and some simplifying assumptions to solve it are explained in this section. The purpose of the modeling is to obtain the complete dynamic response of the bridge as a function of time and location on the bridge. It involves the simultaneous travel of two trains traversing the bridge in opposing directions, as in Fig. 1.

The following assumptions are used to solve for the moving load conditions, (Yang *et al*. 2004).

1) The beam is of Euler-Bernoulli type that is homogeneous with constant cross section. The beam plane cross sections remain plane after deformations.

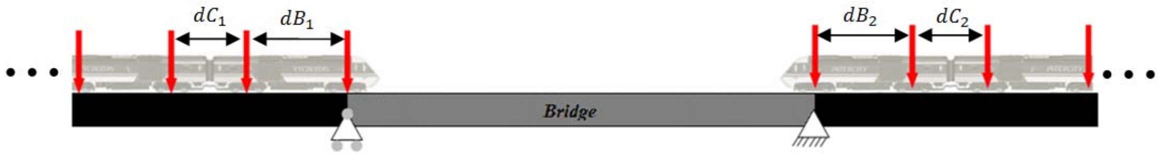


Fig. 1 Schematic representation of two trains simultaneously traversing a bridge in opposite directions

2) Only gravitational effect of the load is considered and its inertia effect is neglected (the same fundamental assumption for the moving load model).
3) The vehicle's speed of travel is constant.
4) The damping of the beam is of Rayleigh type.
5) The beam initial conditions are set to zero (the beam initially at rest). Meaning that the points along the bridge have neither deflection nor speed at time zero.
6) There is no assumption for the road surface stiffness.
7) The second train enters the bridge at the same time as the first train at x = l.
8) The vehicles in the two trains are similar and have the same axle loads and the same distance between the bogies.
9) Variations in the distances between the wagons are neglected.
10) The effect of the twist in the bridge is ignored in the analytical procedure.
11) Only vertical forces are included and the longitudinal and lateral vibrations of the bridge are not considered.

In order to further elaborate on the 10th assumption, it needs to be emphasized that this research is concerned with the dynamic response of the bridge under two opposing moving trains. Under such conditions, and in comparison with the case of a single moving train, it is expected that the two sides of the bridge each carrying a train tolerate much of the torsion effects. Therefore, in comparison with the deteriorating effects of deflection it is justifiable to ignore the lesser effects of torsion in the beam cross sections. The basis for such an assumption is also presented in the results of a numerical modeling reported by Wu *et al*. (2001).

Based on such assumptions and by including the internal damping coefficient of (C_i) and the external damping coefficient of (C_e) the motion equation for the beam can be written as in Eq. (1), (Yang *et al*. (2004)).

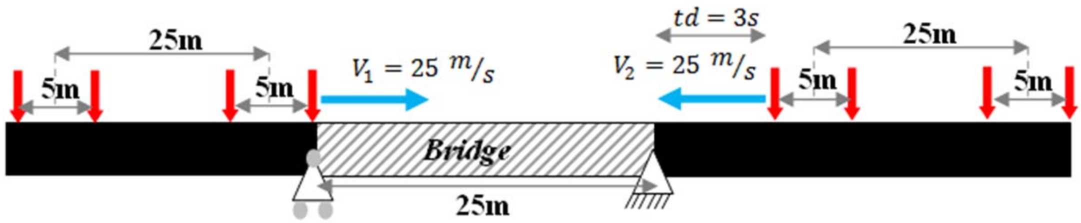
$$m\ddot{u} + C_e\dot{u} + C_i I\left(d^4\dot{u}\big/dx^4\right) + EI\left(d^4 u\big/dx^4\right) = f(x,t) \tag{1}$$

In this equation, m is the mass per unit length and EI is the flexural rigidity of the bridge. $u(x,t)$ is the time dependant bridge deflection at location x. $\dot{u},\ddot{u}$ are the 1st and the 2nd derivatives of u in time. $f(x,t)$ is the bridge excitation force at location x at time t. The bridge excitation forces come from the moving trains and are represented in Eq. (2)

$$f(x,t) = \sum_1^{4N} F_1(x,t) + \sum_1^{4N} F_2(x,t) \tag{2}$$

In this equation F_1 represents the excitation caused by the moving loads of the 1st train and F_2 represents the excitation forces caused by the 2nd moving train. In this case, each vehicle is carried on four wheelsets. For consistency, it is assumed that the 1st moving train enters the bridge from the left end and the 2nd moving train enters from the right end. Generally, subscript 1 is used for the train entering the bridge from the left side and subscript 2 is used for the train that enters the bridge from the right side. All excitation forces on the bridge are of the impact type. To serve the purpose, the entry and the exit time for each load of each train are defined as the events in the corresponding time matrix. An event represents the entry or exit of each load from the left or the right sides of the bridge. The model involves two trains that interact with the bridge. Each train consists of N number of vehicles. The model vehicle is carried on two bogies each equipped with two axles. Each axle load is represented as a moving load that is identifiable with two events. Therefore, the total number of events for such a moving train traversing a bridge adds up to $8\times N$ events. With the prior knowledge about the position of the wheelsets in the trains that is associated with the train makeup and the sizes of the vehicles one can start codifying for the so called events. Adding the train speed of travel to such a code specifies the time of occurrence of the specific events of entry and exit of the individual loads on the bridge. Fig. 2 schematically presents the entry of two railway vehicles onto the bridge. The vehicles are 30 m in length and approach the bridge from opposing directions. The vehicles speed of travel is 25 m/s hence each axle needs 1 sec to clear the bridge. It takes each vehicle 1.2 sec to travel along the bridge. The time matrix codified for the entry and the exit events for the sample case is also presented in Fig. 2. For this case, there are 16 entry and exit events and the time matrix holds 16 columns. Each row in the time matrix defines the entry or the exit events for the rail vehicle on two bogies and four axles. Each column in the time matrix can hold one nonzero element. Such nonzero element represents the time of the occurrence of the event that is associated with the corresponding column number. The first row in the time matrix that is marked with letter “A” represents the entry of the vehicle onto the bridge from the left side.

The four wheelsets of this vehicle enter the bridge at time “0”, “0.2”, “1” and “1.2” in seconds, respectively. The second row in the time matrix that is marked with letter “B” presents the entry of the four wheelsets of the opposing train that enter at time “3”, “3.2”, “4” and “4.2” seconds, respectively. The third raw in the time matrix marked with letter “C” identifies the exit events of the wheelsets of the left entering vehicle. They leave the bridge at time “1”, “1.2”, “2” and “2.2”



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	0	0.2	0	1	0	1.2	0	0	0	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0	0	3	3.2	0	4	0	4.2	0	0
C	0	0	1	0	1.2	0	2	2.2	0	0	0	0	0	0	0	0
D	0	0	0	0	0	0	0	0	0	0	4	0	4.2	0	5	5.2

Fig. 2 The sample time matrix codified for the entry and the exit events for two rail vehicles while each carried on two bogies and each bogie equipped with two axles

seconds, respectively. The fourth raw in the time matrix that is marked with letter "D" identifies the exit events of the wheelsets of the right entering vehicle at time "4", "4.2", "5" and "5.2" seconds, respectively. The use of such a time matrix creates a systematic tracking code for the entry and the exit of the loads onto the bridge and facilities the solution procedure. Obviously the size of the time matrix depends on the train configuration. The case presented in Fig. 2 is only a demonstration that is useful for the special case under consideration. When trying to solve for the system motion equations, there will be two excitation forcing functions namely $F_1(x,t)$, $F_2(x,t)$ as in Eqs. (3)-(4)

$$F_1(x,t) = H(t - T_1(n_e)) \times H(T_1'(n_e') - t) \times P_1 \delta(x - V_1(t - T_1(n_e))) \tag{3}$$

$$F_2(x,t) = H(t - T_2(n_e)) \times H(T_2'(n_e') - t) \times P_2 \delta(x - (l - V_2(t - T_2(n_e)))) \tag{4}$$

