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A comparative study of three collocation point methods for odd order stochastic response surface method

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Abstract. This paper aims to compare three collocation point methods associated with the odd order stochastic response surface method (SRSM) in a systematical and quantitative way. The SRSM with the Hermite polynomial chaos is briefly introduced first. Then, three collocation point methods, namely the point method, the root method and the without origin method underlying the odd order SRSMs are highlighted. Three examples are presented to demonstrate the accuracy and efficiency of the three methods. The results indicate that the condition that the Hermite polynomial information matrix evaluated at the collocation points has a full rank should be satisfied to yield reliability results with a sufficient accuracy. The point method and the without origin method are much more efficient than the root method, especially for the reliability problems involving a large number of random variables or requiring complex finite element analysis. The without origin method can also produce sufficiently accurate reliability results in comparison with the point and root methods. Therefore, the origin often used as a collocation point is not absolutely necessary. The odd order SRSMs with the point method and the without origin method are recommended for the reliability analysis due to their computational accuracy and efficiency. The order of SRSM has a significant influence on the results associated with the three collocation point methods. For normal random variables, the SRSM with an order equaling or exceeding the order of a performance function can produce reliability results with a sufficient accuracy. The order of SRSM should significantly exceed the order of the performance function involving strongly non-normal random variables.

Keywords: stochastic response surface method; collocation points; reliability analysis; probability of failure; performance function

1. Introduction

The response surface method (RSM) was often employed to quantify uncertainty propagation (Bucher and Bourgund 1990, Li *et al*. 2010, Basaga *et al*. 2012). Recently, the stochastic response

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surface method (SRSM) was originally used to quantify uncertainty propagation (Isukapalli *et al*. 1998, Isukapalli 1999). Its basic idea is to approximate the model output responses in terms of random variables, such as standard normal variables, by a Hermite polynomial chaos expansion. One key step underlying the SRSM is the determination of unknown coefficients in the polynomial chaos expansion. The unknown coefficients are often determined using a probabilistic collocation method (Webster *et al*. 1996, Tatang *et al*. 1997, Isukapalli 1999, Huang *et al*. 2007). One key issue associated with the probabilistic collocation method is the choice of collocation points. In practice, as discussed by several studies (Webster *et al*. 1996, Tatang *et al*. 1997, Isukapalli 1999), the choice of the collocation points is derived from the same idea as the Gaussian quadrature method. The available collocation points for uncertain parameters are the results of all possible combinations of the roots of the one-dimensional Hermite polynomial of the next higher order. For a third order Hermite polynomial chaos expansion, the four roots of the fourth order Hermite polynomial, $\pm\sqrt{3\pm\sqrt{6}}$, are selected for generating the collocation points. Additionally, since the origin corresponds to the highest probability for a standard normal random variable, it is suggested to be added to the collocation points associated with the third order Hermite polynomial chaos expansion. As a conclusion, the origin is usually selected for the odd order Hermite polynomial chaos expansion (Isukapalli 1999, Huang *et al*. 2007).

In the literature, there exist two different ways to deal with the origin. One is to take the origin as a collocation point (referred to as the point method hereafter) (Isulapalli *et al*. 1998, Isukapalli 1999, Isulapalli *et al*. 2000, Mollon *et al*. 2011, Mao *et al*. 2012). By this way, for the two-dimensional and third order Hermite polynomials, the total number of available collocation points are $17 = (3+1)^2+1$. The other one is to take the origin as a root of the even order Hermite polynomials (referred to as the root method hereafter) (Huang *et al*. 2007, Phoon and Huang 2007, Huang *et al*. 2009, Li *et al*. 2011). Similarly, the total number of available collocation points are 25 $= (3+1+1)^2$ for the two-dimensional and third order Hermite polynomials. However, this subtle but important difference between the point method and root method has not been highlighted in previous works.

The aforementioned two collocation point methods were often used for reliability analysis (Anile *et al*. 2003, Phoon and Huang 2007, Huang *et al*. 2009, Mollon *et a*l. 2011, Li *et al*. 2011, Mao *et al*. 2012, Li *et al*. 2012a). For example, Anile *et al*. (2003) studied the reliability of the tolerance analysis in microelectronics using the SRSM with the point method. Phoon and Huang (2007) employed the SRSM for reliability analysis of a simple laterally loaded pile by the use of the root method. Huang *et al*. (2009) investigated the application of the SRSM with the root method to the reliability analysis of an infinite slope stability problem. Mollon *et al*. (2011) conducted a probabilistic analysis of pressurized tunnels against face stability using the collocation-based stochastic response surface method (CSRSM) with the point method. Li *et al*. (2011) proposed a SRSM with the root method for the reliability analysis of rock slope stability involving correlated non-normal variables. Mao *et al*. (2012) performed the probabilistic analysis and design of strip foundations using the CSRSM with the point method. Although these two collocation point methods are widely used for reliability problems, a systematical comparison of the accuracy and efficiency between the point method and the root method applied to reliability analysis has not been explored. In addition, the effect of the origin located in the central part of the probabilistic space on the probability of failure is not highlighted. Thus the resulting another collocation point method that the origin is not taken as a collocation point (referred to as the without origin method hereafter) should be also investigated. Also, the optimal order of SRSM for

n random variables following arbitrary distributions with different correlations is not sufficiently investigated.

The objective of this study is to compare the aforementioned three collocation point methods in a systematical and quantitative way. To achieve this goal, this article is organized as follows. In Section 2, the SRSM is briefly introduced for completeness. In Section 3, the three collocation point methods are presented in detail. In Section 4, three numerical examples focusing on reliability analyses are presented to compare the three methods. For validation, the reliability results obtained from the direct Monte Carlo simulation (MCS) are also provided. The criteria for selecting the number of collocation points and an optimal order of SRSM with the three collocation point methods are also discussed.

