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Optimal design of Base Isolation System considering uncertain bounded system parameters

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Abstract. The optimum design of base isolation system considering model parameter uncertainty is usually performed by using the unconditional response of structure obtained by the total probability theory, as the performance index. Though, the probabilistic approach is powerful, it cannot be applied when the maximum possible ranges of variations are known and can be only modelled as uncertain but bounded type. In such cases, the interval analysis method is a viable alternative. The present study focuses on the bounded optimization of base isolation system to mitigate the seismic vibration effect of structures characterized by bounded type system parameters. With this intention in view, the conditional stochastic response quantities are obtained in random vibration framework using the state space formulation. Subsequently, with the aid of matrix perturbation theory using first order Taylor series expansion of dynamic response function and its interval extension, the vibration control problem is transformed to appropriate deterministic optimization problems correspond to a lower bound and upper bound optimum solutions. A lead rubber bearing isolating a multi-storeyed building frame is considered for numerical study to elucidate the proposed bounded optimization procedure and the optimum performance of the isolation system.

Keywords: stochastic earthquake; Base Isolation; bounded uncertainty; optimization

1. Introduction

During the last decades various seismic protection techniques have been emerged as viable alternatives to the traditional aseismic design. The traditional design relies on the energy dissipation by inelastic deformations of structural elements for mitigating the damaging effects of earthquakes through the introduction of flexibility and/or energy absorption capability within the structural system itself. In contrast to such traditional means, the basis of protective technique is limiting or eliminating inelastic action and damage to the structures, reduction of forces for design of foundation and, under certain conditions, reductions of accelerations and protection of non-structural components.Extensive research works have been done in the area of vibration control to mitigate the vibration effect of structures (Housner *et al.* 1997, Soong and Dargush 1997, Baratta and Corbi 2002, 2003, Spencer and Nagarajaiah 2003).Various control devices include tuned mass damper, fluid viscous damper, viscoelastic damper, friction dampers, base isolation (BI) system,

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metallic yield devices, tuned liquid mass damper etc. Amongst these, the seismic isolation is one of the most widely used and successfully utilized design schemes. In fact, it has been found widespread implementation and global acceptance by the profession as an effective technology to control the vibration effects on structures since late 1970s. The main characteristics of seismic isolators consist of horizontal flexibility and energy dissipation. The seismic protection is achieved by shifting the structural natural period far from the range of frequencies which are expected to have maximum amplification effects due to ground motion. Thereby, the shear forces transmitted to the base from the superstructure are reduced considerably. These devices adopt different materials and design methodologies in order to disconnect the superstructure motion from the ground. Many devices, such as Rubber Bearings, Lead Rubber Bearings (LRB), High Damping Rubber Bearings and Friction Pendulum, resilient friction bearing isolator etc. are available nowadays for seismic protection of buildings and bridges.

The effectiveness of BI systems and their performances have been extensively studied in the past (Kelly 1986, Buckle and Mayes 1990, Jangid and Datta1995, Symans and Constantinou 1999, Karabork 2011). The studies on stochastic response of BI system under random earthquakes providing insight into the behaviour of such systems are notable (Constantinou and Tadjbakhsh 1985, Lin *et al.* 1990, Jangid 2010). It is well established that the performance of BI system is largely dependent on the characteristics of the isolator parameters. Attempts are made in order to characterize such optimal parameters to ensure desired performance (Baratta and Corbi 2004, Matsagar and Jangid 2004, Jangid 2010).The most commonly used approach of designing BI system is to consider the earthquake load as the only source of randomness assuming all other system parameters as deterministic in nature.The standard optimization problem is formulated to minimize the stochastic response obtained by random vibration theory, referred as stochastic structural optimization (SSO). A major limitation of such deterministic approach is that the uncertainty in the performance-related decision variables cannot be included in the stochastic response analysis and the related optimization procedure. It has been demonstrated that the interplay among the parameter uncertainty and loading uncertainty (Jensen 2005) can markedly change the response of a system and thereby the safety of structure (Chaudhuri and Chakraborty 2004). The optimal design is also observed to be changed significantly by system uncertainty (Schuëller and Jensen 2008). In case of seismic vibration mitigation, the sources of uncertainty include both the structural system and the seismic actions. The frequency of the mechanical model representing the stiffness and mass distribution may be afflicted by significant variation during the service life of a structure for example in civil buildings or bridges. It is often difficult to predict the frequency accurately. In modelling of dynamic system, the proper characterization of energy dissipation process during the dynamic motion of a system is very difficult and depends on various interacting complex parameters. One would always expect to consider the presence of uncertainty in the damping properties of the structure. On the contrary, the stochastic spectra are traditionally used to consider the effect of random nature of seismic motion. The load model parameters are normally derived from few analyses on specific accelerograms which were subsequently generalized to a generic class of soils, such as rigid, medium and soft, simply referred to a single studied seismic event. But, in practical applications, the operators usually use lexical and formal criteria for their identification. It can be reasonably affirmed that proper evaluation of these parameters and the related uncertainty are indeed an essential topic for professional engineers. Thus, the seismic vibration mitigation utilizing BI system considering uncertain parameters is attracting growing interests in seismic safety study.

The developments in the field of passive vibration control considering system parameter uncertainty are notable (Papadimitriou and Katafygiotis 1997, Taflanidis *et al*. 2008a, Debbarma and Chakraborty 2010, Jensen and Sepulveda 2011). Juhn and Manolis (1992) have indicated that the effect of uncertainty with regard to BI parameters and the ground motion filter parameters cannot be ignored for accurate estimation of responses. Kawano *et al*. (2002) have studied the effects of uncertain parameter on the nonlinear dynamic response of BI system in the framework of Monte Carlo Simulation (MCS) and observed that the uncertain parameters have significant roles on the maximum response of BI system. Nagai and Nishitani (2005) studied the nonlinear vibration of BI system considering fluctuations in the parameters involved in such hysteresis system. The equivalent linearization technique combined with the perturbation approach is adopted for response statistic evaluation to estimate the safety and reliability of isolated buildings. Scruggs *et al*. (2006) proposed a probability based active control synthesis for seismic isolation of an eight-storey benchmark structure considering uncertain model parameters. Zhou *et al*. (2006), Zhou and Wen (2008) presented adaptive back stepping control algorithms for active seismic protection of building structures considering uncertain hysteretic behaviour, typically observed in BI system. Taflanidis *et al*. (2008) presented a stochastic-simulation-based nonlinear controller design for benchmark building with elastomeric and friction pendulum isolators considering probabilistic description of the ground-motion model parameters. Bucher (2009) presented a computationally efficient method for reliability based design optimisation of friction-based seismic isolation device in the framework of MCS and response surface method. In a recent study, Taflanidis and Jia (2011) presented a simulation-based framework for risk assessment and probabilistic sensitivity analysis of a three-story isolated structure by explicitly incorporating uncertainties in the excitation and or structural model.