In the above equations, H is the unit step function, $T_1(n_e)$ is the n_e^{th} event that includes the entry of the 1st train load. $T_1'(n_e')$ is the exit event corresponding to the entry of the n_e^{th} event. The event corresponding to the entry event of a load needs to be associated with the exit event of the same load. P_1 is the axle load on the bogie of the 1st train and V_1 is the speed of travel of the 1st train. The same parameters when marked with subscript 2 are used for the 2nd train. The correspondence between the entry and the exit events of a wheelset is presented in Eq. (5)

$$T'(n_e') = T(n_e) + {l}/{V_{1or2}} \tag{5}$$

Eq. (1) is the general equation of vibrations of the beam that represents the bridge. With the assumption of simply supported ends for this beam with nil initial displacements, one can write the boundary conditions as

$$u(x,t) = 0, \quad u(l,t) = 0, \quad EI\left(\frac{d^2u(0,t)}{dx^2}\right) = 0, \quad EI\left(\frac{d^2u(l,t)}{dx^2}\right) = 0 \tag{6}$$

and the initial conditions are

$$u(x,0) = 0, \quad \dot{u}(x,0) = 0 \tag{7}$$

By the concept of the modal superposition the deflection of the beam $u(x,t)$ can be expressed as

$$u(x,t) = q(t) \times \phi(x) \tag{8}$$

Where $q(t)$ denotes the generalized coordinate and $\phi(x)$ the shape function of the vibration mode. Substituting Eq. (8) into Eq. (1) and multiplying both sides of the equation by ϕ (that represents the mode shapes) and integrating with respect to x over the length l of the beam, results in the motion equation for the nth mode of the simply supported beam, (Yang *et al*. (2004))

$$\begin{aligned} &\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = \sum\nolimits_{nt=1}^{8N} \frac{2p}{ml} \sin\frac{n\pi V_1}{l}(t - T_1(n_e)) \times H(t - T_1(n_e)) \times H(T_1'(n_e') - t) + \\ &\sum\nolimits_{nt=1}^{8N} \frac{2p(-1)^{n+1}}{ml} \sin\frac{n\pi V_2}{l}(t - T_2(n_e)) \times H(t - T_2(n_e)) \times H(T_{21}'(n_e') - t) \end{aligned} \tag{9}$$

With the assumption of

$$C_e = \alpha_e m, \quad C_i = \alpha_i E$$

The following definitions can be used

$$\zeta_n = \frac{1}{2}\left(\frac{\alpha_e}{\omega_n} + \alpha_i \omega_n\right) \tag{10}$$

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{m}} \tag{11}$$

Where α_i is the coefficient of the internal damping, α_e is the coefficient of the external damping, ω_n is the bridge natural frequency of the nth mode and ξ_n is the total damping coefficient of the nth mode of vibration of the bridge. In order to adjust to the support conditions, it is also considered that for the case of the simply supported beams, the nth modal shape of vibration is

$$\phi(x) = \sin\frac{n\pi x}{l} \tag{12}$$

The above equations present the relations governing the time response of the beam by including the excitation functions of the left and the right entering trains. The buildup of the excitation forces consistent with Eq. (9) is presented in Fig. 3.

It needs to be notified that the dynamic response of the beam given by Yang *et al.* (2004) excluded the effect of damping and considered only the first mode of vibration. Such simplifications in the solution procedure are capable of altering the results and do not exist in the present method. The solution process in the present research is by including damping and considering as many modes of vibration as necessary. The solution starts by seeking the excitation functions in the motion equations, Eq. (9). This equation needs to be solved repeatedly with the number of repetitions equal to the total number of the events. The initial conditions at the start of each event come from the status of the system at the end of the last event. Therefore, the system motion equation is actually solved in a piecewise manner while the initial conditions are continuously updated. It is observed that if the two loads enter the bridge at the same time, the odd terms in the frequency responses accumulate while the even terms in the frequency responses

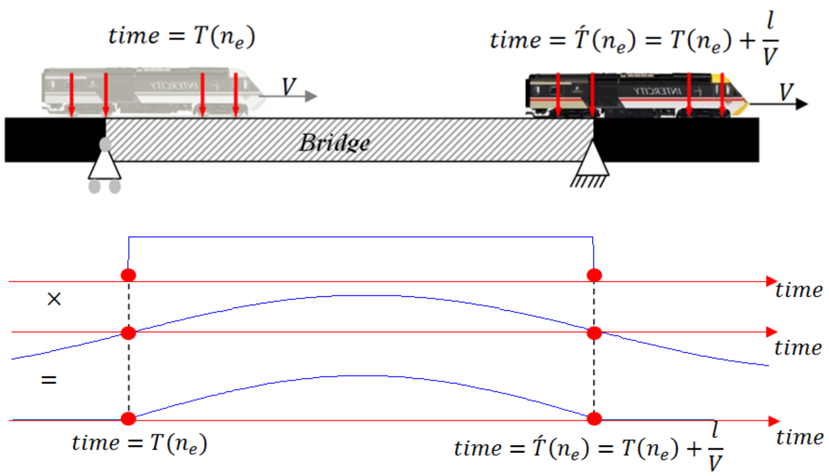


Fig. 3 The buildup of the excitation forces between the entry and the exit of a wheelset

subtract. The parameters defining the entry and the exit events are embedded in the time matrix that is already discussed in the last section. In Eq. (9), the only term that highlights the excitation function of the opposing train is the term $(-1)^{n+1}$. Such a term inverts the even numbered mode shapes. The odd modes are of the same phases while the even modes are at opposing phases.

2.2 Solving for the system motion equations

Consistent with the prior description on the solution procedure, the time response of the system can be expressed as

$$q(t)=\begin{cases} \begin{bmatrix} Bc_1 & Bs_1 \end{bmatrix}\times\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\end{bmatrix}+\begin{bmatrix} Ac_1 & As_1\end{bmatrix}\times e^{-2\xi\omega\tau}\begin{bmatrix}\cos\omega_d t\\ \sin\omega_d t\end{bmatrix} & T_1=0\le t<T(2)\\ \begin{bmatrix} Bc_1 & Bs_1 & Bc_2 & Bs_2\end{bmatrix}\times\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ \cos\Omega(t-T(2))\\ \sin\Omega(t-T(2))\end{bmatrix}+\begin{bmatrix} Ac_1 & As_1 & Ac_2 & As_2\end{bmatrix}\times e^{-2\xi\omega\tau}\begin{bmatrix}\cos\omega_d t\\ \sin\omega_d t\\ \cos\omega_d(t-T(2))\\ \sin\omega_d(t-T(2))\end{bmatrix} & T(2)\le t<T(3)\\ \vdots & \vdots\\ \vdots & \vdots\\ \begin{bmatrix} Bc_1 & Bs_1 & \dots & Bc_{ne} & Bs_{ne}\end{bmatrix}\times\begin{bmatrix}\cos\Omega t\\ \sin\Omega t\\ \vdots\\ \cos\Omega(t-T(ne))\\ \sin\Omega(t-T(ne))\end{bmatrix}+\begin{bmatrix} Ac_1 & As_1 & \cdots & Ac_{ne} & As_{ne}\end{bmatrix}\times e^{-2\xi\omega\tau}\begin{bmatrix}\cos\omega_d t\\ \sin\omega_d t\\ \vdots\\ \cos\omega_d(t-T(ne))\\ \sin\omega_d(t-T(ne))\end{bmatrix} & T(ne)\le t<T(ne+1)\end{cases} \tag{13}$$

