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Power series solution of circular membrane under uniformly distributed loads: investigation into Hencky transformation

Jun-yi Sun[*1,2], Yang Rong[1], Xiao-ting He[1,2], Xiao-wei Gao[1] and Zhou-lian Zheng[1,2]

[1]*College of Civil Engineering, Chongqing University, Chongqing 400045, PR China*
[2]*Key Laboratory of New Technology for Construction of Cities in Mountain Area (Chongqing University), Ministry of Education, Chongqing 400030, PR China*

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Abstract. In this paper, the problem of axisymmetric deformation of the circular membrane fixed at its perimeter under the action of uniformly-distributed loads was resolved by exactly using power series method, and the solution of the problem was presented. An investigation into the so-called Hencky transformation was carried out, based on the solution presented here. The results obtained here indicate that the well-known Hencky solution is, without doubt, correct, but in the published papers the statement about its derivation is incorrect, and the so-called Hencky transformation is invalid and hence may not be extended to use as a general mathematical method.

Keywords: axisymmetric deformation; membrane; power series method; Hencky transformation; Hencky solution

1. Introduction

Membrane structures have found increasing applications in many fields (Chien and Chen 1985, Arjun and Wan 2005, Chucheepsakul *et al*. 2009, Ersoy *et al*. 2009, Xu *et al*. 2009, Zhao *et al*. 2010, Lee and Han 2011, Zheng *et al*. 2011). Hencky (1915) originally investigated the problem of axisymmetric deformation of the circular membrane fixed at its perimeter under the action of uniformly-distributed loads, and presented the solution of the problem, based on the given stress and deflection patterns in power series. A computational error in Hencky (1915) was corrected by Chien (1948) and Alekseev (1953). The problem dealt with by Hencky (1915) is usually called Hencky problem or circular membrane problem for short, and its solution is usually called well-known Hencky solution (or circular membrane solution for short). This solution is the first solution of circular membrane problems and is often cited with regard to this problem. Alekseev (1951) originally studied the problem of axisymmetric deformation of the circular membrane centrally connected with a rigid plate under the action of a centrally concentrated load, and presented only a partial solution of the problem. Also, the problem dealt with by Alekseev (1951) is usually called Alekseev problem (or annular membrane problem for short), and its solution is called annular membrane solution for short. The annular membrane solution is the second solution of circular

∗Corresponding author, Associate Professor, E-mail: sun.junyi@yahoo.com.cn

membrane problems after the well-known Hencky solution. Sun *et al*. (2010) presented the complete solution of Alekseev problem. The third solution of circular membrane problems is about the axisymmetric deformation problem of the circular membrane under the action of uniformly-distributed loads in its central portion, which was presented by Chien *et al*. (1981). The problem of axisymmetric deformation of the circular membrane under the action of concentrated load, which is also called concentrated load problem for short, was dealt with by Chen and Zheng (2003) with the application of a so-called extended Hencky transformation. Also Jin (2008a) dealt with the concentrated load problem under the given restriction condition that the stress component (radial or circumferential) is nonnegative. Without any additional restriction condition, Sun *et al*. (2011) dealt with the concentrated load problem by using limiting method, based on the complete solution of Alekseev problem. Due to the somewhat intractable nonlinear equations, closed-form solutions of membrane problems are available in a few cases, but usually a shooting method is utilized to obtain numerical solutions for displacements, strains, and stresses (Plaut 2008).

The solving process about the well-known Hencky problem is always described as follows: the general solution of the membrane equation ($Z^2Z'' + x^2 = 0$) may be derived from a particular solution of the equation, $Z(x)$, with the application of the transformation $c^{-4/3}Z(cx)$. Here, $c^{-4/3}Z(cx)$ or the transformation from $Z(x)$ to $c^{-4/3}Z(cx)$ is called Hencky transformation. As stated by Chien (1948) and Chien *et al*. (1981), it can easily be proved that if $Z(x)$ is a particular solution of the equation $Z^2Z'' + x^2 = 0$, then $c^{-4/3}Z(cx)$ is also a solution of the equation, in which c is an arbitrary constant. So, this implies that solving a second-order boundary value problem needs only one boundary condition if a particular solution and a so-called transformation can be obtained in advance. It seems to be a novel mathematical method for solving nonlinear differential equation of the second order.