The studies on BI system considering model parameter uncertainty as discussed above primarily use the total probability theory concept to obtain the unconditional response of the system which is subsequently used as the performance measure. Though, the probabilistic methods are powerful, the approach cannot be applied in many real life situations when the required detailed information about the uncertain parameters is limited. In many real situations, the maximum possible ranges of variations expressed in terms of percentage of the corresponding nominal values of the parameters are known and can be only modelled as uncertain but bounded (UBB) type parameters. In such cases, the convex models and interval analysis methods in which the bounds on the magnitude of the uncertain parameters are only required are a viable alternative. The interval analysis problems can be approximated to equivalent deterministic one through Taylor series expansion about the mean values of the uncertain model parameters to yield conservative response bounds (McWilliam 2001, Qiu and Wang 2003). For system possessing small degree of parameter uncertainty, the response can be considered to be monotonic and linear perturbation analysis will be valid. However, the applications of such interval analysis methods deal with response evaluation and optimization under deterministic load (Chen and Zhang 2006, Chen *et al*. 2007) and application to passive vibration control is very limited (Chakraborty and Roy 2011).

The present study focuses on the bounded optimization of BI system to mitigate the seismic vibration effect of structures considering UBB type system parameters. With this intention in view, the conditional stochastic response quantities are obtained in random vibration framework using the state space formulation. Subsequently, with the aid of matrix perturbation theory using first order Taylor series expansion of dynamic response function and its interval extension, the vibration control problem is transformed to appropriate deterministic optimization problems. This requires two separate objective functions correspond to a lower and upper bound optimum solutions. An LRB system isolating a multi-storeyed building frame is considered for numerical study to elucidate the proposed optimization procedure and the effect of optimum performance of BI system.

2. Response of base isolated building frame under random earthquake

For efficient development of the proposed optimization procedure for BI system to mitigate the vibration of structure due to stochastic earthquake load considering UBB type system parameters, the description of the mechanical model of the BI system and the equivalent linear stochastic dynamic analysis by state space formulation in time domain is first briefly introduced in this section.

2.1 Description of the BI system

A two dimensional building frame structure, isolated by LRB is considered in the present study. It is idealized as a shear building type model with attached isolator as shown in Fig. 1(a). The idealized mechanical model of the isolator along with its idealized force-deformation behaviour is depicted in Fig. 1(b) and Fig. 1(c). As the BI system substantially reduces the structural response, the superstructure under consideration can reasonably be assumed to be linear. However, the hysteretic energy dissipation in the LRB occurs through large shear deformation and yielding of the lead core. Consequently, the behaviour of LRB is highly non-linear and modelled accordingly. The equation of motion of the N-storey superstructure subjected to horizontal component of earthquake ground motion ($\ddot{x}_g$) can be written as

$$[\mathbf{M}]\{\ddot{x}\}+[\mathbf{C}]\{\dot{x}\}+[\mathbf{K}]\{x\}=-[\mathbf{M}]\{\mathbf{r}\}\left(\ddot{x}_g+\ddot{x}_b\right) \tag{1}$$

where, [**M**], [**K**] and [**C**] are the matrices of size N representing the mass, stiffness and damping matrices of the superstructure, $\{x\}=\{x_1 \quad x_2 \quad .... \quad x_N\}^T$ is the displacement vector containing the lateral displacement of each floor relative to the isolator, as shown in Fig. 1(a). The influence coefficient vector {**r**} represents the pseudo-elastic deformation of the respective floor under a unit deformation of ground. $\ddot{x}_b$ is the relative acceleration of the isolator with respect to the ground due to ground acceleration.
The governing equation of motion of the isolator mass (Fig. 1(b)) can be expressed as

$$m_b\ddot{x}_b+c_b\dot{x}_b+F_b-c_1\dot{x}_1-k_1x_1=-m_b\ddot{x}_g \tag{2}$$

where, m_b is the mass of the isolator, c_b is the viscous damping of the LRB, k_1 and c_1 are the stiffness and damping of the first storey. F_b is the restoring force of the isolator modelled by the differential Bouc-Wen model (Bouc1967, Wen 1976). The bi-linear force-deformation behaviour of the LRB, adopted herein is expressed as

$$F_b\left(x_b,Z\right)=\alpha k_b x_b+\left(1-\alpha\right)F_Y Z \tag{3}$$

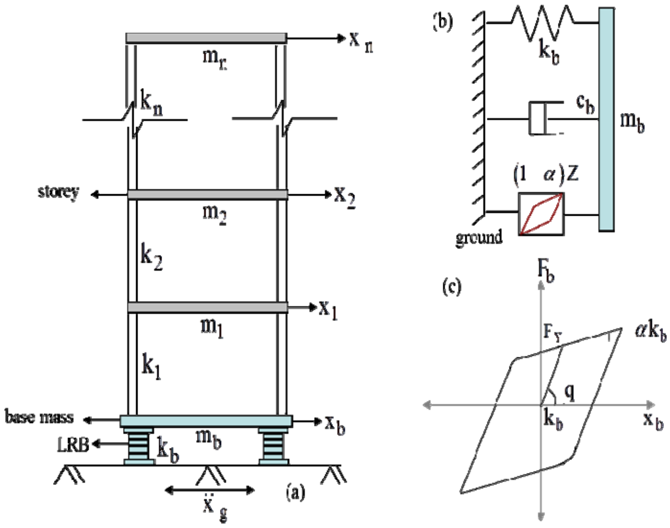

Fig. 1 (a) The idealized building frame with LRB (b) Idealization of the LRB isolator and (c) The bi-linear hysteretic model of the isolator

where, k_b is the initial elastic stiffness, x_b is the displacement of the LRB and α is an index representing the ratio of the post to pre yield stiffness of the LRB, referred as rigidity ratio. F_y is the yield strength of the isolator. Z is a variable quantifying the hysteretic response of the isolator, expressed through Bouc-Wen model