2. Stochastic response surface method (SRSM)

The stochastic response surface method (SRSM) can be interpreted as an extension of the deterministic response surface method (RSM). The main difference between them is that the inputs are random variables in the former and deterministic quantities in the latter (Isukapalli *et al*. 1998). One of the main ideas underlying SRSM is that square integrable random variables can be expressed as the functions of independent random variables. For simplicity, standard normal variables are usually chosen as standard random variables (SRVs) due to the mathematical tractability (Xiu and Karniadakis 2003, Eldred *et al*. 2008).

The first step in the implementation of the SRSM is to represent all the random inputs in terms of SRVs. Thus, the SRVs are selected from the ith set of independent and identically distributed random variables, $\boldsymbol{U}_i = \{U_{ij}\}_{j=1}^{n}$, in which n is the number of independent random inputs, and each U_{ij} has zero mean and unit variance. When the input random variables are independent, the uncertainty variable in the jth input of the ith set in the physical space, x_{ij}, can be expressed directly as a function of the corresponding independent standard normal random variable, U_{ij}, by using an isoprobabilistic transformation (Isulapalli 1999, Li *et al*. 2012b).

In the SRSM, a specific representation of the output response is in terms of an expansion of a series of SRVs. A widely used functional approximation is the Hermite polynomial chaos expansion (PCE) (Ghanem and Spanos 2003)

$$F(\boldsymbol{U}_i) = a_0 + \sum_{i_1=1}^{n} a_{i_1}\Gamma_1(U_{i_1}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(U_{i_1},U_{i_2}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(U_{i_1},U_{i_2},U_{i_3})$$

$$+\cdots+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_n=1}^{i_{n-1}} a_{i_1 i_2,\cdots,i_n}\Gamma_n(U_{i_1},U_{i_2},\cdots,U_{i_n}) \tag{1}$$

where F is a random output response of the model; $a_{i_1 i_2,\ldots,i_n}$ are unknown coefficients in the expansion to be estimated; n is the number of random variables used to represent the uncertainty in the model inputs; $\boldsymbol{U}_i = (U_{i1}, U_{i2}, \cdots, U_{in})$ is a vector of independent standard normal variables; $\Gamma_n(U_{i_1},U_{i_2},\cdots,U_{i_n})$ is multidimensional Hermite polynomials of degree n given by

$$\Gamma_n(U_{i_1},U_{i_2},\cdots,U_{i_n}) = (-1)^n e^{\frac{1}{2}\mathbf{U}^{\mathrm{T}}\mathbf{U}} \frac{\partial^n}{\partial U_{i_1}\partial U_{i_2}\cdots\partial U_{i_n}} e^{-\frac{1}{2}\mathbf{U}^{\mathrm{T}}\mathbf{U}} \tag{2}$$

For notational simplicity, Eq. (1) is rewritten as

$$F(\boldsymbol{U}_i) = \sum_{j=0}^{N_c-1} c_j \Psi_j(\boldsymbol{U}_i) \tag{3}$$

in which there is a one-to-one mapping between $\Psi_j(\boldsymbol{U}_i)$ and $\Gamma_n(U_{i_1}, U_{i_2}, \cdots, U_{i_n})$, and also between the coefficients c_j and $a_{i_1 i_2,\ldots,i_n}$.

From Eq. (1), the number of the unknown coefficients in Eq. (3), N_c, for a p order Hermite PCE involving n random variables is calculated by (Ghanem and Spanos 2003)

$$N_c = \frac{(n+p)!}{n!\,p!} \tag{4}$$

For Example # 1 involving six random variables in this study, if the PCE of third order is used, then the number of unknown coefficients is 462. After obtaining the Hermite PCE for the output responses, the unknown coefficients in Eq. (1) need to be determined. The probabilistic collocation method (Webster *et al*. 1996, Tatang *et al*. 1997) is often used to determine the unknown coefficients, because it can decouple the deterministic response evaluation and probabilistic analysis. However, the probabilistic collocation method is inherently unstable, especially for the high order Hermite polynomial, because the Hermite polynomial has to pass through all collocation points selected. Thus, any collocation points in the model space could significantly affect the behavior of the Hermite polynomial (Atkinson 1988). To circumvent such limitations, the regression based SRSM is used.

A regression based SRSM proposed by Isukapalli (1999) is often used to determine the unknown coefficients in the Hermite PCE. In the regression based SRSM, the sets of collocation points are selected first. When N sets of collocation points are selected, the corresponding output responses, $\mathbf{F}=[\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_N]^{\mathrm{T}}$, can be obtained through the deterministic analysis at each set of collocation points. Then, the model output responses at the selected points are equated with the estimates from the series approximation, a system of linear equations is constructed as

$$\mathbf{ZC} = \mathbf{F} \tag{5}$$

in which $\mathbf{C}$ is the vector of the unknown coefficients; $\mathbf{Z}$ is a space-independent matrix of dimension $N \times N_c$, consisting of Hermite polynomial evaluated at the selected sets of collocation points (hereafter referred to as the Hermite polynomial information matrix). It is given by

$$\mathbf{Z} = \begin{bmatrix} \Psi_0(\boldsymbol{U}_1) & \Psi_1(\boldsymbol{U}_1) & \cdots & \Psi_{N_c-1}(\boldsymbol{U}_1) \\ \Psi_0(\boldsymbol{U}_2) & \Psi_1(\boldsymbol{U}_2) & \cdots & \Psi_{N_c-1}(\boldsymbol{U}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_0(\boldsymbol{U}_N) & \Psi_1(\boldsymbol{U}_N) & \cdots & \Psi_{N_c-1}(\boldsymbol{U}_N) \end{bmatrix} \tag{6}$$

Then, the unknown coefficients can be readily determined by solving Eq. (5). In the regression based SRSM, Eq. (5) can be further rewritten as

$$\mathbf{Z}^{\mathrm{T}}\mathbf{ZC} = \mathbf{Z}^{\mathrm{T}}\mathbf{F} \tag{7}$$

The vector of the unknown coefficients can be obtained by solving the following system of equations using the singular value decomposition method

$$\mathbf{C} = (\mathbf{Z}^{\mathrm{T}}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}\mathbf{F} \tag{8}$$

Once the unknown coefficients in the Hermite PCE are determined, the model output response can be represented as random variables by an analytical PCE. The statistical properties of the output response (eg., probability density functions (PDF), cumulative distribution functions (CDF), various order statistical moments, and correlations between an output and an input, or between two outputs) can be readily evaluated. For engineers, the probability of failure, p_{f}, may be of great interest, which can also be easily estimated by applying the direct MCS on the analytical PCE for a specified performance function.