As a general mathematical method, however, the theoretical basis of the so-called Hencky transformation cannot be found in existing mathematical theory. By extending the so-called Hencky transformation, i.e., with a so-called extended Hencky transformation, Chen and Zheng (2003) solved the problem of large deformation of circular membrane under the concentrated load. But the works of Jin (2008a, b), Jin and Wang (2008), and Sun *et al*. (2011) indicate that the solution presented by Chen and Zheng (2003) is valid only when Poisson's ratio is equal to 1/3, in spite of the fact that by using modern immovable point theorems, Hao and Yan (2006) attested to the validity of the solution presented by Chen and Zheng (2003). Then, it is inevitable for us to answer some questions, such as whether the so-called Hencky transformation does exist, whether the so-called Hencky transformation can be extended to use as a general mathematical method, especially whether the well-known Hencky solution is correct.

In order to answer the questions mentioned above, we resolved the well-known Hencky problem by exactly using power series method. The results obtained here indicate that the well-known Hencky solution is, without doubt, correct, but in the published papers the statement about its derivation is incorrect, and the so-called Hencky transformation is invalid and hence may not be extended to use as a general mathematical method.

2. Membrane equation and its power series solution

The problem dealt with by Hencky (1915) is shown in Fig. 1. An initially flat, linearly elastic, rotationally symmetric, taut circular membrane with Young's modulus of elasticity E, Poisson's ratio v, thickness h, and radius a is clamped at its perimeter. A transverse uniformly-distributed

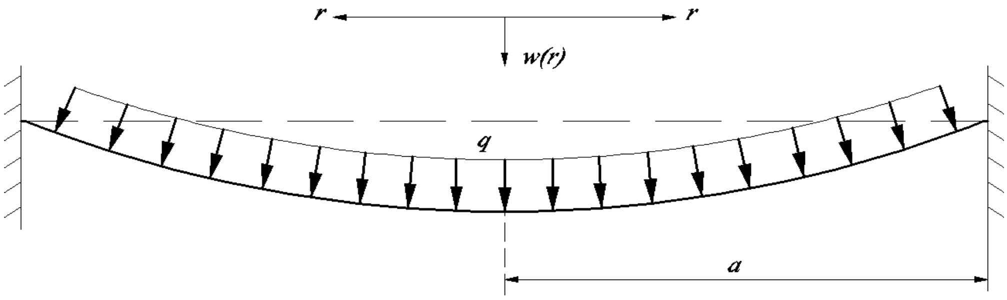

Fig. 1 Hencky problem

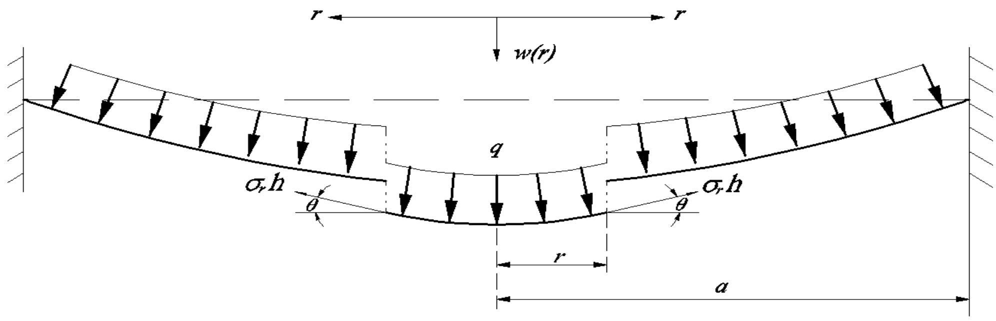

Fig. 2 The equilibrium diagram of the central portion ($r \leq a$) of the circular membrane

loads q is applied quasi-statically onto the film surface. Suppose that the radial coordinate is r, the radial strain is e_r, the circumferential strain is e_t, the radial displacement is $u(r)$, and the transversal displacement is $w(r)$. Let us take a piece of the central portion of the circular membrane whose radius is $0 \leq r \leq a$, with a view of studying the static problem of equilibrium of this membrane under the joint action of the uniformly-distributed loads q and the membrane force $\sigma_r h$ acted on the boundary, just as it is shown in Fig. 2.