$$q\dot{Z} = -\gamma|\dot{x}_b|Z|Z|^{\eta-1} - \beta\dot{x}_b|Z|^{\eta} + \delta\dot{x}_b \tag{4}$$

where, q is the yield displacement of the isolator. The five parameters β,γ,η,α and δ appears in Eq. (4) characterize the shape of the hysteretic loop. Parameter η controls the transition from the elastic to plastic phase; with $\eta \to \infty$ (infinity) the behaviour becomes bilinear. β controls the nature of the model e.g. $\beta > 0$ implies hardening and $\beta < 0$ results softening. The parameters adopted in the present study are $\alpha = 0.05, \beta = \gamma = 0.5, \delta = 1$ and $\eta = 1$, which corresponds to the bi-linear force deformation characteristics as shown in Fig.1c. The post-yield stiffness αk_b of the isolator is selected in order to provide specific isolation time period, $T_b = 2\pi\sqrt{M/\alpha k_b}$, M is the total mass of the isolator-superstructure system, given by the sum of all the floor mass (m_i)and the mass of the LRB system (m_b). The viscous damping of the isolator is given by, $c_b = 2\xi_b M\omega_b$ in which, ξ_b is the viscous damping ratio and ω_b is the frequency of the isolator. The yield strength is conveniently normalized with respect to the total weight of the structure ($W = Mg$) and normalized yield strength is denoted as, $F_0 = F_Y/W$, g is the gravitational acceleration.

The present work is intended to study the bounded stochastic optimization of BI system in mitigating the seismic vibration effect of structures considering UBB type system parameters characterizing the mechanical model of the BI system and the stochastic earthquake load model.

The nonlinear force-deformation characteristic of the LRB as represented by Eq. (4) is too complicated to be readily incorporated in the state-space formulation for evaluating the response of the BI system accounting for the fluctuation involve due to system parameter uncertainty. The statistic response evaluation is conducted by utilizing the techniques of statistical linearization (Roberts and Spanos 2003, Hurtado and Barbat 2000). The equivalent linear form of the nonlinear Eq. (4) can be obtained as

$$q\dot{Z} + C_e\dot{x}_b + K_e Z = 0 \tag{5}$$

where, C_e and K_e are the equivalent damping and stiffness obtained by the least square error minimization between the linear and nonlinear terms of Eqs. (5) and (4). For $\eta = 1$, the equivalent damping and stiffness of the isolator can be obtained in closed form as

$$C_e = \sqrt{\frac{2}{\pi}}\left\{\gamma\frac{E[\dot{x}_b Z]}{\sqrt{E[\dot{x}_b^2]}} + \beta\sqrt{E(Z^2)}\right\} - \delta,\ K_e = \sqrt{\frac{2}{\pi}}\left\{\gamma\sqrt{E[\dot{x}_b^2]} + \beta\frac{E[\dot{x}_b Z]}{\sqrt{E[Z^2]}}\right\} \tag{6}$$

where, E[ ] is the expectation operator. In stochastic linearization, the responses $(x_b, \dot{x}_b)$ of the system are assumed to be jointly Gaussian. This does not result in serious error so far the stochastic response evaluation is concerned (Roberts and Spanos 2003). It is noted that even though the differential Bouc-Wen model equation of the isolator is stochastically linearized (Eq. (5)) for easy incorporation in the state space equations, the relevant equivalent damping (C_e) and stiffness (K_e) are still functions of the system response. This implies that the nonlinearity of isolator is still present in the response equation.

2.2 Response covariance analysis

In principle, for realistic seismic reliability analysis of structure subjected to random earthquake requires the records of the ground motions at a site. In absence of sufficient statistical data, available stochastic models for earthquake loading are usually utilized. The well-known Kanai-Tajimi stochastic model (Kanai 1957, Tajimi 1960) which characterizes the input frequency content for a wide range of practical situations is adopted in the present study. The process of excitation at base can be expressed as

$$\ddot{x}_f + 2\xi_f\omega_f\dot{x}_f + \omega_f^2 x_f = -\ddot{w},\ \ \ddot{x}_g = \ddot{x}_f + \ddot{w} = -2\xi_f\omega_f\dot{x}_f - \omega_f^2 x_f \tag{7}$$

where, $\ddot{w}$ is the white noise intensity at the rock bed with PSD S_0, ω_f and ξ_f are the frequency and damping of the ground representing the soil strata over the rock bed and underlying the building. $\ddot{x}_f$, $\dot{x}_f$ and x_f are the acceleration, velocity and displacement response of the Kanai-Tajimi filter.

The superstructure, isolator and the filter equations are now rearranged to express those in state space form suitable for stochastic response evaluation. Substituting Eq. (3) in Eq. (2) and normalizing with respect to m_b, the equation of isolator Eq. (2) can be written as

$$\ddot{x}_b + \frac{c_b}{m_b}\dot{x}_b + \alpha\frac{k_b}{m_b}x_b + \frac{(1-\alpha)F_y}{m_b}Z - \frac{c_1}{m_b}\dot{x}_1 - \frac{k_1}{m_b}x_1 = -\ddot{x}_g \tag{8}$$

Multiplying both sides of Eq. (1) with $[\mathbf{M}]^{-1}$and substituting the expression of $\left(\ddot{x}_g+\ddot{x}_b\right)$ from the above, Eq. (1) can be rewritten as

$$\{\ddot{x}\}=-[\mathbf{M}]^{-1}[\mathbf{C}]\{\dot{x}\}-[\mathbf{M}]^{-1}[\mathbf{K}]\{x\}+\{\mathbf{r}\}\left(\frac{c_b}{m_b}\dot{x}_b+\alpha\frac{k_b}{m_b}x_b+\frac{(1-\alpha)F_y}{m_b}Z-\frac{c_1}{m_b}\dot{x}_1-\frac{k_1}{m_b}x_1\right) \tag{9}$$

Substituting the expression of $\ddot{x}_g$ from filter Eq. (7) in Eq. (8), the equation of base mass/isolator of Eq. (2) can be finally obtained as

$$\ddot{x}_b=-\frac{c_b}{m_b}\dot{x}_b-\alpha\frac{k_b}{m_b}x_b-\frac{(1-\alpha)F_y}{m_b}Z+\frac{c_1}{m_b}\dot{x}_1+\frac{k_1}{m_b}x_1+2\xi_f\omega_f\dot{x}_f+\omega_f^2x_f \tag{10}$$

