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Analysis of structural dynamic reliability based on the probability density evolution method

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Abstract. A new dynamic reliability analysis of structure under repeated random loads is proposed in this paper. The proposed method is developed based on the idea that the probability density of several times random loads can be derived from the probability density of single-time random load. The reliability prediction models of structure based on time responses under several times random loads with and without strength degradation are obtained by using the stress-strength interference theory and probability density evolution method. The resulting differential equations in the prediction models can be solved by using the forward finite difference method. Then, the probability density functions of strength redundancy of the structures can be obtained. Finally, the structural dynamic reliability can be calculated using integral method. The efficiency of the proposed method is demonstrated numerically through a speed reducer. The results have shown that the proposed method is practicable, feasible and gives reasonably accurate prediction.

Keywords: probability density evolution; random loads; structure; dynamic; reliability

1. Introduction

The structural reliability is an important indicator to evaluate the structural performance. Classic reliability of civil and structure have been described in many papers, such as the first order reliability method (FORM) (Dilip and Tanmoy 2001, Au *et al.* 2007, Katafygiotis and Zuev 2008, Kmet *et al*. 2011, Knut and Gunner 2000) and Monte Carlo simulation (MCS) (Chen 1994, Schueller 2009, Paik *et al*. 2009, Basage *et al*. 2012, Chen *et al*. 2001). In addition, dynamic reliability of structures has been studied as well (Basaga *et al*. 2012, Gao *et al*. 2003, Benfratello *et al*. 2006, Moustafa and Mahadevan 2011, Song and Lv 2009). In these methods, structures under single load were analyzed without considering the relationship between strength, load and time, and thus the estimated reliability of the structures were not accurate. From time to time, research has explored new methods to precisely compute structural reliability. A fully probabilistic analysis

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method provides a new way to solve this problem and accurately determine the structural reliability. The fully probabilistic analysis has been developed based on the integral method in reliability analysis when the complete information of random responses can be obtained using the distribution of the simulated random variables and the transfer function of the relationship between the load and response. In recent years, static and dynamic random responses of the structures have been studied using the probability density evolution method (PDEM) (Chen and Li 2004, Li and Chen 2004, Chen and Li 2004, Li and Chen 2008). The probability density functions of the structures under different static loading levels have been obtained by PDEM with the complete information of random variables. In fact, in the service period of a structure, not only it has been subjected to the random loads over time but also its strength has been degraded with time due to corrosion, vibration, and other factors. Therefore, the dynamic probabilistic reliability of structures and systems using the first order second moment (FORM) method was studied (Fang *et al*. 2013, Wang *et al*. 2010). Structural reliability from the time response prediction models under common loads using the FORM method has also been considered (Fang and Chen 2012) but the FORM method is not accurate when it is used to calculate structural reliability.

From time to time, new methods have been proposed to precisely compute structural reliability. A fully probabilistic analysis method provides a new way to solve this problem and accurately determine the structural reliability. The fully probabilistic analysis has been developed based on the integral method in reliability analysis when the complete information of random responses is obtained by using the distribution of the simulated random variables and the transfer function of the relationship between the load and response. In this paper, the fully probabilistic structural reliability from time response prediction model under several times random loads are established by using the PDEM. The proposed method is developed accordance to the stress-strength interference theory when the probability distribution of random loads and the structural strength are known. Two cases with and without strength degradation over time are considered. The structural reliability from time responses is estimated by solving the resulting differential equations of the reliability prediction model. Finally, it is demonstrated that the proposed method is feasible, accurate and practicable by two worked examples.

2. The probability density evolution equation (PDEE) of the structural reliability from time response

2.1 The PDEE of the structural reliability from time response without strength degeneration

The structure has been beard several times random loads but its strength is not degraded in its service period. The structural a load S is a random variable, its stress s is a random variable too where its cumulative distribution function and the probability density function are $G_{(S)}$ and $g_{(S)}$, respectively. During the structural service period, the structure is not subjected to single continuous load, but several times random loads. If the structure does not fail under the maximum load of several time random loads, the structure will also not fail under the several times random loads. Let the maximum value of n times of random loads is $S_{\max}$ and it is assumed that the structural reliability under n times random loads is equivalent to the reliability under the maximum random load. For this reason, from a conservative point of view, the structural reliability under n times the maximum load $S_{\max}$ can be used the structural reliability under n times random loads. That is to say, $S_{\max}$ be used the equivalent load to predict the structural reliability.