Here there are two vertical forces, i.e., the total force $\pi r^2 q$ (in which $0 \leq r \leq a$) of the uniformly-distributed loads q and the total vertical force $2\pi rh\sigma_r$, $\sin\theta$, which is produced by the membrane force $\sigma_r h$, in which θ is the slope angle. The equilibrium condition is

$$2\pi rh\sigma_r \sin\theta = \pi r^2 q \tag{1}$$

Generally, $\theta < 15^\circ$, as far as physical phenomenon is concerned. Then we may have

$$\sin\theta \cong -\frac{dw}{dr} \tag{2}$$

Substituting Eq. (2) into Eq. (1), we obtain the equilibrium equation (which is perpendicular to the plane of the circular membrane)

$$h\sigma_r \frac{dw}{dr} = -\frac{1}{2} rq \tag{3}$$

In the plane of the circular membrane, there are the actions of the radial membrane force $\sigma_r h$ and the circumferential membrane force $\sigma_t h$, the equilibrium equation is

$$\frac{d}{dr}(rh\sigma_r) - h\sigma_t = 0 \tag{4}$$

Then there are the relations of the strain and displacement of the large deflection problem

$$e_r = \frac{du}{dr} + \frac{1}{2}(\frac{dw}{dr})^2, \quad e_t = \frac{u}{r} \tag{5a, b}$$

The relations of the stress and strain are

$$\sigma_r = \frac{E}{1-\nu^2}(e_r + \nu e_t), \quad \sigma_t = \frac{E}{1-\nu^2}(e_t + \nu e_r) \tag{6a, b}$$

Substituting Eq. (5a, b) into Eq. (6a, b), we may obtain

$$h\sigma_r = \frac{Eh}{1-\nu^2}[\frac{du}{dr} + \frac{1}{2}(\frac{dw}{dr})^2 + \nu\frac{u}{r}] \tag{7a}$$

and

$$h\sigma_t = \frac{Eh}{1-\nu^2}[\frac{u}{r} + \nu\frac{du}{dr} + \frac{\nu}{2}(\frac{dw}{dr})^2] \tag{7b}$$

By means of Eqs. (7a, b) and (4), we may obtain

$$\frac{u}{r} = \frac{1}{Eh}(h\sigma_t - \nu h\sigma_r) = \frac{1}{Eh}[\frac{d}{dr}(rh\sigma_r) - \nu h\sigma_r] \tag{8}$$

If we substitute the u of Eq. (8) into Eq. (7a), then

$$r\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(r^2 h\sigma_r)] + \frac{Eh}{2}(\frac{dw}{dr})^2 = 0 \tag{9}$$

The detailed derivation from Eq. (4) to Eq. (9) may be obtained from any general theory of plates and shells. It is not necessary to discuss this problem here.

Eqs. (3) and (9) are two equations for the solutions of e_r and dw/dr. Let us introduce the following nondimensional variables

$$Q = \frac{a^4 q}{h^4 E}, W = \frac{w}{h}, S_r = \frac{a^2\sigma_r}{Eh^2}, S_t = \frac{a^2\sigma_t}{Eh^2}, x = \frac{r^2}{a^2} \tag{10a, b, c, d, e}$$

and transform Eqs. (9), (3) and (4) into

$$\frac{d^2}{dx^2}(xS_r) + \frac{1}{2}(\frac{dW}{dx})^2 = 0 \tag{11}$$

$$\frac{dW}{dx}S_r = -\frac{Q}{4} \tag{12}$$

and

$$S_t = S_r + 2x\frac{dS_r}{dx} \tag{13}$$

The boundary conditions, under which Eqs. (11) ~ (13) may be solved, are

$$S_r = \text{ the finite value at } x = 0 \tag{14a}$$

and

$$u/r = 0,\ W = 0 \text{ at } x = 1 \tag{14b, c}$$

Eliminating dW/dx from Eqs. (11) - (12), we may obtain an equation which contains only S_r