The linearized equation for the hysteretic isolator, obtained through stochastic linearization depicted by Eq. (5) can be rewritten as

$$\dot{Z}=-\frac{C_e}{q}\dot{x}_b-\frac{K_e}{q}Z \tag{11}$$

Eq. (7) can be rewritten as

$$\ddot{x}_f=-2\xi_f\omega_f\dot{x}_f-\omega_f^2x_f-\ddot{w} \tag{12}$$

Now, Eqs. (9) to (12) are expressed in the state space form. Introducing the state vector containing the variables as: $\{Y\}=\left[\{x\}\quad x_b\quad Z\quad x_f\quad \{\dot{x}\}\quad \dot{x}_b\quad \dot{x}_f\right]^T$, the state space equation can be obtained as

$$\frac{d}{dt}\{Y\}=[A]\{Y\}+\{w\} \tag{13}$$

where, [A] is the augmented system matrix and $\{w\}=\left[\{0\}\quad 0\quad 0\quad 0\quad \{0\}\quad 0\quad -\ddot{w}\right]^T$. {Y} has the length of (2N + 5), N is the number of structural degrees of freedom. The details of the augmented [A] matrix and {W} are furnished in the appendix.

The response of the system can be evaluated by solving Eq. (13) by numerical Runge-Kutta integration method. In stochastic analysis, rather than the response, the statistics such as covariance of responses are evaluated. Assuming the stochastic response processes to be Markovian, the evolution equation for the response covariance matrix $[C_{YY}]$ of the state vector {Y} can be readily obtained as (Lutes 1997)

$$\frac{d}{dt}[C_{YY}]=[A][C_{YY}]^T+[C_{YY}][A]^T+[S_{ww}] \tag{14}$$

The elements of $[C_{YY}]$, having dimension of (2N + 5, 2N + 5) is given by $C_{YiYj}=E[Y_iY_j]$. $[S_{WW}]$ is the covariance matrix of the rock bed white noise intensity. Following the structure of {W}, the matrix $[S_{WW}]$ has all terms zero except the last diagonal, given by $2\pi S_0$. It may be noted that the equivalent linear stiffness and damping are functions of the responses itself. Thus, for the solution of Eq. (14) by Runge-Kutta integration technique, these terms are required to be updated in each iteration following the response statistics of the previous step until convergence. Assuming statistical independency of the state variable {Y} and rock bed white noise excitation vector {W}, the response statistics of the derivative process can be obtained from

$$\left[C_{\dot{Y}\dot{Y}}\right]=[A][C_{YY}][A]^{T}+[S_{ww}] \tag{15}$$

The root mean square (rms) responses are obtained from the covariance of the response as

$$\sigma_{Y_i}=\sqrt{C_{Y_iY_i}} \tag{16}$$

The absolute floor acceleration $\left(\ddot{u}_N\right)$ is the summation of the relative floor, base and ground acceleration. Thus, the absolute rms acceleration (rmsa) at the top floor can be obtained as,

$$\sigma_{\ddot{u}_N\ddot{u}_N}=\sqrt{C_{\dot{Y}\dot{Y}}\left(2N+3,2N+3\right)+C_{\dot{Y}\dot{Y}}\left(2N+4,2N+4\right)+C_{\dot{Y}\dot{Y}}\left(2N+5,2N+5\right)} \tag{17}$$

3. Optimal design of BI system: conventional stochastic structural optimization

The objective function typically considered in the conventional SSO is the rms responses (displacement, acceleration, stress etc.) or exceedance of some predefined serviceability or strength limit state by the structural performance variables. In the present study, the top floor rmsa of the building as defined by Eq. (17) is used as the objective function. From the description of the isolator model it is apparent that the isolation time period (T_b), viscous damping coefficient (ξ_b) and the normalized yield strength (F_0) are the characteristic design variables of the BI system. However, it is seen that the responses monotonically vary with the first two parameters i.e. isolator time period (T_b) and damping (ξ_b); whereas the isolator normalized yield strength (F_0) possesses optimum value to ensure minimum responses (Baratta and Corbi 2004, Jangid 2010).Thus, F_0 is taken as the design variable in the optimization study.

The response of the BI system being a nonlinear function of the design variables, it requires the solution of a nonlinear optimization problem. The SSO for optimal design of BI system subjected to stochastic ground motion is thus transformed into a standard nonlinear programming problem (Nigam 1972) and can be stated as,

$$\text{Find } F_0 \text{ to minimize } \sigma_{\ddot{u}_N}\left(\overline{\theta}_i\right) \tag{18}$$

It may be noted from above that the objective function depends on the system parameters. In the conventional stochastic optimization as presented above assumes those system parameters ($\overline{\theta}_i$) as deterministic. Thus, it is obvious that the optimum BI configuration obtained by solving above is conditional.

4. Optimal design of BI system under system parameter uncertainty

The system matrix [**A**] as described by Eq. (A.1) in appendix involves system parameters which include the characteristic of the building, isolator and the ground motion model parameters describing the stochastic earthquake load. The response statistic evaluated under stochastic earthquake load intuitively assumes that these system parameters are deterministic. But, the uncertainties may cause significant deviations of the various system parameters from their assumed deterministic values. As a result of which, the commonly used SSO procedures disregarding the presence of system parameter uncertainty may lead to an improper design and catastrophic consequences in many cases (Zhao *et al*. 1999, Chaudhuri and Chakraborty 2006). Therefore, apart from the stochastic nature of the earthquake load**,** uncertainty with regard to the system parameters are expected to have influences in the optimum design of BI system and should be considered properly in the design. Consideration of such parameter uncertainty in the analysis procedure will involve sensitivity analysis of stochastic dynamic system (Chaudhuri and Chakraborty 2004, Jensen 2005). In the present section the related formulations are briefly presented to elucidate the proposed optimal study of BI system considering UBB type system parameters.

