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# A new method to calculate the equivalent stiffness of the suspension system of a vehicle

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**Abstract.** The stiffness of a suspension system is provided by the bushings and the stiffness of the wheel center controls the suspension's elasto-kinematic (e-k) specification. So the stiffness of the wheel center is very important, but the stiffness of the wheel center is very hard to measure. The paper give a new method that we can use the stiffness of the bushings to calculate the equivalent stiffness of the wheel center, which can quickly and widely be used in all kinds of suspension structure. This method can also be used to optimize and design the suspension system. In the example we use the method to calculate the equivalent stiffness of the wheel center which meets the symmetric and positive conditions of the stiffness matrix.

**Keywords:** stiffness of the wheel center; stiffness of bushing; suspension structure

## 1. Introduction

The suspension is a structure which connect the vehicle's frame (or body) with the axle (or the wheel). The main role of the suspension is passing all the forces and torques between the wheel and the body, such as the supporting force, braking force, driving force and cornering force, easing the impact force which is caused by the uneven road and passed by the body, attenuating the vibration which caused by the impact force, ensuring the occupant comfort and reducing the dynamic load of the vehicle. Therefore the suspension is very important for a vehicle. There are so many researchers investigating the suspension characteristics (Fialho and Balas 2002, Long *et al.* 2011). A vehicle suspension is shown in Fig. 1.

The suspension controls the vibration of the vehicle. Many researchers have studied the vibration control issues (Li *et al.* 2004, Li and Liu 2011, Yun and Li 2011), and some researchers have investigated the suspension structure optimization (Chen and Huang 2005, Nguyen and Choi 2009, Crews *et al.* 2011, Kang *et al.* 2011). Some methods have been found to optimal the parameters of suspension with constant harmonic excitations (Metallidis *et al.* 2003, Verros *et al.* 2005, Georgiou *et al.* 2007, Crews *et al.* 2011). These researchers use the equivalent stiffness of the suspension to analyze the vibration of vehicle.

The suspension system also controls the elasto-kinematic performance and the handling. Gerrard considers the suspension system is a single elastic system connecting the wheel carrier to the ground

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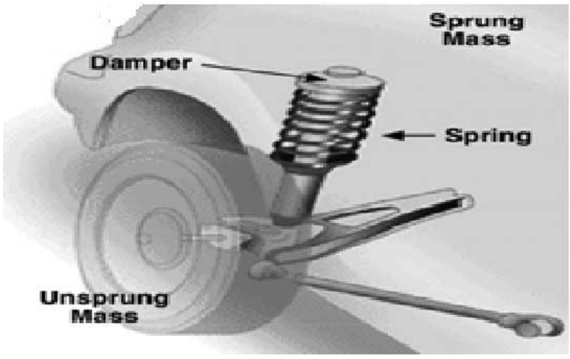


Fig. 1 Vehicle suspension

and the compliant behavior of the elastic system can be represented by a single stiffness matrix. He finds the Equivalent Elastic Mechanism (EEM) method to analysis and design the compliant suspension linkages (Gerrard 2005). Nishimura and Nozawa use the EEM theory to analysis and design the suspension system and the results match with the actual measurement (Nishimura and Nozawa 2007). We can also use this method to calculate the toe angle, castor angle and camber angle which determine the good or bad vehicle handling (Gerrard 2005). So it's very important to know the equivalent stiffness matrix of the wheel center.

There are so many articles about the analysis of the stiffness matrix, such as JOSIP find that there is a normal form for a generic compliance matrix when the stiffness assumes a normal form (JOSIP LONCARIC 1987), Li and Schimmels analyze the stiffness of a 3-PUU parallel kinematic machine through decomposition of the stiffness matrix (Huang and Schimmels 2000, Li and Xu (2008), and Lipkin use the stiffness matrix to find the compliant axes (Patterson and Lipkin 1993). But until now all the stiffness matrix used in these articles is experimentally measured or simulated in software, there is no such a method to solve the equivalent stiffness of a complex structure.