$\Omega=\dfrac{n\pi V_{1or2}}{l}$ is the excitation frequency and $T(n_e)$ is the time of the occurrence of the event number "n_e". The time response $q(t)$ comprises of a number of sub-elements. The number of such elements is equivalent to the total number of the events that is already defined in the time matrix. Each sub-element carries two associated coefficient matrices *A&B*. *As* and *Ac* are the elements of matrix *A* and are the coefficients of the Sine and the Cosine terms in the homogeneous part of the solution. *Bs* and *Bc* are the elements of matrix *B* and are the coefficients of the Sine and the Cosine terms in the particular solution in the bridge dynamic response. Obviously, the elements of the matrices *A&B* need to be calculated, repeatedly. It is at this stage that the proposed solution procedure seeks computer assistance in order to calculate and fill in for all the elements of the coefficient matrices *A&B*. It is then fair to call the procedure as a computer assisted analytical solution. In order to reduce the number of such calculations, all harmonic functions in the time response carry a time shift equal to $T(n_e)$. The value of the elements in the coefficient matrix *B* (corresponding to the excitation) is only a function of the side of the entry of the load for the specified event. For the exit events, the corresponding element in matrix *B* drops to zero and according to Eq. (5) the value of the associated (entry) event needs to be zero, as well. At such a moment no load enters the bridge and the effect of the bygone load need to be suppressed. *A* is a matrix of coefficients corresponding to the free vibrations of the bridge. The elements of this matrix are calculated by applying the conditions of continuity in the response and its first derivative for any specific event in comparison to its prior event. In fact it is possible to write a computer program that uses series of back tracking equations in order to extract all elements of the

A & B coefficient matrices. For the purposes of this research, all such processes are performed by including the first five natural modes of vibrations of the bridge in the final solution. By summing up the results, the time response in the bridge displacement function is completely calculated. The solution to the system motion equation is assisted by writing a computer program in MATLAB environment. This program receives two sets of input in order to perform the primary calculations and to prepare the system idealized time matrix. The program solves for the system motion equations and outputs the bridge deflection. The flow of events corresponding to this computer program is presented in Fig. 4. Definitions for the symbols that are used in this flowchart are available in appendix 1. The consistency in the modeling procedure and the adequacy of the results are checked by using the computer software for solving some known problems that are already reported in the corresponding literature.

3. Validation of the proposed vehicle-bridge dynamic interaction model and the solution procedures

The issue of the dynamic response of the bridges under moving loads while the loads enter and traverse from the same side of the bridge is already practiced by some researchers. Such problems have been analytically and numerically investigated. There are reports by Wu *et al*. (2001) and Nguyen *et al*. (2009) about numerical solution to the vehicle bridge interaction dynamics including a complete model of a train and the bridge. Also, there are reports about the two special limited cases when the trains enter the bridge from different sides only at a specified speed and with certain delay time. Wu *et al*. (2001) also studied the case of opposing moving trains by using the Finite Element model under the assumptions that the trains travel at similar constant speed. However, they did not include the train geometrical specifications in the model. Also, the effect of varying the speed of travel or introducing the delay time between the trains on the dynamic response of the bridge was not considered. There is hardly any report regarding the response of the railway bridge under the effect of two trains entering from the opposite directions of the bridge while including different speed of travel or the train geometrical specifications. Such issues have been attended to in this research. The sample cases include two moving loads entering the bridge from the same and then from the opposite directions. The dynamic response of the bridge under such conditions is obtained. In what follows the results of such analysis are discussed.

3.1 The case of a single train traversing the bridge

Fig. 5 is a comparison between the results of the computer assisted analytical solution from this research and the analytical solution reported by Yang *et al*. (2004). It considers a concrete beam with length l=20 m, moment of inertia I=3.81 m^4, modulus of elasticity E=29.43 GPa, mass per unit length m=34088 kg/m for which the first frequency of vibration solved is ω_l=44.75 rad/sec. The train is assumed to be consisted of N=5 cars of identical length d=24 m. The two bogies of each car are separated by a distance of 18 m. The mass per bogie is 22000 kg. For the bridge with this configuration the resonance speed found to be 34 m/s and the speed of cancellation equal to 26 m/s, (Yang *et al*. (2004)).

It was reported that for a train at the speed of 34 m/s the midpoint response of the beam tends to increase steadily as there are more loads passing the beam, which is indicative of the resonance phenomenon. For a train at the speed of 26 m/s, the response of the beam appears merely as a

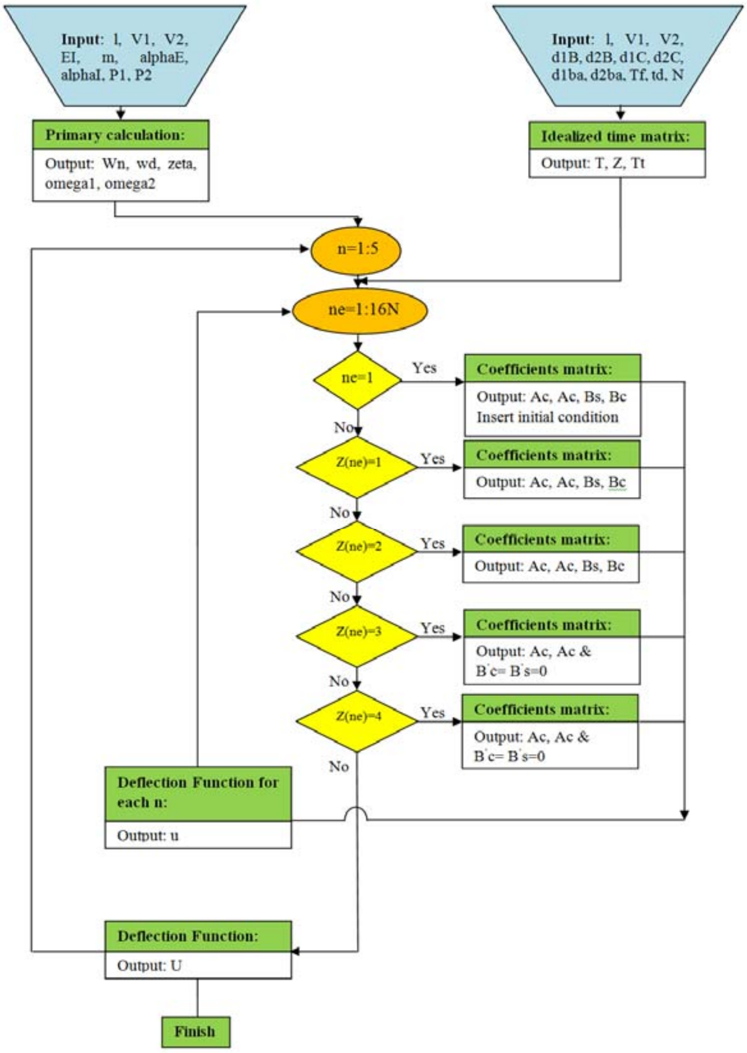


Fig. 4 The computer program flowchart

periodic function. No effect of amplification observed during the passage of the wheel loads on this bridge. As long as all the wheel loads depart from the beam, no residual response remains on the beam, which is a typical cancellation phenomenon. The comparisons are for the two cases when the speed of travel is equal to 26 m/s and 34 m/s. The case for the opposing movements of the loads cannot be considered, yet. This was not exercised in the work reported by Yang *et al*. (2004). Naturally, under such conditions, the bridge and the rolling stock are the only parties in implying the excitation and bringing up the responses. One may also expect to observe the resonance and the cancellation conditions.

It is also needed to be emphasized that the solution method in this research is capable of including the bridge damping into the calculations. However, in order to compare the results with the results that are reported by Yang et al. (2004) the effect of the bridge damping is not included.