4.1Stochastic sensitivity analysis

The uncertainty considered in the present study in the parameters of the structure, isolator and the stochastic earthquake load model are denoted as

$$\{\theta\} = \begin{bmatrix} k & c & k_b & c_b & F_Y & \xi_g & \omega_g & S_0 \end{bmatrix}^T \tag{19}$$

where, k is the stiffness of each storey, c is the damping in each storey of the superstructure. k_b is the stiffness, c_b is the damping and F_Y is the yield strength of the LRB. The mass, stiffness and damping ratios of each floor of the building are assumed to be identical, for simplicity. However, the applicability of the present formulation is not restricted to such simplification and can tackle different combination of these parameters.

The evolution equation for the first order sensitivity of response is obtained by differentiating Eq. (14) with respect to the i-th parameter θ_i. On rearranging the terms, the equation can be written as

$$\frac{d}{dt}\left[\frac{\partial C_{YY}}{\partial \theta_i}\right] = [A]\left[\frac{\partial C_{YY}}{\partial \theta_i}\right]^T + \left[\frac{\partial C_{YY}}{\partial \theta_i}\right][A]^T + [B] \tag{20a}$$

where

$$[B] = \left[\frac{\partial A}{\partial \theta_i}\right][C_{YY}]^T + [C_{YY}]\left[\frac{\partial A}{\partial \theta_i}\right]^T + \left[\frac{\partial S_{ww}}{\partial \theta_i}\right] \tag{20b}$$

In the above, $\partial C_{YY}/\partial \theta_i$ is the sensitivity of the response covariance (C_{YY}) with respect to the parameter θ_i. It may be noted here that, Eq. (15) has the same form as that of Eq. (14) and the sensitivity of the time derivative process (e.g., acceleration) can be obtained similarly i.e.

$$\left[\frac{\partial C_{\dot{Y}\dot{Y}}}{\partial \theta}\right] = [A]\left[\frac{\partial C_{YY}}{\partial \theta}\right][A]^T + [B_1] \tag{21a}$$

where

$$[B_I] = [A][C_{YY}]\left[\frac{\partial A}{\partial \theta_i}\right]^T + \left[\frac{\partial A}{\partial \theta_i}\right][C_{YY}][A]^T + \left[\frac{\partial S_{ww}}{\partial \theta_i}\right] \tag{21b}$$

The system parameter matrix [A] defined in the Appendix, is explicit function of uncertain model parameters $\{\theta\}$. Thus, the derivatives can be directly obtained by differentiating [A] with respect to these uncertain parameters. However, the formulation does not impose any limit to the number of elements or degrees of freedom employed in the analysis. But, with increasing number of elements, the matrices will be of bigger size resulting increasing computational requirement. For more complex super-structural system, involving finite element modelling; the matrix [A] cannot be obtained explicitly. In such cases, for implicitly generated element mass, stiffness and damping matrix of the system, the differentiation need to be carried out through sequence of calculations or alternatively, by finite difference approximation.

The sensitivity of any response quantity can be obtained by differentiating appropriate expression. For example, by differentiation of Eq. (16) with respect to the ith parameter will provide the following

$$\frac{\partial \sigma_{Y_m}}{\partial \theta_i} = \frac{1}{2}\frac{1}{\sqrt{C_{Y_m Y_m}}}\frac{\partial C_{Y_m Y_m}}{\partial \theta_i} \tag{22}$$

where, σ_{Ym} is the rms of the response Y_m. $\partial\sigma_{Ym}/\partial\theta i$ is the first order sensitivity of response σ_{Ym} with respect to the parameters θi.

4.2 Bounded optimization of BI system

In many cases, even though some experimental data are available about the system parameters, it is not enough to construct the probability density function reliably. The available data can be used, particularly in combination with engineering experience, to set some tolerances or bounds on the uncertain. If $\overline{\theta}_i$ is the nominal value of the ith UBB parameter viewed as the mean value and $\pm\delta\theta_i$ represents the maximum deviation from the nominal value, then the UBB parameter value deviates from the nominal value can be expressed as (McWilliam 2001)

$$\begin{aligned} &\theta_i^I = [\theta_i^l, \theta_i^u] = [\overline{\theta}_i - \delta\theta_i, \overline{\theta}_i + \delta\theta_i] = \overline{\theta}_i + \delta\theta_i[-1,1] = \overline{\theta}_i + \delta\theta_i e_\Delta \\ &where\, \overline{\theta}_i = \frac{\theta_i^l + \theta_i^u}{2}, e_\delta = [-1,\ 1] \end{aligned} \tag{23}$$

Thus, the ith interval variable can be written as: $\theta_i = \overline{\theta}_i + \Delta\theta_i$, where, $|\Delta\theta_i| \le \delta\theta_i, i = 1,2,..,m$

The performance function i.e., the rmsa as defined by Eq. (17) is also a function of the uncertain parameters. The unconditional rmsa can be expanded in first order Taylor series as the mean and the fluctuating part as following

$$\sigma_{\ddot{u}_N} = \sigma_{\ddot{u}_N}\left(\overline{\theta}_i\right) + \sum_{i=1}^{nv}\frac{\partial \sigma_{\ddot{u}_N}}{\partial \theta_i}\Delta\theta \tag{24}$$

Now, by making use of the interval extension in interval mathematics assuming monotonic responses, the interval extension of the above expression can be obtained as

$$\sigma^{I}_{\ddot{u}_N} = \sigma_{\ddot{u}_N}\left(\bar{\theta}_i\right) + \sum_{i=1}^{nv} \frac{\partial \sigma_{\ddot{u}_N}}{\partial \theta_i} \delta\theta_i e_{\delta} \tag{25}$$

The interval region of the function involving the UBB variables can be then separated out to the upper and lower bound as below

$$\sigma^{U}_{\ddot{u}_N} = \sigma_{\ddot{u}_N}\left(\bar{\theta}_i\right) + \sum_{i=1}^{nv} \frac{\partial \sigma_{\ddot{u}_N}}{\partial \theta_i} \delta\theta_i \tag{26a}$$

$$\sigma^{L}_{\ddot{u}_N} = \sigma_{\ddot{u}_N}\left(\bar{\theta}_i\right) - \sum_{i=1}^{nv} \frac{\partial \sigma_{\ddot{u}_N}}{\partial \theta_i} \delta\theta_i \tag{26b}$$

The optimization problem now involves two separate objective functions correspond to the lower and upper bound solutions. The formulation presented here involves linear perturbation based approximation of the responses around the mean values of the UBB parameters. The acceptability of the approximation approach for engineering problems has been justified (Chen and Zhang 2006). The study of accuracy of perturbation based approach and interval extension for evaluation of nominal response and its dispersion in this regard may be found in Chen *et al.* (2007) where it is numerically shown that the error in estimation goes up as the relative uncertainties of the interval variables increases.For larger level of uncertainty, alternative approach to linear perturbation analysis e.g., stochastic simulation should be applied. However, stochastic simulation will require the assumption of probability distribution. One may choose conservative uniform distribution for this purpose.