3. Three collocation point methods associated with odd order SRSM

The choice of collocation points has a significant influence on the results obtained from the SRSMs (Isukapalli 1999, Li and Zhang 2007, Li *et al*. 2011). One particular scheme is that, by analogy with Gaussian quadrature, the choice of the collocation points corresponds to the choice of the PCE order. The available collocation points for the PCE order p are the results of all possible combinations of the roots of the one-dimensional Hermite polynomial of the PCE order (p+1) (Webster *et al*. 1996, Tatang *et al*. 1997, Isukapalli 1999). For instance, the available collocation points for a second order Hermite PCE, are all possible combinations of the three roots of the third order Hermite polynomial, 0, and $\pm\sqrt{3}$. For a third order Hermite PCE, all possible combinations of the four roots of the fourth order Hermite polynomial, $\pm\sqrt{3\pm\sqrt{6}}$. For a fourth order Hermite PCE, all possible combinations of the five roots of the fifth order Hermite polynomial, 0, and $\pm\sqrt{5\pm\sqrt{10}}$. Similarly, for a fifth order Hermite PCE, all possible combinations of the six roots of the sixth order Hermite polynomial, ±0.617, ±1.889, and ±3.324.

As suggested by several studies (Isukapalli 1999, Li and Zhang 2007, Huang *et al*. 2007, Huang *et al*. 2009, Li *et al*. 2011), another potential criterion for selecting the collocation points is that the collocation points selected should capture the regions of high probability. Thus the collocation points are selected successively according to increasing norm. It is well known that the origin corresponds to the region of highest probability for a standard normal variable. Thus, the origin is suggested to be included in the collocation points selected for the SRSMs. Since the origin has been included in the roots of the odd order Hermite polynomials, there is no need to add the origin again for the even order Hermite PCE. However, since the roots of the even order Hermite polynomials do not have the origin, it should be included in the collocation points for the odd order Hermite PCE. For example, the four roots of the fourth order Hermite polynomial do not have the origin as mentioned earlier, so the origin should be included in the collocation points for the third order Hermite PCE. Similarly, the origin should be included in the collocation points for the fifth order Hermite PCE as well. In this way, there exist two methods to add the origin to the collocation points associated with the odd order SRSMs. The first method is to take the origin as a collocation point (Isukapalli 1999, Mollon *et al*. 2011), which is referred to as the point method as mentioned previously. The second method is to take the origin as a root of the next even order Hermite polynomial, which is referred to as the root method herein. To investigate whether the origin is absolutely necessary, another without origin method is proposed for comparison. As discussed previously, the roots of one-dimensional Hermite polynomial of the PCE order p are (p+1). Therefore, for the point method, the total number of available collocation points, N_a, for the p order SRSM involving n random variables can be easily calculated by

$$N_a = (p+1)^n + 1 \tag{9}$$

For the root method, an additional root, namely the origin, is added to the (p+1) roots of one-dimensional Hermite polynomial of the PCE order p. Consequently, there are (p+2) roots for generating the collocation points. The total number of available collocation points is given by

$$N_a = [(p+1)+1]^n \tag{10}$$

If the origin underlying the point method is not taken as a collocation point, the point method becomes the same as the without origin method, the corresponding total number of available collocation points is

$$N_a = (p+1)^n \tag{11}$$

It can be seen from Eqs. (9)-(11) that the total number of available collocation points associated with the root method is significantly greater than that associated with the other two methods, especially for the high dimensional PCE. For the third order SRSMs, the total numbers of available collocation points for two-dimensional Hermite PCE are 17, 25 and 16 for the point method, root method, and without origin method, respectively. Similarly, for a six-dimensional Hermite PCE, the corresponding numbers of available collocation points are 4097, 15625 and 4096, respectively. The number of available collocation points for the root method is about four times those for the other two methods. In the subsequent sections, the accuracy and efficiency of the three collocation point methods underlying the odd order SRSMs are further investigated and discussed through three numerical examples.

4. Illustrative examples

4.1 Example # 1: an explicit performance function involving six random variables

To investigate the effect of the condition of full rank matrix on the accuracy of the SRSMs with the three collocation point methods, an explicit performance function involving six random variables is used first. This example was analyzed by Nguyen *et al*. (2009), resulting from a problem of stress distribution in a steel joint, and addresses elevated temperatures and fatigue phenomena. The performance function is strongly nonlinear, which is expressed as

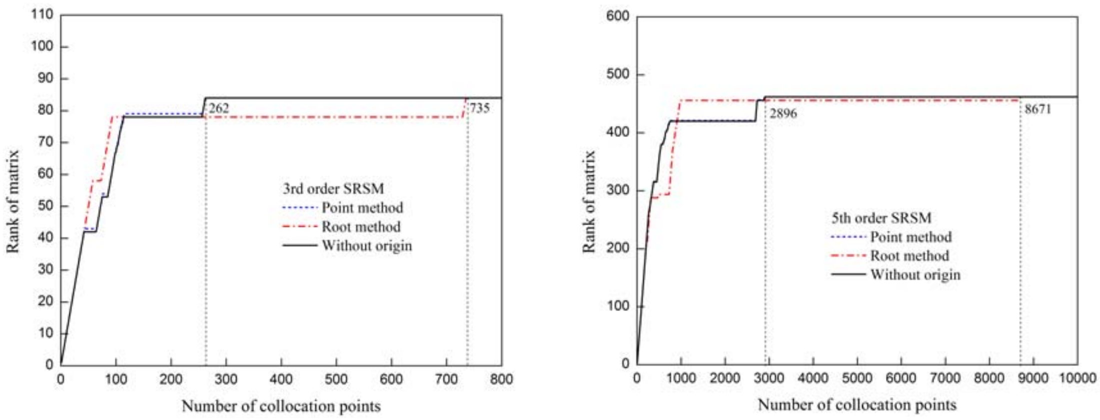
$$G(x) = x_1 - 10^4 \left[\frac{x_2 (x_4 x_5)^{1.71}}{x_3} + \frac{(1-x_2)(x_4 x_5)^{1.188}}{x_6} \right] \tag{12}$$

The statistical parameters of the basic random variables are listed in Table 1. The probability of failure is defined as the probability of $G(x) < 0$.