The equivalent stiffness of the wheel center is very important for the suspension system. As we know that the stiffness of the wheel center is provided by the bushings which connect the vehicle frame with the suspension and the stiffness of the bushing can be easily measured. We can change the stiffness of the suspension system through changing the stiffness of the bushing or the location of the bushing. But until now we don't know the relationship between the stiffness of the wheel center and the stiffness of the bushings. The paper will give such a method which will find the relationship between the stiffness of the wheel center and the stiffness and the location of the bushings. We also can use the method to optimize the stiffness of the suspension system.

The paper describes a method which calculates the stiffness of the wheel center by the bushing's stiffness. We can use this method to quickly calculate the stiffness of the wheel center from the stiffness of the bushings. This method can not only be used in the automotive industry but also be used in the other machinery industry.

In this paper, first we give the method to calculate the equivalent stiffness of the wheel center. We start from one bushing to Multi-bushings and we get a recursive formula. Second, we provide a example of practical application of the method.

## 2. Calculating the equivalent stiffness matrix of a suspension

First, we will describe the assumptions throughout this paper:
(1) The linkages of the suspension is rigid;
(2) The stiffness of the suspension systems provides only by the bushings;
(3) The position of the linkage is known;
(4) All the unknown force direction are set to coordinate with the positive direction of the Global Co-ordinate System (GCS);
(5) We only consider the case of small rotations displacement and assume that all stiffness is linear.
(6) The linkages connect directly the wheel center to the bushings.
These assumptions do not introduce significant error when we consider typical loads applied in the wheel center.

Suspensions typically have numerous bushings and linkages that connect the wheel carrier with the vehicle body. In other words, we are able to consider the suspension to be a simple system connecting the wheel carrier to vehicle body with the stiffness. So we can express a suspension system by its stiffness **K**, which describes the relationship between force **F** and displacement **X** at the wheel carrier

$$F = KX \tag{1}$$

First we consider the simplest case:

### *2.1 One bushing and one linkage*

There is only a bushing and a linkage, as illustrated in Fig. 2. Point 0 is the wheel center, there is one bushing at the point 1 and the bushing fixed on the wheel carrier. $\theta_1$ is the angle between the rod 0-1 projection in 0-*xy* plane and the positive direction of *x*-axis and $\beta_1$ is the angle between the rod 0-1 and the positive direction of *z*-axis. The length of the rod 0-1 is $\mathbf{l}_1$ and the length of the bushing is neglected, the stiffness of the bushing is $\mathbf{K}_1$. We assume that the wheel carrier don't move. There is a force ($F = (F_x \ F_y \ F_z \ M_x \ M_y \ M_z)^T$) at the point 0.

Using the theoretical mechanics knowledge we can calculate the force of the bushing, written in matrix form