5. Numerical study

A five storied shear building model is taken up to illustrate the proposed bounded optimization procedure for BI system in seismic vibration mitigation of structures characterized by UBB type parameters. The stiffness and mass parameters for each storey are selected for desired value of time period of the superstructure intended to study. Unless specifically mentioned, the mean value of the damping ratio and the time period of the building frame are assumed as 2% and 0.5 sec, respectively. The mass ratio (m_b/m_i) of the isolator is taken as 1. The time period and the viscous damping of the LRB are taken as 2 sec and 5%, respectively. The yield strength (q) of the isolator is considered to be 0.025m. The mean values of the parameters characterizing the stochastic earthquake load are taken as: ω_f= 5π rad/sec, ξ_f= 0.6 and S_0 = 0.05m^2/s^3. With these numerical data, the top floor rmsa of the building without BI is 22.6502 m/sec^2 and rms displacement (rmsd) is 0.1408 m. The uncertain parameters considered in the study are mentioned in the vector {θ} of Eq. (19). The uncertainty of any such parameter (θ_i) is represented by the maximum possible dispersion ($\delta\theta_i$) expressed in terms of the percentage of corresponding nominal value $(\bar{\theta}_i)$.

Using the proposed optimization procedure considering the upper and lower bound performance functions represented by Eqs. (26a) and (26b), the optimum isolator yield strengths and the associated responses are obtained. The variation of the optimum yield strengths with increasing level of uncertainty of all the parameters is shown in Fig. 2. The associated top floor rmsa of the building frame is shown in Fig. 3. For comparison, the results obtained by the conventional SSO procedure as described by Eq. (18) are also shown in the same plot. Though the results follow the general trend of the SSO case, there is a definite change in the lower and the upper bound optimum solutions with the results of the deterministic case. It may be noted that upper bound solution obtained by solving Eq. (26a) gives higher rmsa i.e., the performance is sacrificed. The improved performance from the lower bound solution as obtained by solving Eq. (26b) with respect to the SSO case is obvious as it needs higher values of yield strength. As expected, the width of the bounded solution increases as the level of uncertainty increases.

To study the sensitivity of various parameters involved in the proposed bounded stochastic optimization procedure, further results are developed. The optimum yield strength of the isolator with varying building time period is shown in Fig. 4. The corresponding optimum value of the top floor rmsa is shown in Fig. 5. To develop these plots, the structural damping is taken as 2% and the uncertainty level of various system parameters as mentioned in Eq. (19) are considered to be 10% of the respective nominal values i.e., $\delta\theta_i$ is considered to be 10% of $\overline{\theta}_i$ .It may be noted that the effect of uncertainty on the optimum yield strength of the isolator and its performance (top floor rmsa as considered herein) is more when superstructure time period is in the range of 0.4 to 0.6 sec. In this regard it may be pointed out here that the BI systems are applied typically for reduction of vibration level of structure (thereby improving the safety level) having the time period typically in this range. Thus, neglecting the effect of uncertainty could be a critical issue for intended performance of BI system.

The variations of the optimum yield strength and the corresponding optimum value of the rmsa at top floor with varying structural damping are shown in Figs. 6 and 7, respectively. The structural time period is taken as 0.5 sec and the uncertainty level of various system parameters are considered to be 10% of associated nominal values. Similar results are shown in Figs. 8 and 9, for varying intensity of earthquake. The effect of uncertainty on the optimal solution for the BI system is more prominent for comparatively higher seismic intensity level. It may be pointed out here that BI systems are typically intended for reduction of higher vibration level due to strong motion earthquakes and the effect of uncertainty should be properly considered in the optimum design.

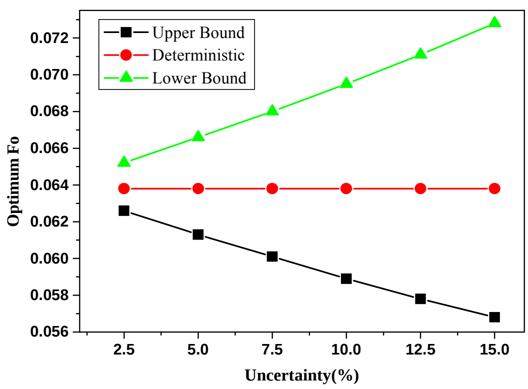

Fig. 2 The variation of the isolator normalised yield strength with varying level uncertainty

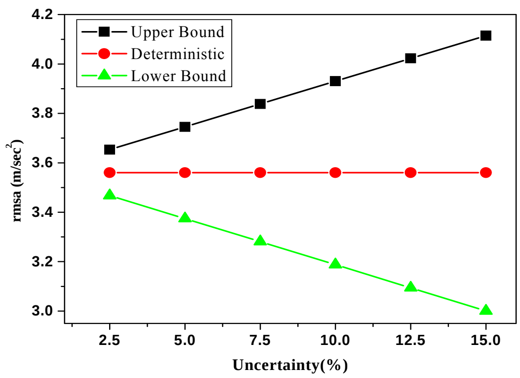

Fig. 3 The variation of the top floor rmsa with varying level uncertainty

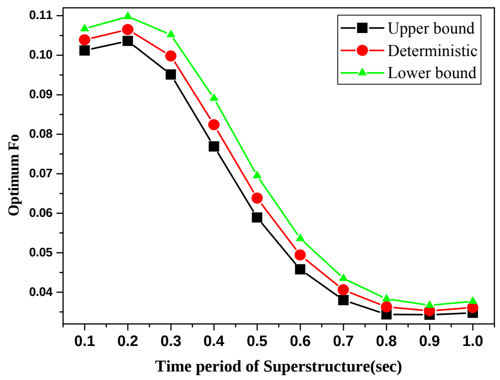

Fig. 4 The variation of isolator normalised yield strength with time period of the superstructure