To investigate the relationship between the rank of Hermite polynomial information matrix **Z** in Eq. (5) and the number of collocation points, the variation of the rank of matrix with the number of collocation points for the 3rd and 5th order SRSMs are plotted in Figs. 1(a) and (b), respectively.

Table 1 Statistical parameters of basic random variables for Example #1

Random variables	Mean	Standard deviation	Distribution
x_1	1.044	0.3132	Lognormal
x_2	0.7	0.07	Normal
x_3	0.2391	0.09564	Lognormal
x_4	1.011	0.15165	Lognormal
x_5	0.0005	0.00008	Type I largest
x_6	1.802	0.7208	Lognormal



(a) 3rd order SRSM (b) 5th order SRSM

Fig. 1 Comparison of the rank of matrix with the number of collocation points among different collocation point methods

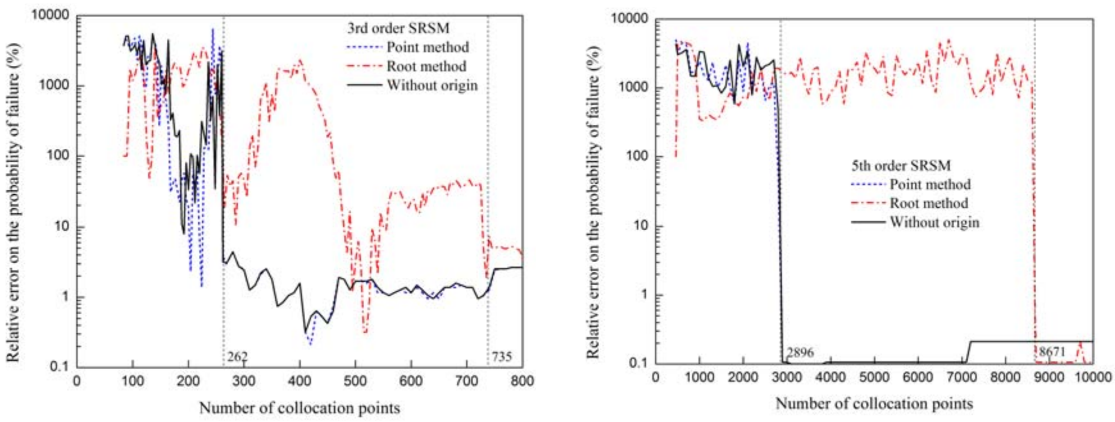
It can be seen from Fig. 1(a) that, in order to satisfy the condition of full rank matrix, the numbers of collocation points selected, N_p, for the 3rd order SRSM with the point method, root method, and without origin method should be more than 262, 735 and 262, respectively. The number of collocation points selected for the root method is 2.8 times those for the other methods. Similarly, for the 5th order SRSM, the number of collocation points selected, N_p, should be more than 2896 for the point method and the without origin method, which is significantly smaller than 8671 for the root method. It is evident that the SRSMs with the point method and the without origin method are much more efficient than the SRSM with the root method, especially for the reliability problems involving many random variables. The point method and the without origin method can achieve the same computational efficiency, which indicates that the condition of full rank matrix can still be satisfied and no more computational effort is needed when the origin is not selected as a collocation point.

Table 2 compares the probabilities of failure using the SRSMs with the three collocation point methods. Taking the probability of failure, 9.42×10^{-3}, obtained from the direct MCS with 10^5 samples as the exact solution, the relative errors in the probability of failure are also provided in Table 2. Note that when the number of collocation points selected satisfies the condition of full rank matrix, all the three collocation point methods can produce sufficiently accurate results. These results indicate that the condition of full rank matrix is actually enough to achieve a sufficient accuracy. To further validate the condition of full rank matrix, Fig. 2 shows the relative errors in the probability of failure associated with different collocation point methods. It is evident that only when the number of collocation points selected can lead to a full rank matrix, the accuracy of the SRSMs is enough.

Since the CDF curves of the performance function can directly reflect the level of probability of failure, the log-scale CDF curves of the performance function obtained from the 5th order SRSM with the point method and from the direct MCS with a sample size of 10^5 are plotted in Fig. 3. Note that the CDF curve for the MCS is obtained by running a Monte-Carlo simulation directly with the actual performance function shown in Eq. (12). This curve is used to benchmark the accuracy of the other curves obtained from the 5th SRSM with different numbers of collocation points. The numbers of collocation points N are 2, 3, 4, 5, 6 and 7 times the number of unknown coefficients N_c for the 5th order SRSM. It can be observed that only when the condition of full rank (R = 462) for the matrix **Z** is satisfied, the CDF curve obtained from the 5th order SRSM agrees well with the exact solution. The CDF curves associated with R = 421 and 457 are significantly different from the exact solution. Such results further indicate that the condition of full rank matrix can produce reliability results with a sufficient accuracy. In addition, although more collocation points are selected, it cannot always improve the accuracy of SRSM if the condition of full rank matrix is not satisfied. For example, the results for the 5th order SRSM with 2310 collocation points are not better than those with 924 collocation points.