The cumulative distribution of the stress s_{max} of S_{max} under n times random loads which is equivalent to the maximum load can be written as follows

$$F(s_{max}) = [G(s_{max})]^n \tag{1}$$

The arising random loads are considered to obey a Poisson distribution with mean parameter λ (Fang *et al.* 2013). Thus, the probability distribution of the stress at time t is given as follows

$$\begin{aligned} P(s(t)) &= \sum_{n=0}^{+\infty} P(N(t) - N(0) = n)[G(s_{max})]^n \\ &= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}[G(s_{max})]^n \\ &= e^{\lambda t[G(s_{max})-1]} \end{aligned} \tag{2}$$

Here $N(0)$ is the number of the loads at time 0, $N(t)$ is the number of the loads at time t.
Its probability density function of the stress at time t is

$$f(s(t)) = \lambda t e^{\lambda t[G(s_{max})-1]} g(s_{max}) \tag{3}$$

The probability density of the structure strength redundancy $r(t) = \delta$- $s_{max}(t)$ under n times random loads can be obtained as follows

$$p_{r(t)} = f(\delta)\int_0^{\delta} \lambda t e^{\lambda t[G(s_{max})-1]} g(s_{max}(t)) ds_{max} \tag{4}$$

where the strength of the structure is δ and its probability density function is $f(\delta)$.

Let s_{max} is substituted by $s(t)$, Eq. (4) can be rewritten as follows

$$\begin{aligned} p_{r(t)} &= f(\delta)\int_0^{\delta} \lambda t e^{\lambda t[G(s(t))-1]} g(s(t)) ds(t) \\ &= f(\delta)(e^{\lambda t[G(\delta)-1]} - e^{\lambda t[G(0)-1]}) \end{aligned} \tag{5}$$

The probability density of the structure strength redundancy under several times random loads can be denoted as follows

$$p(\delta, t_0) = f(\delta)(e^{\lambda t_0[G(\delta)-1]} - e^{\lambda t_0[G(0)-1]}) \tag{6}$$

Based on the stress-strength interference theory, strength and stress are the only random variables and it is assumed that there are no other random variables in the whole service period of the structure. In other words, other variables are deterministic. Hence, the probability conservation principle is conformed as follows

$$\frac{\partial p(\delta,t)}{\partial t} + a(t)\frac{\partial p(\delta,t)}{\partial \delta} = 0 \tag{7}$$

where $a(t)$ is a Dirac function.

Eq. (7) is a classical Liouville differential equation and its initial condition is given as

$$p(\delta,t)|_{t=t_0} = f(\delta)(e^{\lambda t[G(\delta)-1]} - e^{\lambda t[G(0)-1]}) \tag{8}$$

The probability density function *p(δ, τ)* of the structure strength redundancy *r(t)* can be obtained by solving Eq. (8). The structural reliability from time response *R*(*t*) under several times random loads for the case without strength degeneration can be calculated as follows

$$R(t) = \int_{\delta > S} p(\delta,t)d\delta \tag{9}$$

2.2 The PDEE of the structural reliability from time response with strength degeneration

In fact, the structural strength is degraded over time due to vibration and corrosion etc in its service period. The remaining strength *δ(τ)* of the structure at time *τ* is assumed to obey a Weibull probability distribution (Schaff and Davidson 1997).

Its cumulative distribution function can be calculated as follows

$$H[\delta(\tau)] = 1 - \exp[-(\frac{\delta(\tau)-\gamma(\tau)}{\eta(\tau)})^{\beta(\tau)}] \tag{10}$$

where $\beta(\tau)$, $\eta(\tau)$, and $\gamma(\tau)$ are the Weibull distribution parameters over time, τ is the structural service time in its service period T, $\beta(\tau)$, $\eta(\tau)$, and $\gamma(\tau)$ are the shape parameter, the scale parameter and the position parameter of the functions of the parameters of Weibull distribution in the time τ, respectively.

The probability density function $h[\delta(\tau)]$ of the $\delta(\tau)$ can be obtained by Eq. (10) as follows

$$h[\delta(\tau)] = \frac{\beta(\tau)}{\eta(\tau)}(\frac{\delta(\tau)-\gamma(\tau)}{\eta(\tau)})^{\beta(\tau)-1}\exp[-(\frac{\delta(\tau)-\gamma(\tau)}{\eta(\tau)})^{\beta(\tau)}] \tag{11}$$

The probability density of the structural strength redundancy $r(t) = \delta(t) - s_{\max}(t)$ under several times random loads with the strength degeneration is obtained as follows

$$q_{r(t)} = f(\delta(\tau))(e^{\lambda t[G(\delta(\tau))-1]} - e^{\lambda t[G(0)-1]}) \tag{12}$$

The probability density of the structural strength redundancy under several times random loads with the strength degeneration can be denoted as follows

$$q(\delta(0),t_0) = f(\delta(0))(e^{\lambda t_0[G(\delta(0))-1]} - e^{\lambda t_0[G(0)-1]}) \tag{13}$$