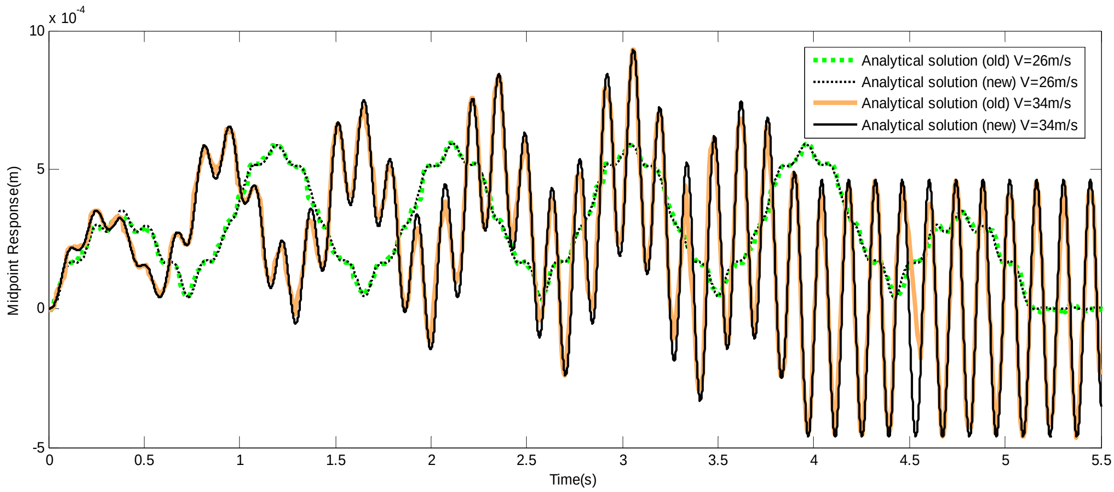


Fig. 5 A comparison between the results from this research and the results reported by Yang *et al*. (2004), a single vehicle traversing the bridge at the speed of 26 m/s and 34 m/s

There is good agreement between the results from the two methods. In the work reported by Yang *et al*. (2004) the mathematical series representing the sequential loading is replaced with an equivalent sinusoidal series. However, only the 1st mode of vibration was included and the effect of damping ignored. That is while the method of solution that is proposed in this research correlates the position of the individual wheelset loads directly to their time matrix. The solution procedure is direct and does not contain any simplifications. It can handle as many modes of vibrations as felt necessary.

3.2 The case of two opposing trains traversing the bridge

The accuracy of the proposed method is further proved by comparing the results with that reported by Wu *et al*. (2001). A three dimensional model of the train with 15 wagons is presented. The bridge elements are also modeled and the interaction between such elements is numerically solved. The data in Table 1 is used for the modeling purposes. This is the case of a two-way railroad bridge with two trains on two different tracks crossing the bridge at the same or at different speed. The trains enter the bridge at similar speed of 100 m/s. Each of the two trains consists of 15 identical cars. The trains meet at the midspan of the bridge (referred to as the symmetric crossing movement). The maximum displacement of the bridge is found to be equal to 7.8 mm. Wu *et al*. (2001) introduced the system damping in a matrix form. However, as per their suggestion and for the purposes of this research the damping coefficient is equal to 3.5%.The other input data to the model are the same as in Table 1. In this attempt, the dynamic response of the bridge at its mid point is calculated in 2 cases. Case (a) is for the simultaneous entry of the two trains onto the bridge. This case is considered as the symmetric crossing movement. In case (b) it is assumed that the 2nd train enters the bridge at a time when the midpoint of the 1st train passes the bridge midpoint (asymmetric crossing). All such cases are solved by the computer assisted analytical method that is proposed in this research and the results are compared with the numerical results reported by Wu *et al*. (2001). The results are presented side by side, in Fig. 6.

Table 1 The bridge and the rolling stock particulates as reported by Wu *et al*. (2001)

EI=22.149×10^{10} $N.m^2$	P_1 = 0 & 27475 N	d_1B = 17.5 m	td = 0 & 1.9 Sec
l = 30 m	P_2 = 274750 N	d_2B = 17.5 m	tf = 4.2 Sec
m = 41740 kg/m	V_2 = 360 km/h	d_1C = 7.5 m	TS = 0.01 Sec
N = 15	V_1 = 360 km/h	d_2C = 7.5 m	ξ_n = 0.035

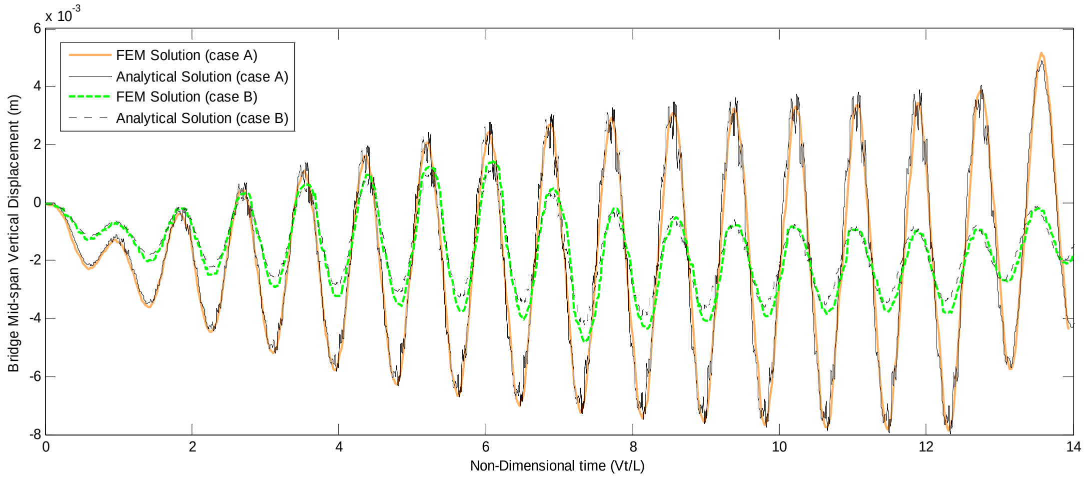


Fig. 6 A comparison between the computer assisted analytical results from this research and the results from the numerical solutions reported by Wu *et al*. (2001)

The close proximity of the results of the two methods can clearly be noted specially in predicting the maximum values and the frequency of vibrations for the two cases considered.

Nguyen *et al*. (2009) also used the same model as Wu *et al*. (2001) with the same specifications for the bridge and the rolling stock as in Table 1. They also used a numerical procedure to solve for the dynamic response of the bridge and ended up with the same results as in Fig. 6. Comparisons with the results from some numerical methods, assisted in verifying the method of solution that is proposed in this research. Having established the method, it is intended to highlight the differences in the dynamic responses when the trains enter the bridge from the same sides or from the opposite sides.

4. The effect of the axle distances in the dynamic response of the bridge

In many pioneer studies dealing with the rail vehicle-bridge interactions, the distance between the bogie axles is not considered, (Yang *et al*. (2004)). In such cases, the bogie is modeled as a single load with the amount of load equal to twice of the axle load or simply equal to half of the total mass of the vehicle. It was actually a remedy to avoid increasing the number of the unit step functions in defining the entry and exit of the load onto the bridge. The effect of such simplification (limitation) for one way entry or opposing entry onto the bridge is investigated in this section of the research. In order to reach to the purpose and by using the time matrix, initially half of the bogie load enters the bridge and after some time delay the second half of the bogie load enters the bridge. Such time delay is equal to the ratio of the longitudinal axle box distance (*ABD*) and the vehicle speed of travel (*V*). Such practice is applied to all bogies of the 15 wagons that were included in the model. The data in Table 1 is used for the bridge and the rolling stock. The results are presented in Fig. 7.