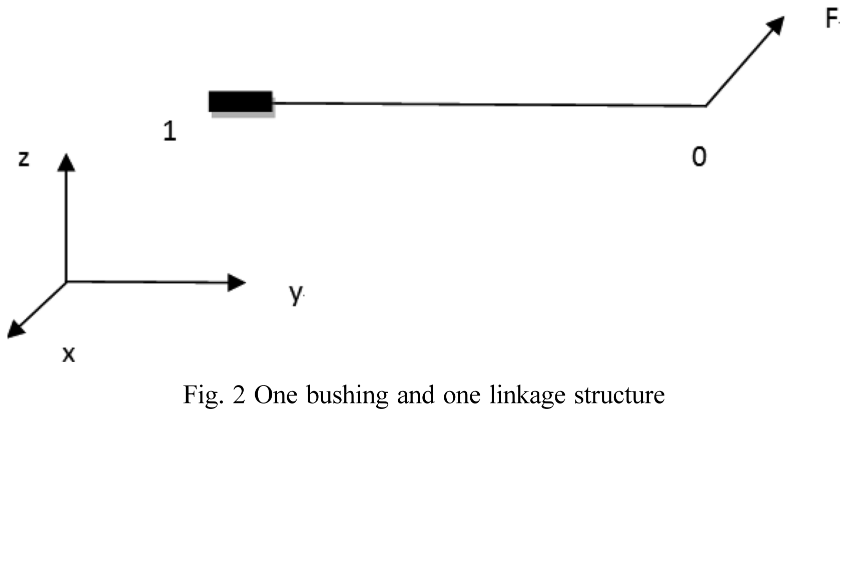


Fig. 2 One bushing and one linkage structure

$$F_1 = \begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ M_{1x} \\ M_{1y} \\ M_{1z} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_1 \cos\beta_1 & l_1 \sin\beta_1 \sin\theta_1 & 1 & 0 & 0 \\ l_1 \cos\beta_1 & 0 & -l_1 \sin\beta_1 \cos\theta_1 & 0 & 1 & 0 \\ -l_1 \sin\beta_1 \sin\theta_1 & l_1 \sin\beta_1 \cos\theta_1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{pmatrix} = AF \tag{2}$$

Using the formula (1) we can calculate the deformation (or displacement) of bushing, namely the displacement of point 1

$$d_1 = (d_{1x} \quad d_{1y} \quad d_{1z} \quad \theta_{1x} \quad \theta_{1y} \quad \theta_{1z})^T = K_1^{-1} F_1 \tag{3}$$

Using the rigid body kinematics knowledge and the method described in Appendix I, we can calculate the displacement of the point 0, namely the displacement of the wheel center. Written in matrix form

$$d_0 = \begin{pmatrix} d_{0x} \\ d_{0y} \\ d_{0z} \\ \theta_{0x} \\ \theta_{0y} \\ \theta_{0z} \end{pmatrix} = \begin{pmatrix} d_{1x} + l_1 * (\sin\beta_1 \cos(\theta_1 + \theta_{1z}) \cos\theta_y + (\cos\beta_1 \cos\theta_{1x} + \sin\beta_1 \sin(\theta_1 + \theta_z) \sin\theta_{1x}) \sin\theta_{1y} - \sin\beta_1 \cos\theta_1) \\ d_{1y} + l_1 * (\sin\beta_1 \sin(\theta_1 + \theta_{1z}) \cos\theta_{1x} - \cos\beta_1 \sin\theta_{1x} - \sin\beta_1 \sin\theta_1) \\ d_{1z} + l_1 * ((\cos\beta_1 \cos\theta_{1x} + \sin\beta_1 \sin(\theta_1 + \theta_{1z}) \sin\theta_{1x}) \cos\theta_{1y} - \sin\beta_1 \cos(\theta_1 + \theta_{1z}) \sin\theta_{1y} - \cos\beta_1) \\ \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \end{pmatrix} \tag{4}$$

Based on the assumptions (5), we have the following equations

$$\begin{aligned} 1 - \cos\theta_{1x} = 0, &\quad \sin\theta_{1x} = \theta_{1x} \\ 1 - \cos\theta_{1y} = 0, &\quad \sin\theta_{1y} = \theta_{1y} \\ 1 - \cos\theta_{1z} = 0, &\quad \sin\theta_{1z} = \theta_{1z} \end{aligned} \tag{5}$$

Therefore, we can write the formula (4) into the following simplified form

$$d_0 = \begin{pmatrix} d_{0x} \\ d_{0y} \\ d_{0z} \\ \theta_{0x} \\ \theta_{0y} \\ \theta_{0z} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & l_1 \cos\beta_1 & -l_1 \sin\beta_1 \sin\theta_1 \\ 0 & 1 & 0 & -l_1 \cos\beta_1 & 0 & l_1 \sin\beta_1 \cos\theta_1 \\ 0 & 0 & 1 & l_1 \sin\beta_1 \sin\theta_1 & -l_1 \sin\beta_1 \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \\ \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \end{pmatrix} = Dd_1 \tag{6}$$