$$\frac{d^2}{dx^2}(xS_r) + \frac{Q^2}{32S_r^2} = 0 \tag{15}$$

Let us substitute $Z(x)$ for xS_r, i.e., let

$$xS_r = \frac{1}{2}(\frac{Q}{2})^{2/3} Z \tag{16}$$

Substituting Eq. (16) into Eq. (15), we may obtain a nonlinear equation

$$\frac{d^2 Z}{dx^2} + \frac{x^2}{Z^2} = 0 \tag{17}$$

Based on the given condition of Eq. (14a), Z can be expanded to the power series of the x

$$Z(x) = \sum_{n=0}^{\infty} b_n x^n \tag{18}$$

It is well known that the power series solution of Eq. (17) should have two undetermined constants, i.e., b_0 and b_1. After substituting Eq. (18) into Eq. (17), it can easily be seen that $b_1 \equiv 0$. We here substitute b for b_1. Then, the general solution of Eq. (17) may be written as

$$\begin{aligned} Z(x) = bx &- \frac{1}{2b^2}x^2 - \frac{1}{6b^5}x^3 - \frac{13}{144b^8}x^4 - \frac{17}{288b^{11}}x^5 - \frac{37}{864b^{14}}x^6 - \frac{1205}{36288b^{17}}x^7 \\ &- \frac{219241}{8128512b^{20}}x^8 - \frac{6634069}{292626432b^{23}}x^9 - \frac{51523763}{2633637888b^{26}}x^{10} \\ &- \frac{998796305}{57940033536b^{29}}x^{11} - \frac{118156790413}{7648084426752b^{32}}x^{12} - \ldots\ldots \end{aligned} \tag{19}$$

or

$$\begin{aligned} Z(x) = bx(1 &- \frac{1}{2b^3}x - \frac{1}{6b^6}x^2 - \frac{13}{144b^9}x^3 - \frac{17}{288b^{12}}x^4 - \frac{37}{864b^{15}}x^5 - \frac{1205}{36288b^{18}}x^6 \\ &- \frac{219241}{8128512b^{21}}x^7 - \frac{6634069}{292626432b^{24}}x^8 - \frac{51523763}{2633637888b^{27}}x^9 \\ &- \frac{998796305}{57940033536b^{30}}x^{10} - \frac{118156790413}{7648084426752b^{33}}x^{11} - \ldots\ldots) \end{aligned} \tag{20}$$

If we let $b = c^{-1/3}$, we can easily transform Eq. (20) into

$$\begin{aligned}Z(x)=c^{-4/3}cx[1&-\frac{1}{2}cx-\frac{1}{6}(cx)^2-\frac{13}{144}(cx)^3-\frac{17}{288}(cx)^4-\frac{37}{864}(cx)^5-\frac{1205}{36288}(cx)^6\\&-\frac{219241}{8128512}(cx)^7-\frac{6634069}{292626432}(cx)^8-\frac{51523763}{2633637888}(cx)^9\\&-\frac{998796305}{57940033536}(cx)^{10}-\frac{118156790413}{7648084426752}(cx)^{11}-......]\end{aligned} \tag{21}$$

So, we may further write Eq. (21) into

$$Z(x)=c^{-4/3}(cx)f(cx) \tag{22}$$

in which $f(x)$ is

$$\begin{aligned}f(x)=1&-\frac{1}{2}x-\frac{1}{6}x^2-\frac{13}{144}x^3-\frac{17}{288}x^4-\frac{37}{864}x^5-\frac{1205}{36288}x^6-\frac{219241}{8128512}x^7\\&-\frac{6634069}{292626432}x^8-\frac{51523763}{2633637888}x^9-\frac{998796305}{57940033536}x^{10}-\frac{118156790413}{7648084426752}x^{11}-......\end{aligned} \tag{23}$$

Thus, from Eqs. (16) and (22), we may obtain

$$S_r=(\frac{Qc}{2})^{2/3}\frac{1}{2c}f(cx) \tag{24}$$

According to Eq. (8), we have

$$\begin{aligned}\frac{u}{r}&=\frac{1}{Eh}[\frac{d}{dr}(rh\sigma_r)-\nu h\sigma_r]=\frac{h^2}{a^2}[2x\frac{dS_r}{dx}+(1-\nu)S_r]\\&=\frac{h^2}{2a^2c}(\frac{Qc}{2})^{2/3}[2cxf'(cx)+(1-\nu)f(cx)]\end{aligned} \tag{25}$$