The differential equation can be obtained by using the probability conservation principle as follows

$$\frac{\partial q(\delta(\tau),t)}{\partial t} + a(t)\frac{\partial q(\delta(\tau),t)}{\partial \delta(\tau)} = 0 \tag{14}$$

Eq. (14) is a classical Liouville differential equation and its initial condition is given

$$q(\delta(0),t)|_{t=t_0} = h(\delta(0))(e^{\lambda t[G(\delta(0))-1]} - e^{\lambda t[G(0)-1]}) \tag{15}$$

The probability density function $q(\delta(\tau),t)$ of the structural strength redundancy $r(t)$ can be obtained by solving Eq. (14). By integrating the probability density function, the reliability $R(t)$ of the structural reliability from time response under several times random loads for the case with structural strength degeneration can be obtained as follows

$$R(t) = \int_{\delta(\tau)>s} q(\delta(\tau),t)d\delta(\tau) \tag{16}$$

3. Solving the PDEE

Eqs. (7) and (14) are the probability density evolution differential equations for the cases without and with strength degeneration, respectively. These differential equations have analytical solutions (Chen *et al.* 2001). Nevertheless, in practice, the probability density evolution differential equation is normally solved using numerical approaches. In this study, Eqs. (7) and (14) are solved using the forward finite difference (FFD) method (Li and Chen 2004). This is because not only compatibility of the algorithm can be ensured, but also nonnegative, completeness of the probability density and accurateness of the reliability can be obtained.

The main steps of the algorithm are given as follows:

Step 1. The initial conditions shown in Eqs. (8) and (15) can be dispersed as follows

$$p(\delta_i,t_0) = f(\delta_i)(e^{\lambda t_0[G(\delta_i)-1]} - e^{\lambda t_0[G(0)-1]}) \tag{17}$$

$$q(\delta_i(0),t_0) = h(\delta_i(0))(e^{\lambda t_0[G(\delta_i(0))-1]} - e^{\lambda t_0[G(0)-1]}) \tag{18}$$

where $\delta_i = i\cdot \Delta\delta$, δ is divided into $\Delta\delta$ for Eq. (17) and $\delta_i(t) = i\cdot \Delta\delta(t)$, $\delta(t)$ is divided into $\Delta\delta(t)$ for Eq. (18), i = 0,1,2, …. Similarly, t is discretized, Δt is a dividing in direction of the t, $t_j = j\cdot \Delta t$, j = 0,1,2, ….

Then, $\lambda_1 = \Delta t/\Delta\delta$ and $\lambda_2 = \Delta t/\Delta\delta(\tau)$. The limit $0 < \lambda_1 a(t), \lambda_2 a(t) \le 1$ is to ensure convergence and stable of the algorithm.

Step 2. Eqs. (7) and (14) can be computed using Eqs. (19) and (20), respectively

$$p_j^{k+1} = p_j^{(k)} - \lambda_1[p_j^{(k)} - p_{j-1}^{(k)}] \tag{19}$$

where $p_j^{(k)} = p(\delta_j, t_k)$

$$q_j^{k+1} = q_j^{(k)} - \lambda_2[q_j^{(k)} - q_{j-1}^{(k)}] \tag{20}$$

where $q_j^{(k)} = q(\delta_j(\tau), t_k)$

The parameters $\beta(\tau)$, $\eta(\tau)$ and $\gamma(\tau)$ of the Weibull distribution in $q_j^{(k)}$ can be computed from $q_{j-1}^{(k)}$.

The probability density functions of Eqs. (7) and (14) can be obtained by the proposed algorithm. Once the probability density functions have been obtained, the structural reliability from time response can be calculated as given in Eq. (9) for the case without strength degeneration or Eq. (16) for the case with strength degeneration.

4. Worked Examples

4.1 Example 1. PDEM without strength degradation

Fig. 1 shows an axial speed reducer (ASR) which is made of A3 steel. The reliability of the most dangerous cross section will be predicted. It is investigated to test the performance of the proposed probability density evolution method. The axial strength of the speed reducer obeys the normal distribution with the mean value of 110MPa and standard deviation of 15MPa. The strength is not degraded over time. Similarly, the stresses on the ASR also obey the normal distribution with the mean value of 50MPa and standard deviation of 15MPa.

For the case without strength degeneration, the probability density function of the speed reducer axial strength redundancy is calculated using Eq. (7) under repeated loads with $\lambda = 1$ and $\lambda_1 a(t) = 0.5$. The results are shown in Fig. 2 different time steps, $t = 0$, 25 and 100. Next, the reliability of the axial speed reducer is computed using 3 different approaches i.e., PDEM, FORM and MCS and the results are shown in Table 1.