Fig. 7 presents the dynamic response of the bridge for three different wheel distances. Fig. 7(a) is for the dynamic response of the bridge midpoint when the train enters from one side only. Fig. 7(b) presents the dynamic response of the bridge when two opposing trains traverse the bridge. It is clear from the results that ignoring the wheel distances has no major effect on the bridge dynamic response. The responses of the bridge are almost the same in both cases with slight phase differences that appear in the dynamic response after simplifications. Upon ignoring the axle distance in a bogie, it is assumed that the entire bogie load enters the bridge at one instant. When considering the axle distance, the bogie load enters the bridge in two instances. This can justify the slight phase differences that appear between the results. The above example is with the assumption that the speed of travel of the two trains is equal.

In what follows the assumptions of one load per bogie or two loads per bogie are further investigated. It is mixed with the possibility for the two trains opposing each other and traveling at unequal speed. The same data as in Table 1 are considered for the bridge and the rolling stock. The problem is solved four times with the results presented in Fig. 8. While the speed of travel of the first train is assumed to be the same, it is assumed that the 2nd train traverses once at a speed of 270 km/hr (V2=0.75V1) and again at a speed of 180 km/hr (V2=0.5V1). The trains enter the bridge from the opposite directions. Regarding the distance between the bogie axles, the problem is also surveyed by including and then ignoring the axle distances. When including the axle distances there are two loads per bogie. When ignoring such distance there is only one load per bogie. Based on such conditions, the dynamic response of the bridge midpoint is calculated and presented in Fig. 8. From these results it is clear that there is no noticeable sensitivity in the bridge midpoint dynamic response to the simplification in reducing the bogie load to one. In other words, when the speed of travel of the opposing trains is not the same, ignoring the axle distance does not generate noticeable error in the results.

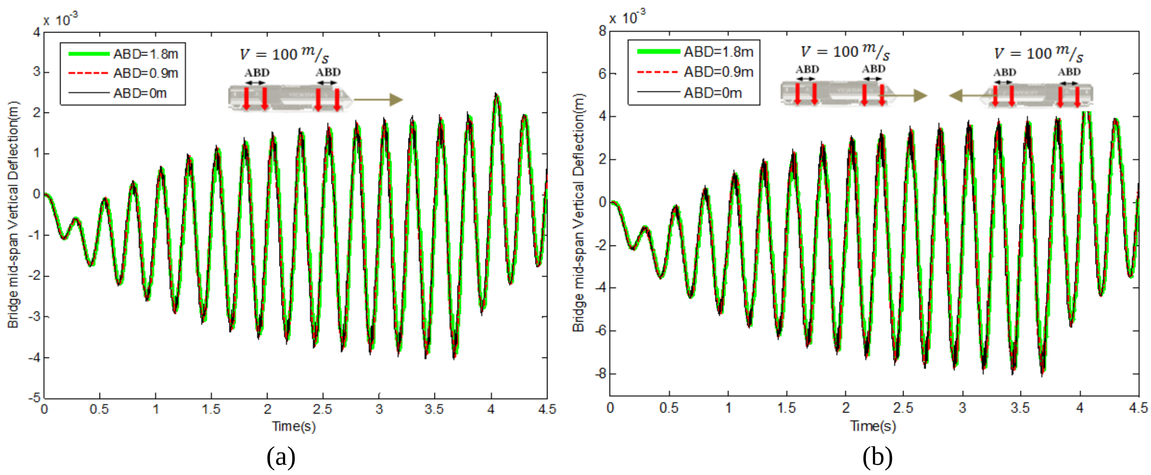


Fig. 7 The effect of including the distance between the bogie axles in the dynamic response of the bridge midpoint (for three different axle distances) (a) Only one train traversing the bridge (b) Simultaneous entry of two opposing trains at similar speed

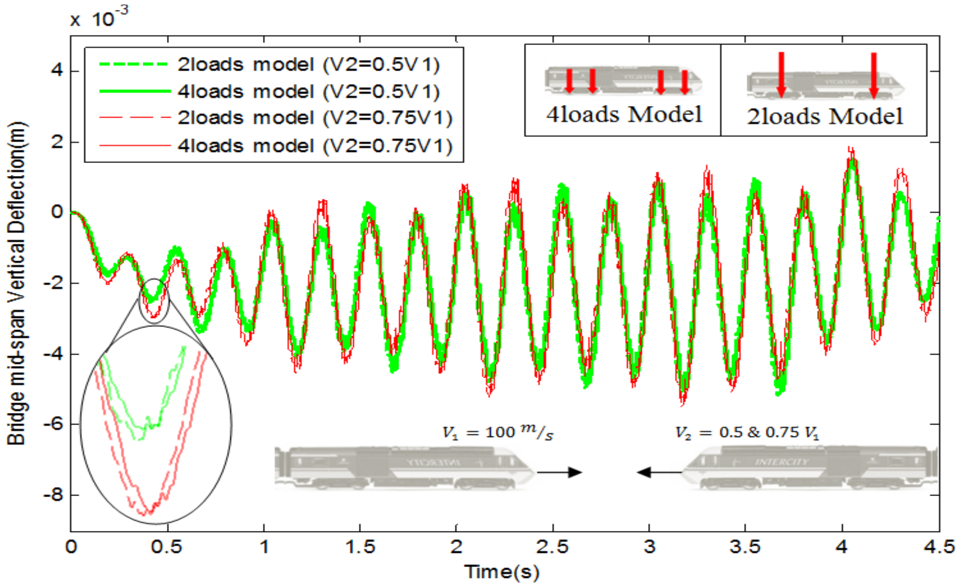


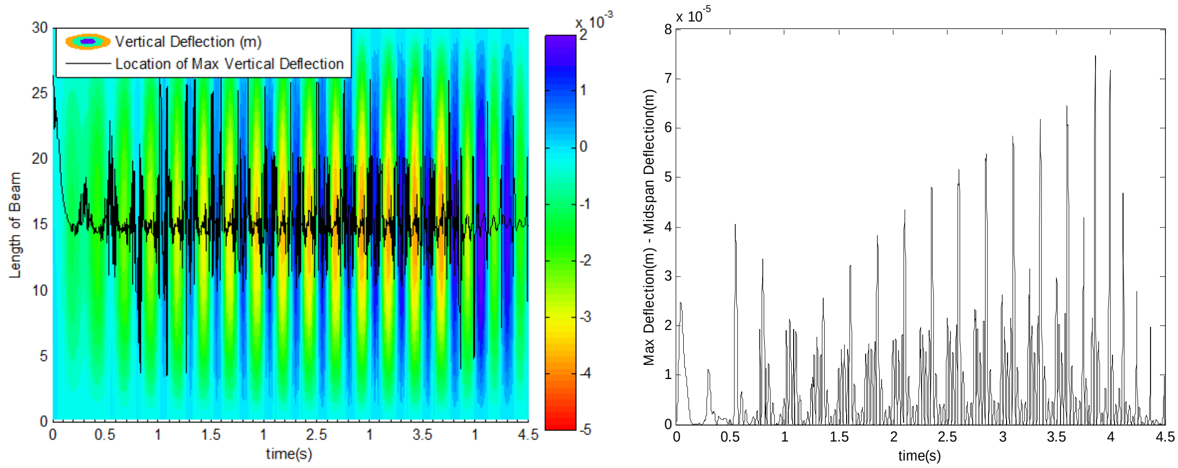
Fig. 8 The bridge calculated midpoint vertical displacement with the assumption of one load per bogie (vehicle carrying 2 loads) and two loads per bogie (vehicle carrying 4 loads) The vehicles traveling at the same or at different speed

5. The location of the maximum bridge deflection

So far, the adequacy of the proposed computer assisted analytical method is established. By comparison, it is accurate and reliable in predicting the dynamic response of the railway bridge under the influence of a combination of moving loads. Now, it is time to study the effect of varying some important parameters that are considered as the input to the model. Such parameters represent the bridge and the vehicle specifications. While studying the bridge dynamic response, it is often a good practice to introduce an index that can clearly define its behavior. In many references and also in the opening section of this paper it is mentioned that the deflection of the bridge at its midpoint is a proper index in order to study its dynamic response. Such an assumption is based on the importance of the odd mode shapes especially the 1st mode shape in comparison with the other modes. Any variation in the input data that may cause an increase in the maximum value of the midpoint displacement is undesirable and vice versa. The adequacy of such a criterion is investigated in this section. This includes the case of one way traverse or the simultaneous two way traverse of the rolling stocks over the bridge.