From Eq. (12), we have

$$\frac{dW}{dx}=-(\frac{Qc}{2})^{1/3}[f(cx)]^{-1}=-(\frac{Qc}{2})^{1/3}h(cx) \tag{26}$$

in which $h(x)$ is

$$\begin{aligned}h(x)=\frac{1}{f(x)}=1&+\frac{1}{2}x+\frac{5}{12}x^2+\frac{55}{144}x^3+\frac{35}{96}x^4+\frac{205}{576}x^5+\frac{17051}{48384}x^6+\frac{2864485}{8128512}x^7\\&+\frac{103863265}{292626432}x^8+\frac{135239915}{376233984}x^9+\frac{42367613873}{115880067072}x^{10}+\frac{14561952041}{39020838912}x^{11}+......\end{aligned} \tag{27}$$

We obtain from the integral of Eq. (26)

$$W=-(\frac{Qc}{2})^{1/3}g(cx)x+A \tag{28}$$

where A is another undetermined integral constant, and $g(x)$ is

$$g(x)=\frac{1}{x}\int_0^x h(x)dx=1+\frac{1}{4}x+\frac{5}{36}x^2+\frac{55}{576}x^3+\frac{7}{96}x^4+\frac{205}{3456}x^5+\frac{17051}{338688}x^6+\frac{2864485}{65028096}x^7 \\ +\frac{103863265}{2633637888}x^8+\frac{27047983}{752467968}x^9+\frac{42367613873}{1274680737792}x^{10}+\frac{14561952041}{468250066944}x^{11}+...... \tag{29}$$

From Eqs. (25) and (28), Eqs. (14b, c) give

$$v=2cf'(c)f^{-1}(c)+1 \tag{30}$$

and

$$A=(Qc/2)^{1/3}g(c) \tag{31}$$

Hence, for the concrete problem in which the value of v is known in advance, all the undetermined constants, c and A, can thus be determined by using Eqs. (30) - (31). So, from Eqs. (10), (28) and (31), we may finally obtain

$$w(r)=(a^4q/Eh)^{1/3}(c/2)^{1/3}[g(c)-g(cr^2/a^2)r^2/a^2] \tag{32}$$

and the maximum deflection of the membrane, at the central point (r = 0), is

$$w_m=w(0)=(a^4q/Eh)^{1/3}(c/2)^{1/3}g(c) \tag{33}$$

The relation between c and v is shown in Fig. 3, and the relation between $w(r)/(a^4q/Eh)^{1/3}$ and r while v takes 0.1, 0.2, 0.3, 0.4 and 0.5, respectively, is shown in Fig. 4.