Comparing Eqs. (2) and (6), we will find the following equation: $A = D^T$

Combining Eqs. (2), (3) and (6) we get the following equation

$$d_0 = DK_1^{-1}AF \tag{7}$$

So the compliance matrix (**C**) of the wheel center can be received

$$C = DK_1^{-1}A \tag{8}$$

Inversing the compliance matrix (**C**) we can obtain the equivalent stiffness ($K_0^1$) of the wheel center

$$K_0^1 = A^{-1} K_1 D^{-1} \tag{9}$$

### *2.2 Two bushings and two linkages*

There are two bushings and two linkages, as illustrated in Fig. 3. Point 0 is the wheel center, there are two bushings which fixed on the wheel carrier at the point 1 and at the point 2. The length of the rod 0-1 is $\mathbf{l}_1$ and the length of the rod 0-2 is $\mathbf{l}_2$. The length of the bushings is neglected and the stiffness of the bushings is $\mathbf{K}_1$ and $\mathbf{K}_2$, respectively. $\theta_1$ is the angle between the rod 0-1 projection in 0-$xy$ plane and the positive direction of $x$-axis and $\beta_1$ is the angle between the rod 0-1 and the positive direction of $z$-axis. $\theta_2$ is the angle between the rod 0-2 projection in 0-$xy$ plane and the positive direction of $x$-axis and $\beta_2$ is the angle between the rod 0-2 and the positive direction of $z$-axis. We assume that the wheel carrier don't move. There is a force ($F = (F_x \;\; F_y \;\; F_z \;\; M_x \;\; M_y \;\; M_z)^T$) at the point 0.

Setting the displacement of the point 0 is $d_0$: $d_0 = (d_{0x} \;\; d_{0y} \;\; d_{0z} \;\; \theta_{0x} \;\; \theta_{0y} \;\; \theta_{0z})^T$

Using the formula (6), we can calculate the displacement of the bushing 1 and bushing 2

$$d_1 = (d_{1x} \quad d_{1y} \quad d_{1z} \quad \theta_{1x} \quad \theta_{1y} \quad \theta_{1z})^T = D_1 d_0 \tag{10}$$

$$d_2 = (d_{2x} \quad d_{2y} \quad d_{2z} \quad \theta_{2x} \quad \theta_{2y} \quad \theta_{2z})^T = D_2 d_0 \tag{11}$$

where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_1 \cos\beta_1 & -l_1 \sin\beta_1 \sin\theta_1 \\ 0 & 1 & 0 & -l_1 \cos\beta_1 & 0 & l_1 \sin\beta_1 \cos\theta_1 \\ 0 & 0 & 1 & l_1 \sin\beta_1 \sin\theta_1 & -l_1 \sin\beta_1 \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

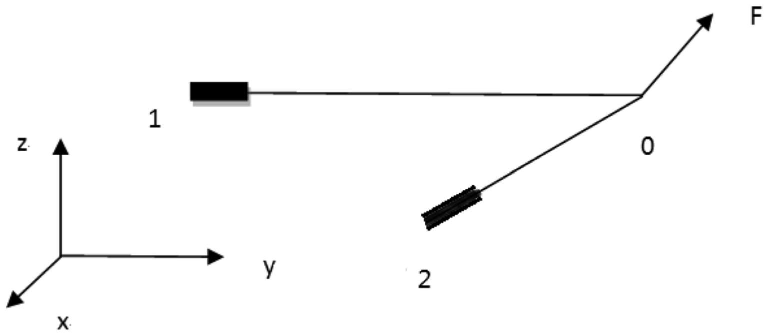


Fig. 3 Two bushings and two linkages structure

$$D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_2 \cos\beta_2 & -l_2 \sin\beta_2 \sin\theta_2 \\ 0 & 1 & 0 & -l_2 \cos\beta_2 & 0 & l_2 \sin\beta_2 \cos\theta_2 \\ 0 & 0 & 1 & l_2 \sin\beta_2 \sin\theta_2 & -l_2 \sin\beta_2 \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the formula (1), we can calculate the force of the bushing 1 and bushing 2

$$F_1 = (F_{1x} \quad F_{1y} \quad F_{1z} \quad M_{1x} \quad M_{1y} \quad M_{1z})^T = K_1 d_1 \tag{12}$$

$$F_2 = (F_{2x} \quad F_{2y} \quad F_{2z} \quad M_{2x} \quad M_{2y} \quad M_{2z})^T = K_2 d_2 \tag{13}$$