5.1 Direction of travel and the bridge maximum deflection

With this purpose in mind, the data in Table 1 is used to calculate the displacement of points along the bridge. In this case there is only one train consisting of 15 vehicles that crosses the bridge. It enters the bridge from the left side at time t=0. The results are calculated for 4.5 seconds that gives the train enough time to clear out of the bridge. The results are presented in Fig. 9.



(a) (b)

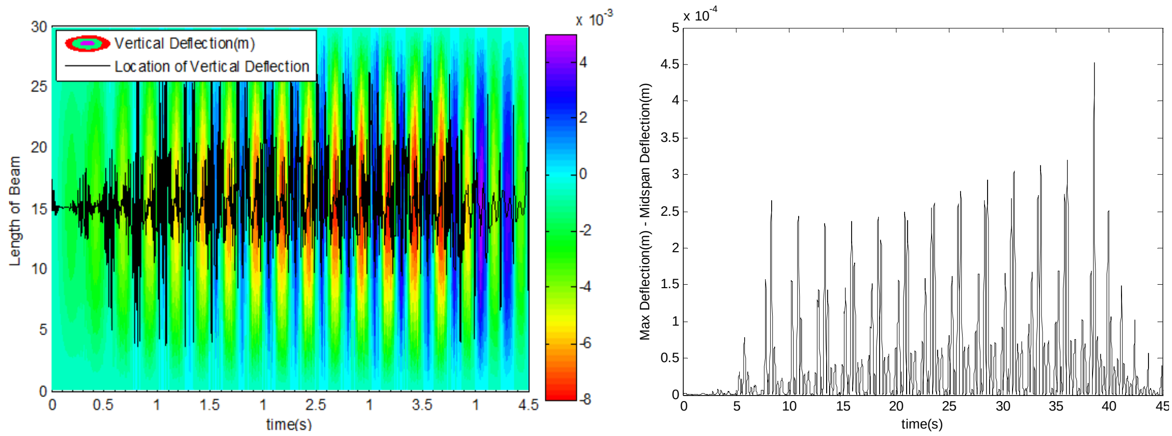
Fig. 9 (a) The calculated bridge deflection and the instantaneous location of the point with maximum deflection for the one sided entry of the train (b) The difference between the maximum absolute vertical displacement and the vertical midpoint displacement

Two parameters of interest are evaluated including the bridge midpoint displacement and the point with the maximum displacement. Such points may not necessarily be the same. Fig. 9(a) presents the spectrum for the vertical displacement of all points on the bridge against the time of the travel. Also the black line in this spectrum presents the time and position for the location of the point with maximum deflection on the bridge. According to many references concerned with this subject, the bridge midpoint displacement is a good indicator and an index of its behavior. In the above example, while the load traverses the bridge, if the maximum vertical displacement was always happening at the midpoint the black line existing in the spectrum in Fig. 9(a) should have stayed as a straight line passing through the bridge midpoint at $x=15$ m. Severe variations of this line around the bridge midpoint are obvious. Therefore, the maximum vertical displacement even in the case of a series of loads entering from one side does not always happen at the bridge midpoint. Such a point of maximum bridge displacement can also happen near the bridge supports. However, yet it is not fair to question the adequacy of the results that are presented in the reference papers. Even though, the points of the maximum bridge vertical displacement sometimes move away from its midpoint it is possible to prove that this happens only when the size of the displacements are very small. Fig. 9(b) presents the difference between the maximum bridge vertical displacement and its midpoint displacement against time. In this case, the maximum for such differences is less than 4% of the maximum vertical displacement during the whole process of the train traversing the bridge. Therefore, when the train enters the bridge from one side only, the point of maximum vertical displacement does not always lie at its midpoint. However, in the moments when the bridge undergoes large deflections the point of maximum deflection lies at the bridge midpoint. Therefore, the bridge midpoint can be a fair representative for its dynamic response during the passage of a series of loads on the one sided entry. To continue on, with the same data as in Table 1, two trains simultaneously enter the bridge. Each train consists of 15 similar vehicles. The results for the dynamic response of the bridge are presented in Fig. 10. It presents a time based spectrum for the vertical displacement of points along the bridge. It also presents the points with maximum displacement and the difference between the vertical displacement of the midpoint and the absolute maximum vertical displacement. Under the conditions of the simultaneous traverse of the two trains entering from the opposite sides, the bridge will be in a symmetrical condition at all times. Therefore, the spectrum for the vertical displacement of the points on the bridge needs to be symmetrical around the midpoint of the bridge at location $x=15$ m. Fig. 10(a) shows that the maximum vertical displacement does not happen at the bridge midpoint, at all time. It may happen at some other points, as well. The difference between the two cases of only one train entering from the left side and opposing movement of the trains presented in Figs. 9-10 lies in the size of the difference between the maximum displacement and the midpoint displacement. Contrary to the one way moving, when there are two trains traversing the bridge the difference between the maximum vertical displacement and the vertical displacement of the midpoint can be noticeable. It happens to be about 6% of the maximum bridge displacement, for this sample case. This is observed during the whole process of the train traversing the bridge. Such a phenomenon can be attributed to the cancelation of the even modes of vibration and the resonance in the odd mode shapes under such moving loads circumstances.

Therefore, during the simultaneous opposing movement of the trains at similar speed, the bridge midpoint cannot be a proper representative to express out the dynamic specifications of the bridge. It will then be necessary to use the maximum displacement to judge about the bridge dynamics.

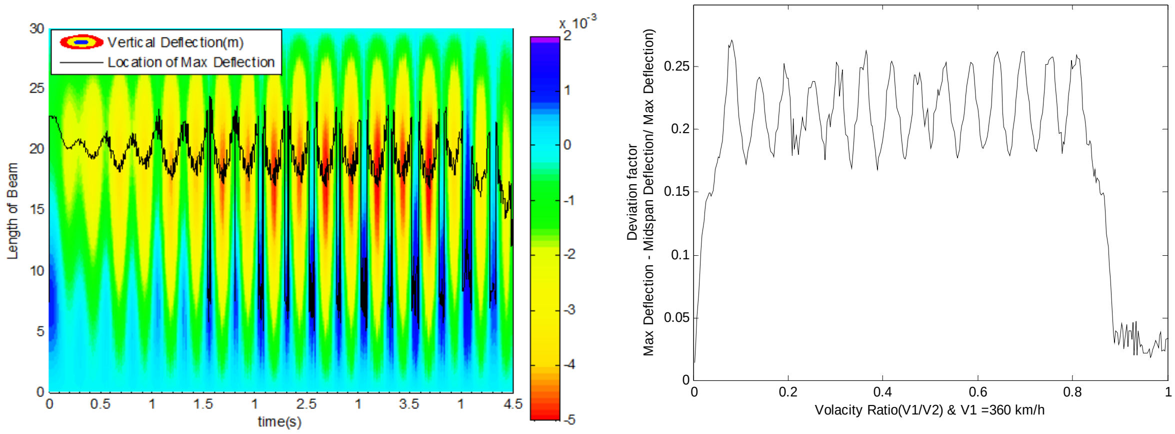
5.2 The vehicles speed ratio and the maximum bridge deflection

During the last exercise, it is also observed that the opposing movement of the trains over the bridge is considerably sensitive to the speed ratio between the two passing trains. Therefore, in this section the spectrum for the vertical displacement of the bridge is presented by varying the speed ratio of the moving vehicles. In the first instance the speed of travel of the 2nd train is half of that for the 1st train (V2=180 km/h). Also presented is the ratio of the difference between the



(a) (b)

Fig. 10 (a) The calculated bridge deflection and the instantaneous location of the point with maximum deflection when two opposing trains traverse the bridge at similar speed (b) The difference between the maximum absolute vertical displacement and the vertical midpoint displacement



(a) (b)

Fig. 11 (a) The vertical displacement of all points on the bridge and the location of the maximum deflection during the simultaneous entry of two trains at V_2=0.5V_1=180 km/h (b) The bridge deviation factor (*DF*) at V_1 = 360 km/h

maximum vertical displacement and the midpoint displacement to the maximum bridge displacement. This is calculated and reported as a deviation factor (*DF*) that is a function of the 2nd train's velocity, according to Eq. (14)

$$DF(V_2) = \max\left(\frac{Max \quad deflection - Mid \quad span \quad deflection}{Max \quad deflection}\right) \tag{14}$$

The corresponding results are presented in Fig. 11.