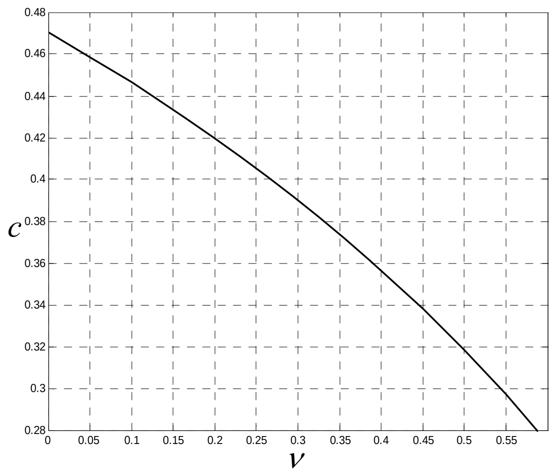

Fig. 3 Variation of c with v

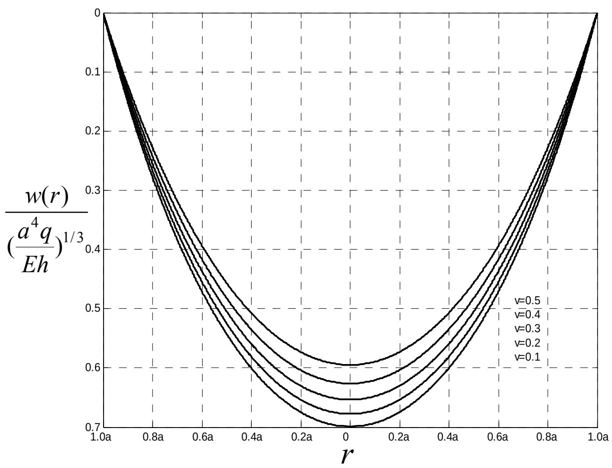

Fig. 4 Variation of $w(r)/(a^4q/Eh)^{1/3}$ with r

3. Results and discussions

The solution obtained above is usually called well-known Hencky solution. It should include the works of Hencky (1915), Chien (1948) and Alekseev (1953). But the derivation here, from Eq. (18) to Eq. (24), is firstly presented by this paper. In the published papers, for example in Chien *et al*. (1981), it was always stated as follows.

In order to satisfy the condition of Eq. (14a) at $x = 0$, Z must be expanded to the power series of the x

$$Z(x) = xf(x) \tag{I}$$

in which $f(x)$ is

$$f(x) = 1 - \frac{1}{2}x - \frac{1}{6}x^2 - \frac{13}{144}x^3 - \frac{17}{288}x^4 - \frac{37}{864}x^5 - \frac{1205}{36288}x^6 - \frac{219241}{8128512}x^7 \ldots\ldots \tag{II}$$

We can easily prove that if $Z(x)$ is a solution of Eq. (17), then $c^{-4/3}Z(cx)$ is also a solution of Eq. (17), in which c is the undetermined integral constant. Thus the general solution (satisfies the boundary condition (14a)) of Eq. (17) is

$$S_r = (\frac{Qc}{2})^{2/3}\frac{1}{2c}f(cx) \tag{III}$$

The statement above is based on the following derivation. From Eq. (17) we can have

$$\frac{d^2Z(cx)}{d(cx)^2} + \frac{(cx)^2}{Z^2(cx)} = 0 \tag{34}$$

Hence, substituting $c^{-4/3}Z(cx)$ into the left of Eq. (17), we may have

$$c^{-4/3}[c^2\frac{d^2Z(cx)}{d(cx)^2}] + \frac{x^2}{c^{-8/3}Z^2(cx)} = c^{2/3}[\frac{d^2Z(cx)}{d(cx)^2} + \frac{(cx)^2}{Z^2(cx)}] \equiv 0 \tag{35}$$

Eqs. (34) - (35) mean that if $Z(x)$ is a solution of Eq. (17), then $c^{-4/3}Z(cx)$ is also a solution of the equation.

However, the statement above seems to emphasize that the particular solution of an equation can be transformed into the general solution of the equation by using a so-called transformation. Here, the expression $c^{-4/3}Z(cx)$, or the transformation from $Z(x)$ to $c^{-4/3}Z(cx)$, was usually called Hencky transformation by the subsequent scholars. But the question is, from Eqs. (34) - (35) we cannot derive a conclusion that if $Z(x)$ is a particular solution of Eq. (17), then $c^{-4/3}Z(cx)$ is the general solution of the equation.

It should emphatically be pointed out that, based on the power series method for differential equation, after being expanded to the power series of the x the expression of the $Z(x)$ in Eq. (17) should be the Eq. (18), rather than Eq. (I). Please notice that Eq. (18) is the general solution of Eq. (17), and yet Eq. (I) is only a particular solution of Eq. (17). However, this mistake in the statement above has not been pointed out, and misled the subsequent scholars. In fact, the derivation here from Eq. (18) to Eq. (24) clearly shows that the general solution of Eq. (17) is no other than the expression $c^{-4/3}Z(cx)$ itself.