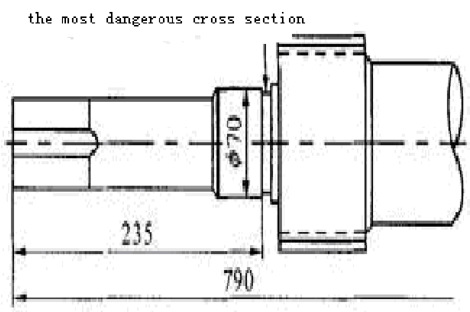

Fig. 1 An axial speed reducer

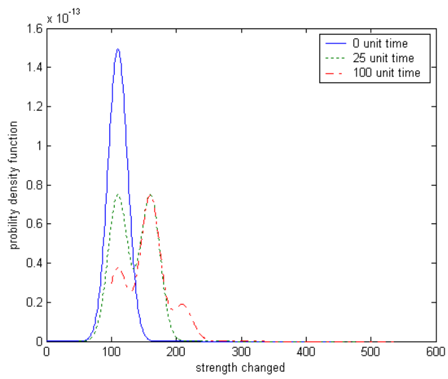

Fig. 2 The curves of the probability density at $t = 0$, 25, 100 without strength degeneration

Table1 Reliability of the speed reducer obtained from 3 different approaches without strength degeneration

time	PDEM	FORM	MCS($n = 10^6$)
0	0.9978	0.9978	0.9978
25	0.9960	0.9952	0.9960
100	0.9785	0.9602	0.9786

4.2 Example 2. PDEM with strength degeneration

The similar axial speed reducer with the same initial strength and stresses where both obey the normal distribution is also used to validate the proposed approach for the case of strength degeneration. The probability density function of the speed reducer axial strength redundancy is computed using Eq. (14) with $\lambda = 1$ and $\lambda_1 a(t) = 0.5$. The curves are shown in Fig. 3 at $t = 0$, 25, 100 unit time. Its reliability from time response is obtained using the PDEM, FORM, MCS and the results at $t = 0$, 25, 100 unit time are shown in Table 2.

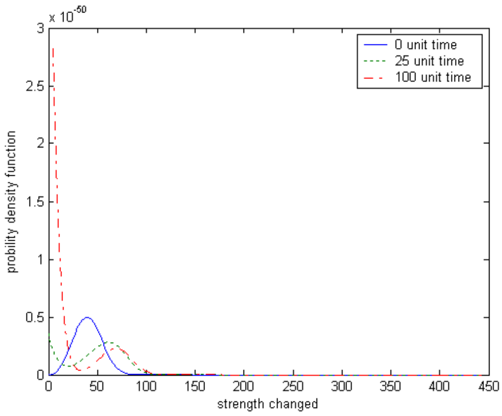

Fig. 3 The curves of the probability density at $t = 0$, 25, 100 with strength degeneration

Table 2 Reliability of the speed reducer obtained from 3 different approaches with strength degeneration

time	PDEM	FORM	MCS($n = 10^6$)
0	0.9978	0.9978	0.9978
25	0.9665	0.9543	0.9665
100	0.9050	0.8987	0.9051

4.3 Discussions

As shown in Figs. 2 and 3, the identified probability density function is a normal distribution when $t = 0$, but this is not the case when $t = 25$ and 100. Thus, the probability density curves are becoming more and more complex over time. The width and peak of the probability density curves are increased, and the fluctuation are significantly enhanced. From Tables 1 and 2, the reliability at $t = 0$ from the 3 different approaches are the same because the random strength and stresses obey the normal distribution. However, when the random strength and stresses do not obey the normal distribution at $t = 25$ and 100, the reliability from time response values obtained from the proposed PDEM method are more accurate compared to the FORM and are almost the same with the values from numerical approach (MCS). In the evolutionary process, when the probability density curves of the structural strength redundancy under repeated random loads do not obey the normal distribution, the dynamic reliability obtained from the FORM is not accurate and occasionally produce an obvious error.

Table 1 shows that even if strength does not degenerate, the structural reliability gradually decreases over time due to the random loads acting on the structure. When strength degenerates, the reliability decreases over time more rapidly as shown in Table 2. This makes sense due to the combination effects of random loads and strength degeneration. The probability density curves of the structural strength redundancy under repeated random loads and the real-time changing reliability with and without the structural strength degeneration can be accurately obtained from the proposed PDEM method.

5. Conclusions

A new probability density evolution method has been developed for modeling the structural reliability from time responses under several times random loads. Two cases with and without strength degeneration are considered. The differential equation in the model is solved using the forward finite difference method. The probability density curves of the structural strength redundancy can be obtained at any time during the service period. The results have shown that the proposed method is simple, practicable, efficient, easy to implement and gives reasonable accurate prediction. The proposed approach can be applied to determine the reliable operation of service life and the maintenance schedule. Thus, it is very useful in the structural life cycle management and dynamic design.

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