Table 2 Comparison between the results obtained from the SRSMs with different collocation point methods compared with the direct MCS

Collocation point methods	3rd order SRSM			5th order SRSM		
	N_p	p_f	Relative errors	N_p	p_f	Relative errors
Point method	262	9.11×10^{-3}	3.3%	2896	9.43×10^{-3}	0.1%
Root method	735	9.24×10^{-3}	1.9%	8671	9.47×10^{-3}	0.5%
Without origin	262	9.12×10^{-3}	3.2%	2896	9.43×10^{-3}	0.1%



(a) 3rd order SRSM (b) 5th order SRSM

Fig. 2 Comparison of relative errors in the probability of failure with the number of collocation points among different collocation point methods

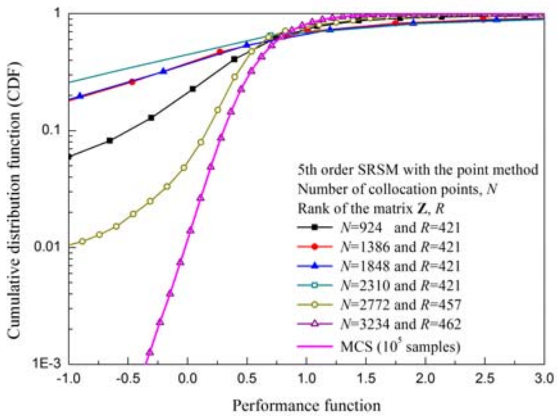
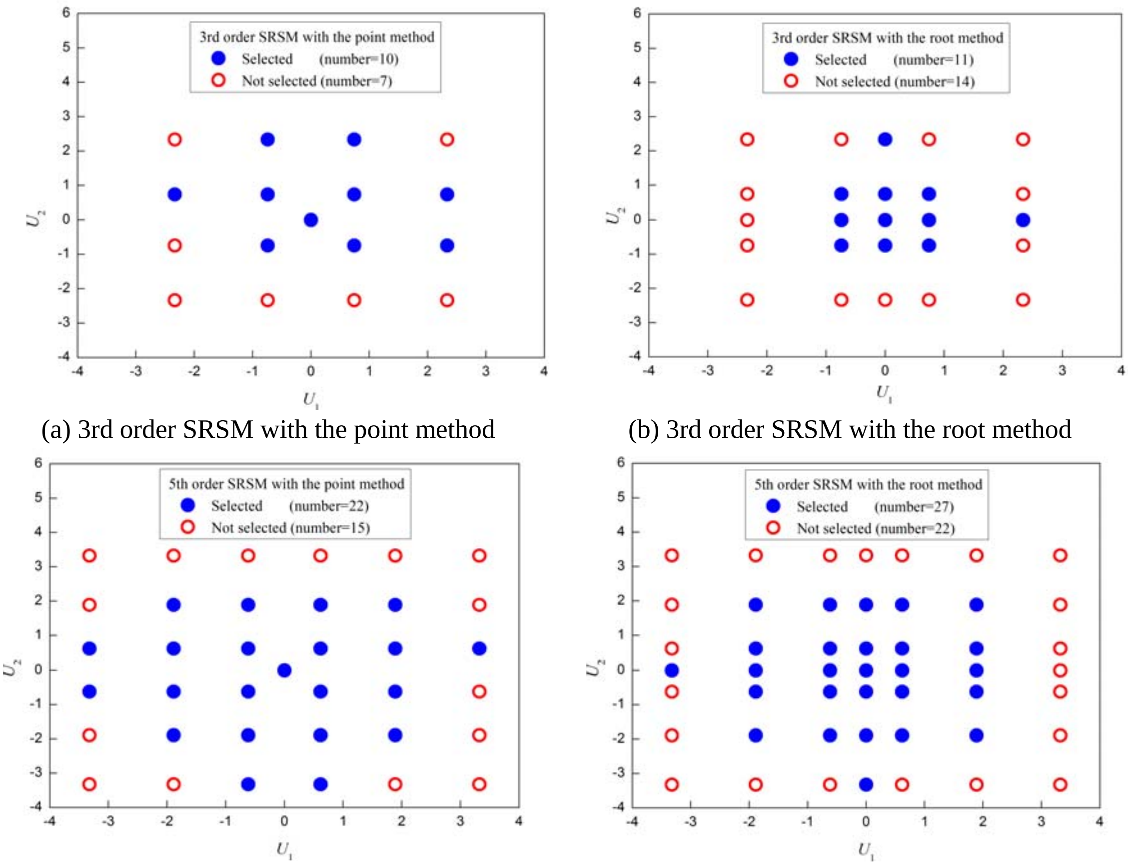


Fig. 3 Comparison among CDF curves of the performance function obtained from the 5th order SRSM based on the point method with different numbers of collocation points

4.2 Example # 2: various order performance functions involving two random variables

For illustrative purposes, two-dimensional collocation point problems are investigated herein. Fig. 4 visualizes the available collocation points for the 3rd and 5th order SRSMs with the point and root methods involving two random variables. It should be pointed out that the results associated with the without origin method are similar to those associated with the point method. The only difference between them is that the origin is not considered in the former. Consequently, for the without origin method, another collocation point is selected to replace the origin for satisfying the condition of full rank matrix. Therefore, the results for the without origin method are not presented herein due to space limitation. The total numbers of available collocation points associated with the 5th order SRSM are 37, 49, and 36 for the point method, root method and without origin method, respectively. More collocation points resulting from the root method are concentrated in the region around the origin in comparison with the other methods. To obtain sufficiently accurate results, the collocation points selected should satisfy the condition that the information matrix **Z** as shown in Eq. (5) is a full rank matrix (Li and Zhang 2007, Sudret 2008, Li *et al*. 2011, Mao 2012, Li *et al*. 2012a), which also be validated by Eample # 1, as demonstrated earlier. Applying this criterion, the minimum number of collocation points N_p can be determined, as illustrated in Fig. 4. The range covered by the collocation points selected for the point method is significantly larger than that for the root method although the number of the collocation points selected for the former is smaller than that for the later. For the 5th order SRSM with the point method and the without origin method, only 22 collocation points need to be selected to meet the condition of full rank matrix. In contrast, 27 collocation points need to be selected for the root method. It should be pointed out that the number of collocation points selected for reliability analysis of a specified performance function is equal to the numbers of performance function evaluations. In other words, more collocation points will directly result in more computational effort, especially for the reliability problems involving complex finite element analyses.