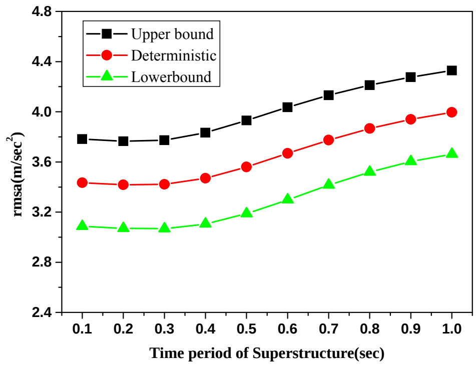

Fig. 5 The variation of the top floor rmsa with time period of the superstructure

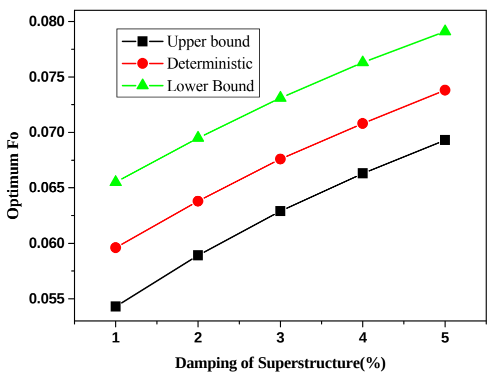

Fig. 6 The variation of isolator normalised yield strength with varying damping ratio of the superstructure

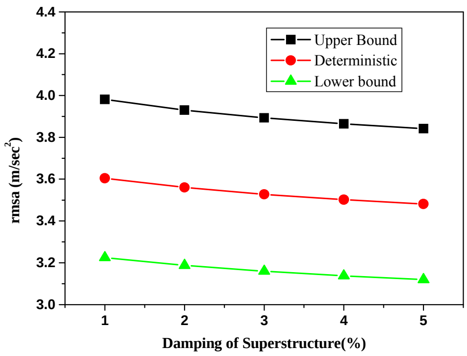

Fig. 7 The variation of the top floor rmsa with varying damping ratio of the superstructure

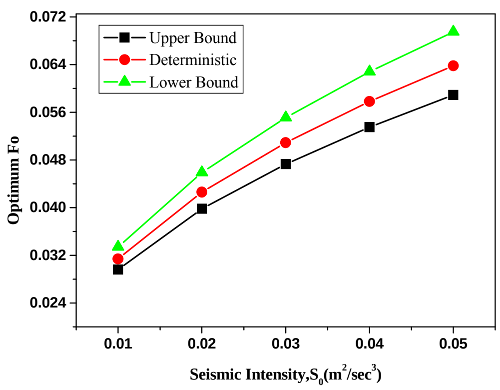

Fig. 8 The variation of isolator normalised yield strength with varying level of seismic intensity

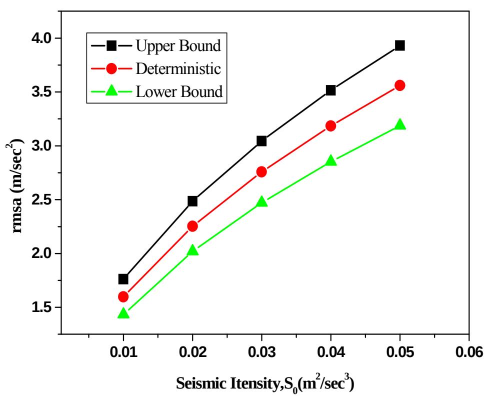

Fig. 9 The variation of the top floor rmsa with varying level of seismic intensity

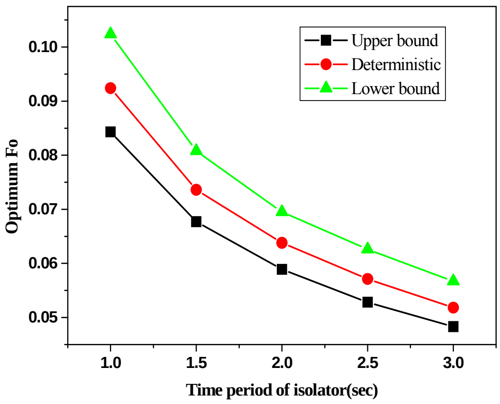

Fig. 10 The variation of the isolator normalised yield strength with varying time period of the isolator

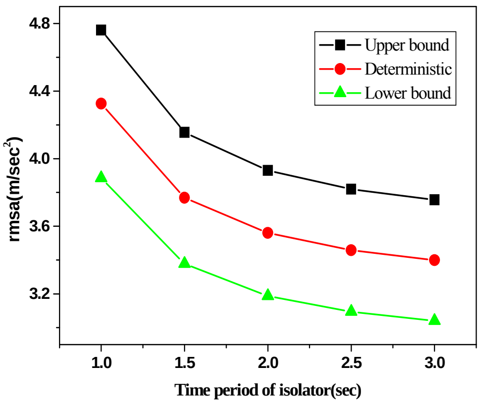

Fig. 11 The variation of the top floor rmsa with varying time period of the isolator

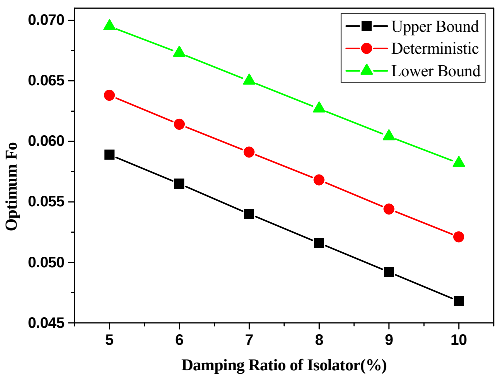

Fig. 12 The variation of the isolator normalised yield strength with varying damping ratio of the isolator

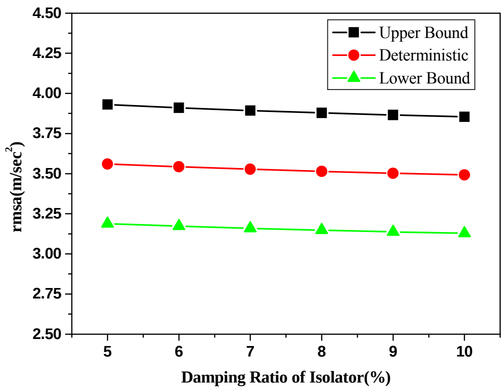

Fig. 13 The variation of the top floor rmsa with varying damping ratio of the isolator

The variations of optimum yield strength and the associated rmsa are further studied with respect to varying time period of the LRB in Figs. 10 and 11 and with varying damping ratio of the LRBin Figs. 12 and 13. It is generally observed that bounded optimum solutions i.e., the nature of variations of the optimum yield strength and associated responses remain similar for wider ranges of structural and BI parameters that characterize the performance of the BI system in seismic vibration mitigation.