Combining Eqs. (10)-(13), we can find the relationship between $\mathbf{F}_1$ and $\mathbf{F}_2$

$$F_1 = K_1 D_1 D_2^{-1} K_2^{-1} F_2 \tag{14}$$

In the Global Co-ordinate System calculating the force and moment balance equations and writing in matrix form

$$A_1 F_1 + A_2 F_2 = F \tag{15}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_1 \cos\beta_1 & l_1 \sin\beta_1 \sin\theta_1 & 1 & 0 & 0 \\ l_1 \cos\beta_1 & 0 & -l_1 \sin\beta_1 \cos\theta_1 & 0 & 1 & 0 \\ -l_1 \sin\beta_1 \sin\theta_1 & l_1 \sin\beta_1 \cos\theta_1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_2 \cos\beta_2 & l_2 \sin\beta_2 \sin\theta_2 & 1 & 0 & 0 \\ l_2 \cos\beta_2 & 0 & -l_2 \sin\beta_2 \cos\theta_2 & 0 & 1 & 0 \\ -l_2 \sin\beta_2 \sin\theta_2 & l_2 \sin\beta_2 \cos\theta_2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining Eqs. (14) and (15) we have the following equation

$$(A_1 K_1 D_1 D_2^{-1} K_2^{-1} + A_2) F_2 = F \tag{16}$$

So we can calculate $\mathbf{F}_2$

$$F_2 = E_2^{-1} F \tag{17}$$

where

$$E_2 = A_1 K_1 D_1 D_2^{-1} K_2^{-1} + A_2 \tag{18}$$

Combining Eqs. (11), (13) and (17) we can find the relationship between the force of wheel center and the displacement of the wheel center as the following equation

$$d_0 = D_2^{-1} K_2^{-1} E_2^{-1} F \tag{19}$$

We can write the Eq. (19) in the following form

$$F = E_2 K_2 D_2 d_0 \tag{20}$$

So the equivalent stiffness of the wheel center ($K_0^2$) is

$$K_0^2 = E_2 K_2 D_2 = A_1 K_1 D_1 + A_2 K_2 D_2 \tag{21}$$

### *2.3 Three bushings and three linkages*

Three bushings and three linkages, as illustrated in Fig. 4. Point 0 is the wheel center, there are three bushings at the point 1, point 2 and point 3 which all fixed on the wheel carrier. The length of the rod 0-1 is $\mathbf{l}_1$, the length of the rod 0-2 is $\mathbf{l}_2$ and the length of the rod 0-3 is $\mathbf{l}_3$. The length of the bushings is neglected. the stiffness of the bushings is $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$, respectively. $\theta_1$ is the angle between the rod 0-1 projection in 0-*xy* plane and the positive direction of *x*-axis and $\beta_1$ is the angle between the rod 0-1 and the positive direction of *z*-axis. $\theta_2$ is the angle between the rod 0-2 projection in 0-*xy* plane and the positive direction of *x*-axis and $\beta_2$ is the angle between the rod 0-2 and the positive direction of *z*-axis. $\theta_3$ is the angle between the rod 0-3 projection in 0-*xy* plane and the positive direction of *x*-axis and $\beta_3$ is the angle between the rod 0-3 and the positive direction of