To make a systematical comparison among the three collocation point methods and determine an optimal order of SRSM, various performance functions with the orders ranging from 1 to 5, often used in the literature, are selected. The five performance functions are summarized in Table 3.



(a) 3rd order SRSM with the point method (b) 3rd order SRSM with the root method

(c) 5th order SRSM with the point method (d) 5th order SRSM with the root method

Fig. 4 Scatter plots of collocation points for the SRSMs with the point and root methods

Table 3 Summary of various order performance functions for Example #2

No.	Performance functions	The first case	The second case	References
1	$g(x_1,x_2)=18-3x_1-2x_2$	X_1: *N* (2.5, 1) X_2: *N* (2.5, 1)	X_1: *Exp* (1, 1) X_2: *Exp* (1, 1)	Noh *et al.* (2009)
2	$g(x_1,x_2)=48-x_1^2-x_1x_2-\frac{x_2^2}{2}$	X_1: *N* (2.5, 1) X_2: *N* (2.5, 1)	X_1: *Exp* (1, 1) X_2: *Exp* (1, 1)	Modified from No. 1
3	$g(x_1,x_2)=x_1^3+x_1^2x_2+x_2^3-18$	X_1: *N* (10, 5.0) X_2: *N* (9.9, 5.0)	X_1: *Exp* (10, 10) X_2: *Exp* (9.9, 9.9)	Kaymaz and McMahon (2005)
4	$g=3-x_2+(4x_1)^4$	X_1: *N* (0.5, 0.1) X_2: *N* (8, 1.6)	X_1: *Exp* (0.5, 0.5) X_2: *Exp* (8, 8)	Au and Beck (1999)
5	$g(x_1,x_2)=x_1^5+x_1^2x_2+x_2^5-18$	X_1: *N* (10, 5.0) X_2: *N* (9.9, 5.0)	X_1: *Exp* (10, 10) X_2: *Exp* (9.9, 9.9)	Modified from No. 3

To take the effect of distributions of random variables into consideration, normal variables and exponential variables often taken as strongly non-normal variables are used. Also, the correlation coefficient, ρ_{x1x2}, between X_1 and X_2 ranging from -0.5 to 0.5 is adopted to account for the effect of correlation on reliability.

Applying the regression based SRSMs with the three collocation point methods, the probabilities of failure for the five performance functions can be readily obtained. For comparison, the results obtained from the direct MCS with 10^6 samples are provided in Table 4. Taking the probabilities of failure for the MCS as the exact solutions, the resulting relative errors in the probability of failure associated with three collocation point methods are determined and also summarized in Table 4. For illustrative purpose, only the results for the three values of ρ_{x1x2}, namely -0.5, 0 and 0.5, are provided in Table 4.

It can be observed from Table 4 that for normally distributed random variables involved in the reliability analysis, all the three collocation point methods can produce the same results as those obtained from the MCS when the order of SRSM is equal to or greater than the order of the performance function. However, when the order of the performance function exceeds the order of SRSM, the SRSM cannot always produce sufficiently accurate results although the normal random variables are involved. For the fifth order performance function with ρ_{x1x2}= -0.5, the relative errors in the probability of failure for the 3rd order SRSM with the three collocation point methods exceed 1.0×10^5%, which are obviously unacceptable. In contrast, for exponentially distributed random variables involved in the reliability analysis, the SRSMs with the three collocation point methods can yield reliability results with a sufficient accuracy only for the low order performance functions such as the first and second order performance functions. For the high order performance functions, the results obtained form the SRSM with the three collocation point methods differ considerably from the exact solutions although the order of SRSM exceeds the order of performance function. For the third order performance function with ρ_{x1x2}= -0.5, all the relative errors in the probability of failure obtained from the 3rd and 5th order SRSMs exceed 1.0×10^3%. Theoretically, the 3rd order SRSM are usually sufficient to tackle most of the performance functions. This inaccuracy is mainly because the Hermite polynomial chaos used by the SRSM is the optimal polynomial chaos for approximating a normal distribution rather than an exponential distribution. Several studies (Phoon 2003, Xiu and Karniadakis 2003, Eldred *et al*. 2008) have indicated that the Laguerre polynomial chaos is the optimal polynomial chaos for an exponential distribution. Thus for the high order performance function involving strongly non-normal variables, in order to produce accurate reliability results, the order of SRSM with the Hermite polynomial chaos should significantly exceed the order of the performance function or the SRSM with other orthogonal polynomial chaos such as Laguerre polynomial or Jacobi polynomial should be adopted. In addition, the results obtained from the SRSM with the without origin method are also almost the same as those obtained from the SRSMs with the point and root methods, which further indicates that the origin is not absolutely necessary.

4.3 Example # 3: a linear frame structure with implicit performance function

Unlike the previous examples with explicit performance functions, the reliability of a linear frame structure with an implicit performance function is investigated to determine the optimal order of SRSM. The example is a linear frame structure with one story and one bay, as shown in Fig. 5. Following Cheng and Li (2009), the cross sectional areas A_i (i = 1, 2) and horizontal load P are treated as random variables. The statistical parameters of basic random variables are listed in Table 5. The sectional moments of inertia are expressed as $I_i = \alpha_i A_i^2$ (i = 1, 2, α_1 = 0.08333, α_2 = 0.1667). The Young's modulus E is treated as deterministic quantity with a value of $E = 2.0\times10^6$ kN/m^2.