6. Conclusions

The bounded optimization of BI system in seismic vibration mitigation is studied considering the UBB type system parameters. There is a definite change in the optimum results obtained by the bounded optimization procedure compare to that of a deterministic solution. The optimum yield strength of the isolator and the controlled response of the primary structure as obtained by the conventional SSO procedure are within the bounded solutions. As expected, the optimum yield strength of the isolator and the associated controlled response of the superstructure is not a unique value, rather provides a bound. It is evident that if the uncertainty which affects the parameters of the system is not considered, the BI system performance is overestimated. The upper bound of the response may be used in such cases for a conservative estimate of the optimum yield strength of the isolator. To address the problem in more formal way, it is possible to formulate the problem as a bi-objective optimization where the mean value of the response and its dispersion will be the two objectives (i.e. the robust optimization approach) and one can achieve desired optimum BI system from a set of pareto solution by ensuring desired level of robustness in the design to minimize the response of the structure. The issue is not studied in the present work and needs further considerations. The degree of parameter uncertainty is assumed to be small in the present formulation so that the linear perturbation analysis is valid. The present study is based on earthquake load modelled as stationary stochastic process and extension to non-stationary earthquake model will be straight forward. However, this will involve time dependent response statistics evaluation and subsequently to deal with time dependent performance function in the optimization procedure.

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Appendix: system matrix

The augmented system matrix [A] has the dimension equal to the number of structural degrees of freedom (N) and {Y} has the length of (2N + 5). The system matrix [A] for a N -storied shear building is given as

$$
\begin{bmatrix}
0 & .. & 0 & 0 & 0 & 0 & 1 & .. & 0 & 0 & 0 \\
: & nxn & : & : & nx3 & : & : & nxn & : & : & : \\
0 & .. & 0 & 0 & 0 & 0 & 0 & .. & 1 & 0 & 0 \\
0 & .. & 0 & 0 & 0 & 0 & 0 & .. & 0 & 1 & 0 \\
0 & 3xn & 0 & 0 & -{k_e}/{q} & 0 & 0 & 3xn & 0 & -{c_e}/{q} & 0 \\
0 & .. & 0 & 0 & 0 & 0 & 0 & .. & 0 & 0 & 1 \\
M^{-1}K_{11}-\frac{k_1}{m_b} & .. & M^{-1}K_{1n} & \alpha\frac{k_b}{m_b} & \frac{(1-\alpha)F_Y}{m_b} & 0 & M^{-1}C_{11}-\frac{c_1}{m_b} & .. & M^{-1}C_{1n} & \frac{c_b}{m_b} & 0 \\
: & M^{-1}K_{ij}-\delta_{i1}\frac{k_1}{m_b} & : & : & : & : & : & M^{-1}C_{ij}-\delta_{i1}\frac{c_1}{m_b} & : & : & : \\
M^{-1}K_{n1}-\frac{k_1}{m_b} & .. & M^{-1}K_{nn} & \alpha\frac{k_b}{m_b} & \frac{(1-\alpha)F_Y}{m_b} & 0 & M^{-1}C_{n1}-\frac{c_1}{m_b} & .. & M^{-1}C_{nn} & \frac{c_b}{m_b} & 0 \\
\frac{k_1}{m_b} & .. & 0 & -\alpha\frac{k_b}{m_b} & -\frac{(1-\alpha)F_Y}{m_b} & \omega_f^2 & \frac{c_1}{m_b} & .. & 0 & -\frac{c_b}{m_b} & 2\xi_f\omega_f \\
0 & .. & 0 & 0 & 0 & -\omega_f^2 & 0 & .. & 0 & 0 & -2\xi_f\omega_f
\end{bmatrix}
\quad (A.1)
$$

All the parameters in the matrices have already been defined in the main text. δ_{ij} is the Kronecker's delta. The matrices (shown in the last two blocks) are functions of the system matrices of N storied shear building. The augmented stiffness ($\mathbf{M}^{-1}\mathbf{K}$) and damping ($\mathbf{M}^{-1}\mathbf{C}$) matrices are shown by indicating them in the respective block of dimension NxN . The mass matrix is diagonal containing the storey mass in each diagonal term. The stiffness and damping matrices for the shear building model have the following form

$$
[\mathbf{K}] = \begin{bmatrix}
k_1+k_2 & -k_2 & .. & .. & .. \\
-k_2 & k_2+k_3 & .. & .. & .. \\
.. & -k_i & k_i+k_{i+1} & -k_{i+1} & .. \\
.. & .. & .. & k_n+k_{n-1} & -k_n \\
.. & .. & .. & -k_n & k_n
\end{bmatrix}
\quad (A.2)
$$

$$
[\mathbf{C}] = \begin{bmatrix}
c_1+c_2 & -c_2 & .. & .. & .. \\
-c_2 & c_2+c_3 & .. & .. & .. \\
.. & -c_i & c_i+c_{i+1} & -c_{i+1} & .. \\
.. & .. & .. & c_n+c_{n-1} & -c_n \\
.. & .. & .. & -c_n & c_n
\end{bmatrix}
\quad (A.3)
$$

where, k_i and c_i are the stiffness and damping of the i-th storey of the building. The damping for the i-th storey can be expressed as, $c_i = 2\xi_s\sqrt{k_i m_i}$, ξ_s is the viscous damping ratio of the superstructure.

The power spectral density (PSD) matrix for the rock bed seismic motion, characterized by the white noise of intensity of s_0 is expressed as

$$\left[\mathbf{S}_{\mathbf{ww}}\right] = \begin{bmatrix} 0 & 0 & .. & 0 & 0 \\ 0 & 0 & .. & 0 & 0 \\ : & : & .. & : & : \\ 0 & 0 & .. & 0 & 0 \\ 0 & 0 & .. & 0 & 2\pi s_0 \end{bmatrix} \tag{A.4}$$

where, $[\mathbf{S}_{ww}]$ is a square matrix of dimension $(2N + 5)$.