Let us give an example to show that the so-called Hencky transformation, $c^{-4/3}Z(cx)$, does not work in fact. We can easily prove that $Z(x) = -(9/4)^{1/3}x^{4/3}$ is a particular solution of Eq. (17), and yet we cannot derive the general solution of Eq. (17) from the transformation $c^{-4/3}Z(cx)$, for

$$c^{-4/3}Z(cx)=c^{-4/3}[-(\frac{9}{4})^{1/3}(cx)^{4/3}]=-(\frac{9}{4})^{1/3}x^{4/3} \tag{36}$$

Obviously, the transforming result of $c^{-4/3}Z(cx)$ does not include an arbitrary constant c. So, the so-called Hencky transformation $c^{-4/3}Z(cx)$ does not work.

As for the problem of large deformation of circular membrane under the concentrated load, which Chen and Zheng (2003) dealt with, the membrane equation is $Z''Z^2 + 1 = 0$. We can easily prove that if $Z(x)$ is a solution of the equation, then $c^{-2/3}Z(cx)$ (obtained by Chen and Zheng (2003)) is also a solution of the equation. We can easily prove that $Z(x) = (3/\sqrt{2})^{2/2}x^{2/3}$ is a particular solution of the equation $Z''Z^2 + 1 = 0$, and yet we cannot derive the general solution of the equation from the transformation $c^{-2/3}Z(cx)$, for

$$c^{-2/3}Z(cx)=c^{-2/3}[(\frac{3}{\sqrt{2}})^{2/3}(cx)^{2/3}]=(\frac{3}{\sqrt{2}})^{2/3}x^{2/3} \tag{37}$$

Also, the transforming result of $c^{-2/3}Z(cx)$ does not include an arbitrary constant c. So, the so-called extended Hencky transformation $c^{-2/3}Z(cx)$ does not also work. Hence, the solution presented by Chen and Zheng (2003) is an incorrect one. Here it is not necessary to discuss whether there were problems in the use of modern immovable point theorems (Hao and Yan 2006).

Let us give an example further. It can be easily proved that if $Z(x)$ is a solution of the equation $Z'' = 5$, then $c^{-2}Z(cx)$ is also a solution of the equation. Also we can easily prove that $Z(x) = 2.5x^2 + c_1x + c_2$ is the general solution of the equation $Z'' = 5$, in which, c_1 and c_2 are the arbitrary constants. So, $Z(x) = 2.5x^2 + 2x + 3$ and $Z(x) = 2.5x^2 + 5x + 7$ should be the two particular solutions of the equation. Then, from $Z(x) = 2.5x^2 + 2x + 3$, $c^{-2}Z(cx)$ gives

$$c^{-2}Z(cx)=2.5x^2+\frac{2}{c}x+\frac{3}{c^2} \tag{38}$$

Obviously, Eq. (38) cannot be used as the general solution of the equation $Z'' = 5$, for it cannot include the case of $Z(x) = 2.5x^2 + 5x + 7$. Further, with a given boundary value, for example $Z(1) = 1$ and $Z(2) = 2$, from $Z(x) = 2.5x^2 + c_1x + c_2$ we can easily obtain the solution of the problem, $Z(x) = 2.5x^2 - 6.5x + 5$. But, if Eq. (38) is used to solve this boundary value problem, it will be seen that no solution can be found in the interval of real number. This is the consequence of using the so-called extended Hencky transformation $c^{-2}Z(cx)$.

All in all, from the discussions above we may see that the particular solution of a differential equation and the so-called transformation are, on the one hand, difficult to be obtained in advance, for example the equation $Z''Z^2 + 1 = 0$ and the transformation $c^{-2/3}Z(cx)$, and on the other hand, even if they can be obtained in advance we cannot still ensure that the transforming result does be the general solution of the equation, for example the equation $Z'' = 5$ and the transformation $c^{-2}Z(cx)$. Then, under such a situation, what on earth can we do with a so-called transformation?

4. Conclusions

Based on the power series solution obtained in Section 2 and the discussions in Section 3, we may conclude as follows:

The well-known Hencky solution is, without doubt, a correct solution, but in the published papers the statement about its derivation is incorrect. It can be used under the condition that the membrane is initially flat, linearly elastic, and with a small rotation. We here firstly presented a correct version about the derivation of the well-known Hencky solution.

The so-called Hencky transformation (or extended) cannot ensure that the transformed result does be the general solution of the solved differential equation, and hence it may not be used as a general mathematical method.

The investigation presented here is significant to solving the allied problems of mechanics.

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