Table 4 Relative errors in the probability of failure obtained from the SRSMs with different collocation point methods

Order of Performance function	$\rho_{x_1x_2}$	Normal variables							Exponential variables						
		$p_{f,\,MCS}$	3rd order SRSM			5th order SRSM			$p_{f,\,MCS}$	3rd order SRSM			5th order SRSM		
			Point method	Root method	Without origin	Point method	Root method	Without origin		Point method	Root method	Without origin	Point method	Root method	Without origin
	-0.5	1.88×10^{-2}	0	0	0	0	0	0	2.78×10^{-3}	11.5	11.5	8.8	1.0	8.9	0.4
1	0	6.39×10^{-2}	0	0	0	0	0	0	7.21×10^{-3}	1.5	1.1	1.8	3.7	4.8	0.5
	0.5	1.04×10^{-1}	0	0	0	0	0	0	1.71×10^{-2}	0.5	1.8	0.6	0.6	0.7	0.1
	-0.5	2.55×10^{-4}	0	0	0	0	0	0	1.12×10^{-3}	63.9	41.7	59.8	3.0	8.0	2.3
2	0	4.76×10^{-3}	0	0	0	0	0	0	2.43×10^{-3}	28.1	31.0	34.4	1.3	1.9	0.7
	0.5	1.56×10^{-2}	0	0	0	0	0	0	6.79×10^{-3}	7.6	21.8	11.6	0.2	0.7	0.3
	-0.5	1.99×10^{-4}	0	0	0	0	0	0	9.89×10^{-4}	3.3×10^{4}	1.9×10^{4}	4.2×10^{4}	3.6×10^{3}	1.6×10^{3}	2.4×10^{3}
3	0	5.84×10^{-3}	0	0	0	0	0	0	4.19×10^{-2}	539.2	624.7	761.8	308.7	212.9	205.9
	0.5	2.05×10^{-2}	0	0	0	0	0	0	9.57×10^{-2}	264.0	211.6	350.2	122.4	110.6	113.7
	-0.5	1.36×10^{-1}	0.5	1.7	1.7	0	0	0	4.48×10^{-1}	8.2	4.3	20.3	4.8	11.5	9.0
4	0	1.07×10^{-1}	1.8	4.1	2.6	0	0	0	3.58×10^{-1}	35.1	31.2	50.2	18.9	8.9	14.0
	0.5	6.99×10^{-2}	4.3	12.4	4.8	0	0	0	2.71×10^{-1}	77.9	72.7	98.0	57.1	36.2	50.5
	-0.5	7.20×10^{-5}	2.1×10^{5}	1.7×10^{5}	1.0×10^{5}	0	0	0	2.67×10^{-4}	1.7×10^{5}	1.7×10^{5}	2.1×10^{5}	1.7×10^{5}	1.4×10^{5}	1.6×10^{5}
5	0	3.69×10^{-3}	4327.0	5216.1	2764.3	0	0	0	2.44×10^{-2}	1503.8	1861.2	1827.7	1819.1	1332.9	1939.4
	0.5	1.50×10^{-2}	983.5	1296.6	708.7	0	0	0	6.58×10^{-2}	648.3	624.5	737.1	577.7	504.5	617.1

*Note: The relative errors are calculated by $|p_{f,SRSM}-p_{f,\,MCS}|/p_{f,\,MCS}\times100\%$, where $p_{f,SRSM}$ denotes the probability of failure using SRSM.

Table 5 Statistical parameters of basic random variables for Example #3

Variable	Mean	Standard deviation	Distribution
A_1 (m^2)	0.36	0.036	Lognormal
A_2 (m^2)	0.18	0.018	Lognormal
P (kN)	20	5.0	Type I largest

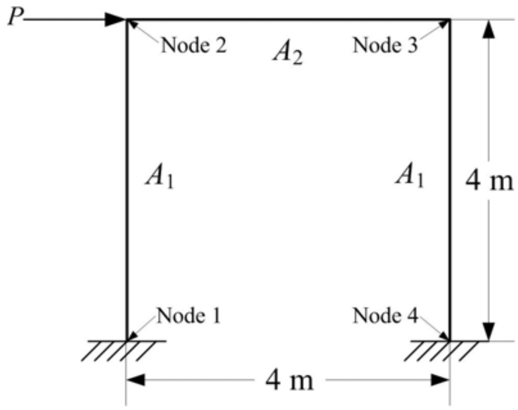


Fig. 5 Linear frame structure

The performance function regarded as the structural safety margin associated with the horizontal displacement at the node 3 is defined as

$$G(A_1, A_2, P, u_{\max}) = u_{\max} - u_3(A_1, A_2, P) \tag{13}$$

where $u_{\max}$ is the maximum allowable horizontal displacement; u_3 (A_1, A_2, P) is the calculated horizontal displacement at the node 3. Following Cheng and Li (2009), the maximum allowable horizontal displacement at the node 3 is taken as $u_{\max} = 10$ mm. The horizontal displacement at the node 3 is 4.3 mm by finite element analysis with the mean values of the three random variables, which is significantly smaller than $u_{\max}$.

It can be seen from Eq. (13) that the performance function cannot be explicitly expressed as random variables and it is also not a polynomial. In this case, there is no an intuitive guideline to choose immediately the optimal order of SRSM that would give an enough accuracy. An effective method proposed by Isukapalli (1999) can be used to check the convergence of the results and to determine the optimal order of SRSM. By successively increasing the order of SRSM, the convergence of SRSM is determined through comparison with the results from two successive order SRSMs. If the CDF curves of SRSM associated with two successive orders agree closely, the SRSM is assumed to have converged. Then, the lower order SRSM can be taken as the optimal order of SRSM. If the CDF curves differ significantly, the next order SRSM is used, and the entire process is repeated until the convergence has been reached. Additionally, when the results obtained from the MCS are available, which can also be used to check the convergence of the SRSM.

Taking the SRSM with the point method as an example, Fig.6 shows the CDF curves on log scale obtained from various order SRSMs. For comparison, the results obtained from the MCS with a sample of 10^6 are also shown in Fig. 6. It can be seen that the CDF curve for the 2nd order

SRSM significantly differs from the CDF curves for the 3rd to 5th order SRSMs. The CDF curves for the 3rd to 5th order SRSMs appear to be the same. Furthermore, the CDF curve for the 3rd order SRSM is almost the same as that obtained from the MCS. According to the aforementioned convergence criterion, the 3rd order SRSM can be taken as the optimal order of SRSM because of its computational accuracy and efficiency. In addition, all the collocation points selected for various order SRSMs can satisfy the condition of full rank matrix, but only the accuracy of the 2nd SRSM is not enough. This inaccuracy could be attributed to the low order of PCE rather than the condition of full rank matrix.