The outcome is contrary to the one way trip or the two way trips of trains at similar speed. In such cases the location of the maximum displacement hovered around the bridge midpoint. However, in the present case the locus of the points with maximum deflection moves away from the bridge support at the side of the entry of the faster train and gets closer to the support at the side of the entry of the slower train. As an example, in this case that the speed of the 2nd train is half of it for the 1st train the locus of the points with maximum displacement drifts around the line going through $x=20$ m. Consequently, the maintenance of two-way bridges needs more rigorous testing. The search for the cracks is needed at points other than the midpoint and nearer to the supports on the side of the entry of the slower trains. Fig. 11(b) is a more general form of presenting the bridge dynamics. The horizontal axis represents the speed ratio of the 2nd train to the 1st train. The vertical axis is the deviation factor (*DF*) as defined in Eq. (14). In this figure, when the speed ratio is small, the conditions are equivalent to the conditions of one way trip. In this case at the worst condition, the maximum bridge displacement has approximately 4% deviation from the bridge midpoint displacement. However, by increasing the speed of the 2nd train such deviation increases sharply to such extents that at some speed the maximum bridge displacement can be 27% more than the maximum midpoint displacement. Since the speed of travel is not the same, the excitation frequencies become variable and all modes of vibrations of the bridge are excited. At this situation, the even modes of vibration may experience resonance and the odd modes may experience cancellation. Therefore, the maximum displacement can happen at any location and it can be higher than the size of the maximum midpoint displacement. As the speed ratio gets closer to 1, meaning that the speed of the 2nd train nears the speed of the 1st train, the size of the deviation factor decreases and eventually reaches to 6%. Such reduction is due to the increase in the symmetry in the excitation that pushes the maximum displacement toward the bridge midpoint and its size nears the midpoint displacement.

6. The conditions of resonance and cancelation

Yang *et al*. (2004) reported attempts to identify the conditions that end up to the resonance phenomenon. In the search for the resonance, he included the vehicles' speed ratio and the ratio of the bridge length to the length of each vehicle in the "train consist".

It is also of interest to this research to study the conditions of resonance and cancellation under the simultaneous traverse of two trains at different speed ratios. The data in Table 1 are used. As an indicator of the geometry of the traversing rolling stock, the distance between the two bogies that represents the length of the vehicle is variable. While the speed of travel of the 1st train is kept constant at 360 km/hr the speed ratio between the trains is variable. The dynamic response of the bridge under such general input conditions is calculated and presented in Figs. 12(a)-(b).

The maximum bridge displacement during the whole process of the train passage is used as a parameter that highlights the resonance and cancellation conditions. Fig. 12 includes the different speed and length ratios. The bridge is 30 m in length and the two trains have the same geometries. The abscissa in Fig. 12(b) is the trains speed ratio and the ordinate is the ratio of the bridge length to the distance between the bogies. The conditions of resonance and cancellation can clearly be noted in this figure. It also shows that there are no distinctions between the cancellation and the resonance zones. With any slight change in the passing trains' geometry or any change in the trains' speed ratio the dynamic response can cross over from the cancellation zone into the resonance zone (moving from the blue zone to the red zone in Fig. 12(b)). Generally by having the results as in Fig. 12(b) and by noticing the geometry of the passing train one can adjust the speed ratio in such a way that minimizes the damage to the bridges during the simultaneous traversing of the trains. The bridge design needs to be such that in case if there is any variation in the speed ratio, the bridge does not fall into the resonance. In the case of this example, the choice of a rolling stock with the ratio of the bridge length to the bogie distances about 0.78 or higher than 1.7 can be helpful. Such results are based on the assumption that the two traversing trains that enter the bridge from the opposite sides are of the same type with the same geometrical specifications.

In order to observe the effect of working with dissimilar traversing trains, another set of results are obtained. It is assumed that the distance between the bogies of the 2nd train is constant at 12 m. With the assumption of 30 m for the bridge length, the distance between the bogies of the 1st train varies with the length to distance l/d ratio. The distribution of the bridge maximum displacement is presented in Fig. 13. It is shown that if the distance between the bogies in a train is kept constant the conditions of resonance and cancellation will be fairly independent from the train speed of travel. It is clearly more dependent on the train geometrical specifications. In this case, design with the distance ratio of 0.78 or higher than 1.7 increases the possibility of falling into the cancellation conditions. Another interesting outcome is the cancellation condition that happens at the speed ratio higher than 0.7. Fig. 13 shows that during the simultaneous traverse of the trains, the excitation on the bridge is so complex that one cannot be confident in claiming that reducing the speed of travel is of any benefit to the bridge.