Fig. 7 further shows the CDF curves obtained from the 3rd and 5th order SRSMs with different collocation point methods. The result for the MCS with a sample of 10^6 is also plotted in Fig. 7, which is taken as the exact solution herein. Note that the CDF curves associated with the three collocation point methods are almost the same as the exact solution. These results indicate that all the three collocation point methods can produce sufficiently accurate reliability results. Based on the results shown in Fig. 7, the probabilities of failure can be determined. Table 6 summaries the probabilities of failure for the three collocation point methods and the corresponding relative errors compared with the probability of failure for the MCS, 2.29×10^{-3}. If the distribution tails and the probability of failure are of interest, the without origin method has a comparable accuracy to the point and root methods underlying the 3rd and 5th order SRSMs. Such results further imply that the origin is not absolutely necessary.

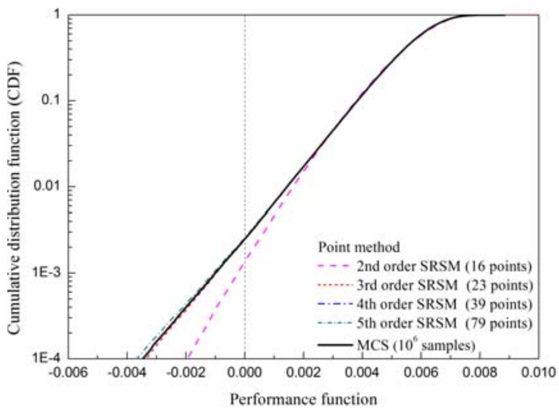
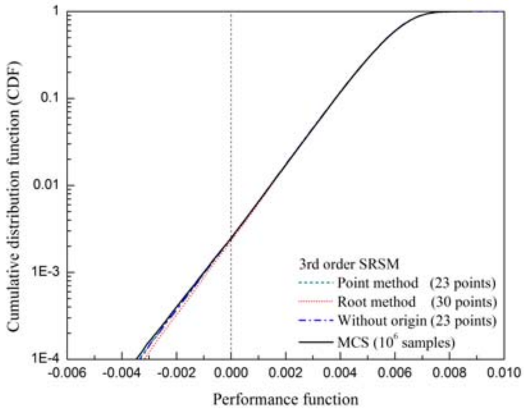


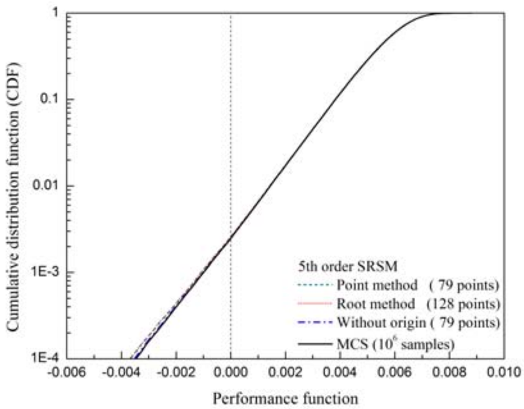
Fig. 6 Comparison of CDF curves of the performance function obtained from various order SRSMs with the point method and the direct MCS

Table 6 Comparison between the results obtained from the SRSMs with different collocation point methods compared with the direct MCS

Collocation point methods	3rd order SRSM			5th order SRSM		
	N_p	p_f	Relative errors	N_p	p_f	Relative errors
Point method	23	2.21×10^{-3}	3.7%	79	2.37×10^{-3}	3.5%
Root method	30	2.07×10^{-3}	9.6%	128	2.39×10^{-3}	4.0%
Without origin	23	2.20×10^{-3}	3.9%	79	2.30×10^{-3}	0.4%



(a) 3rd order SRSM



(b) 5th order SRSM

Fig. 7 Comparison of CDF curves of the performance function obtained from the SRSMs with different collocation point methods

5. Conclusions

Three collocation point methods associated with the odd order SRSMs, namely the point method, the root method and the without origin method, are presented and compared systematically. The regression based SRSMs are employed for conducting reliability analyses. Three numerical examples are investigated to compare the accuracy and efficiency of the three methods comprehensively. Several conclusions can be drawn from this study:

• For the odd order SRSMs with the three collocation point methods, the condition that the Hermite polynomial information matrix evaluated at the collocation points selected has a full rank should be satisfied to yield reliability results with a sufficient accuracy. However, the condition of full rank matrix may not always produce sufficiently accurate reliability results for the high order performance functions involving strongly non-normal variables. This could be attributed to the SRSM itself (i.e., low order of PCE or type of polynomial chaos) rather than the condition of full rank matrix.

• Both the point method and the without origin method can evaluate reliability efficiently. The point method and the without origin method are much more efficient than the root method, especially for the reliability problems involving a large number of random variables or requiring complex finite element analysis.

• The without origin method can also produce sufficiently accurate reliability results in comparison with the point and root methods. The origin often used as a collocation point is not absolutely necessary. Therefore, the odd order SRSMs with the point method and the without origin method are recommended for reliability analysis due to their computational accuracy and efficiency.

• The accuracy of the SRSMs with Hermite polynomial chaos is highly dependent on the form of performance function and the distributions of random variables involved. For normal variables, the SRSM with an order equaling or exceeding the order of the performance function can yield reliability results with a sufficient accuracy. While for strongly non-normal variables, the order of SRSM should significantly exceed the order of the performance function to produce sufficiently accurate results. The SRSM with other orthogonal polynomial chaos such as Laguerre polynomial or Jacobi polynomial should be adopted for the high order performance functions involving strongly non-normal variables.

• In the general cases, there is no an intuitive guideline to select immediately the optimal order of SRSM that can give an enough accuracy for an arbitrary deterministic model. A convergence analysis method can be employed to determine the optimal order of the SRSM and to check the accuracy of reliability results.

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