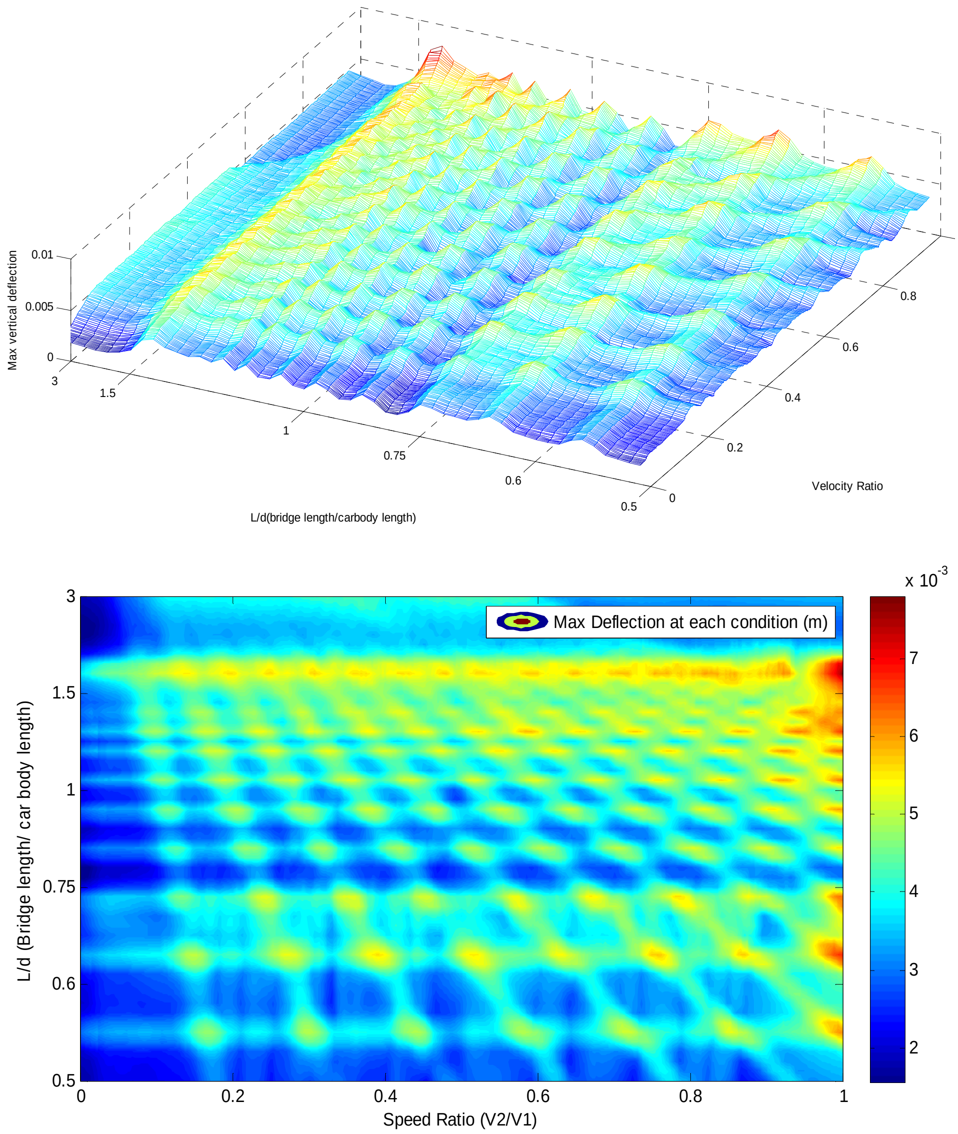


Fig. 12 Variations in the bridge maximum displacement versus the speed and length ratios for the case of similar opposing trains

It needs to be reminded that there are many more interesting situations that are worth investigating. Such are the effects of varying the distances between the axles of the bogies, or the distances between the vehicles. The effect of varying the vehicle axle load on the bridge dynamic response is also important. It is all with the purpose of reaching at proper values for the speed of travel over the bridge and the geometrical parameters that can maximize the bridge life time, as much as possible.

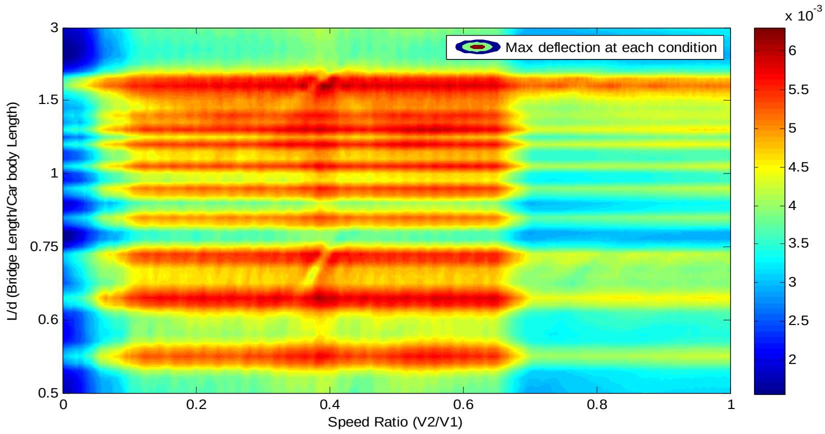


Fig. 13 Variations in the bridge maximum displacement versus the speed and the length ratios for the case of dissimilar opposing trains

7. Conclusions

In this research the dynamic response of the railway bridge during the simultaneous traverse of the series of moving loads investigated. The problem solved by developing a computer assisted analytical method. In comparison with other methods of solution, the proposed method exclusively benefits from offering an analytical closed form solution that also uses computers and some numerical procedures in order to facilitate and speed up its course of actions. It combines the accuracy of the closed form solutions and the speed of calculations that can be provided by computers. Also as another advantage, the effects of damping and many modes of vibrations in the dynamic response of the bridge are included. The proposed algorithm facilitates dynamic analysis of the vehicle-bridge interaction by allowing opposing loading scenarios and considering variations in the length and the velocity ratios. It also provides flexibility in defining the train geometrical particulates. To the knowledge of the authors, none of the available methods provide so much possibilities combined together that can also provide accurate, reliable and fast responses to the issue of the rail vehicle-bridge dynamics.

In the process, the differences in the dynamic response in comparison with the case of one way trip of series of loads discussed. The adequacy of the method established by comparing the results with the results reported in the corresponding literature. It is concluded that during the simultaneous entry of the trains onto the bridge, with the same or at different speed of travel, the assumption of a single load per bogie causes no major error in the results. However, this can be jeopardized if there is any delay between the entries of the two trains.

The location of the point with maximum deflection on the bridge that interacts with the railcar was also investigated. It is concluded that such a point does not always lie at the bridge mid-span. Contrary to the one way moving, when there are two trains traversing the bridge the difference between the maximum vertical displacement and the vertical displacement of the midpoint can be noticeable. The speed ratio of the traversing crossing trains also has noticeable effect on shifting the location of the point with maximum deflection on the bridge, away from its midpoint.

Eventually by simultaneously varying the ratio of the bridge length to the bogie axle distance (*l/d*) the conditions of cancellation and resonance were investigated. It is shown that by proper design for the bridge, the proper choice for the rolling stock and by controlling the speed of travel it is possible to lower the possibility for the resonance phenomenon during the bridge operation.

The railway bridges with two-way traffics are more susceptible to damage. Very often they face the similar type of the fleet though, not guaranteed. It is a fruitful exercise to study the dynamic response of the bridge and its conditions for resonance and cancellation. It will then be possible to choose the speed of travel and the vehicle-bridge geometrical parameters in such a way that lowers the bridge dynamic loading and elongates its life span. Introduction of the time delay between the entries of the opposing trains onto the bridge can alter the results. This is the subject of further studies by the authors and is not reported in this article.

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Appendix 1

Definitions for the terms that are used in the computer program flowchart, presented in Fig. 4.

d1B is the distance between the bogies in a railcar in the 1st train in meters
d2B is the distance between the bogies in a railcar in the 2nd train in meters
d1C is the distance between the railcars in the 1st train in meters
d2C is the distance between the railcars in the 2nd train in meters
d1ba is the distance between the wheelsets of a bogie in the 1st train in meters
d2bs is the distance between the wheelsets of a bogie in the 2nd train in meters
td is the time delay between the entry of the 1st and the 2nd train in seconds
tf is the length of time for calculations in seconds
N is the number of railcars in a train
Z is a row vector of size (1×16*N*) that defines the sequence in the wheels loading of the bridge
Tt is a row vector of size (1×16*N*) associated with Z and defines the time of occurrence of each wheel loading
n_e is the event counter
As & *Ac* are the coefficients of the Sine and Cosine terms in the homogeneous solution of the bridge response
Bs & *Bc* are the coefficients of the Sine and Cosine terms in the particular solution of the bridge response
l is the length of the bridge in meters
V_1 is the speed of travel of the 1st train in km/h
V_2 is the speed of travel of the 2nd train in km/h
EI is the bridge flexural rigidity in $N.m^2$
m is the mass per unit length of the bridge in kg/m
P_1 is the axle load of the 1st train in tonne
P_2 is the axle load of the 2nd train in tonne
Wn is the un-damped natural frequency of the bridge in N.m/sec
wd is the damped natural frequency of the bridge in N.m/sec
alphaE is the external damping coefficient in N.m/sec
alphaI is the internal damping coefficient in N.m/sec
zeta is the total damping coefficient in N.m/sec
omega1 is the excitation frequency of the 1st train in rad/sec
omega2 is the excitation frequency of the 2nd train in rad/sec
n is a natural frequency counter
B'_c & B'_s are the coefficients of the associated cosine and sine terms in the particular solution
u is the multi-facet displacement function at each vibration mode
U is the bridge displacement (combination of the displacements at all vibration modes)