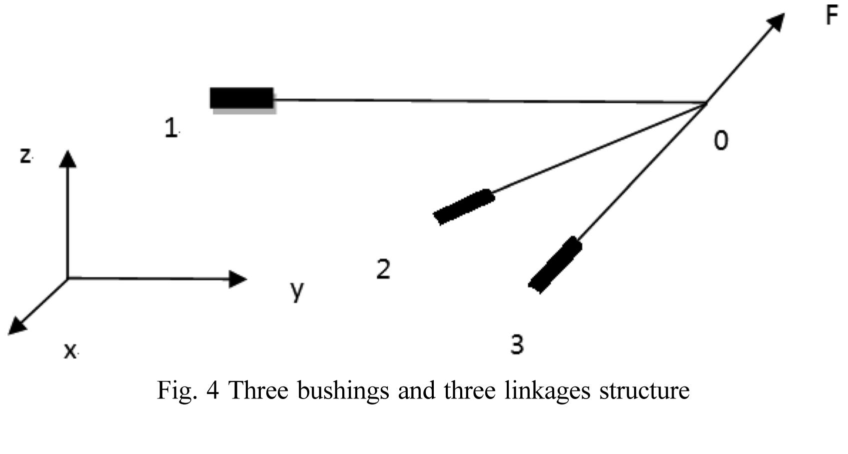


Fig. 4 Three bushings and three linkages structure

$z$-axis. We assume that the wheel carrier don't move. There is a force ($F = (F_x \ F_y \ F_z \ M_x \ M_y \ M_z)^T$) at the point 0.

Setting the displacement of the point 0 is $d_0$: $d_0 = (d_{0x} \ d_{0y} \ d_{0z} \ \theta_{0x} \ \theta_{0y} \ \theta_{0z})^T$

Using the formula (6), we can calculate the displacement of the bushing 1, bushing 2 and bushing 3

$$d_1 = (d_{1x} \quad d_{1y} \quad d_{1z} \quad \theta_{1x} \quad \theta_{1y} \quad \theta_{1z})^T = D_1 d_0 \tag{22}$$

$$d_2 = (d_{2x} \quad d_{2y} \quad d_{2z} \quad \theta_{2x} \quad \theta_{2y} \quad \theta_{2z})^T = D_2 d_0 \tag{23}$$

$$d_3 = (d_{3x} \quad d_{3y} \quad d_{3z} \quad \theta_{3x} \quad \theta_{3y} \quad \theta_{3z})^T = D_3 d_0 \tag{24}$$

where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_1 \cos\beta_1 & -l_1 \sin\beta_1 \sin\theta_1 \\ 0 & 1 & 0 & -l_1 \cos\beta_1 & 0 & l_1 \sin\beta_1 \cos\theta_1 \\ 0 & 0 & 1 & l_1 \sin\beta_1 \sin\theta_1 & -l_1 \sin\beta_1 \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_2 \cos\beta_2 & -l_2 \sin\beta_2 \sin\theta_2 \\ 0 & 1 & 0 & -l_2 \cos\beta_2 & 0 & l_2 \sin\beta_2 \cos\theta_2 \\ 0 & 0 & 1 & l_2 \sin\beta_2 \sin\theta_2 & -l_2 \sin\beta_2 \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_3 \cos\beta_3 & -l_3 \sin\beta_3 \sin\theta_3 \\ 0 & 1 & 0 & -l_3 \cos\beta_3 & 0 & l_3 \sin\beta_3 \cos\theta_3 \\ 0 & 0 & 1 & l_3 \sin\beta_3 \sin\theta_3 & -l_3 \sin\beta_3 \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the formula (1), we can calculate the force of the bushing 1, bushing 2 and bushing 3

$$F_1 = (F_{1x} \quad F_{1y} \quad F_{1z} \quad M_{1x} \quad M_{1y} \quad M_{1z})^T = K_1 d_1 \tag{25}$$

$$F_2 = (F_{2x} \quad F_{2y} \quad F_{2z} \quad M_{2x} \quad M_{2y} \quad M_{2z})^T = K_2 d_2 \tag{26}$$

$$F_3 = (F_{3x} \quad F_{3y} \quad F_{3z} \quad M_{3x} \quad M_{3y} \quad M_{3z})^T = K_3 d_3 \tag{27}$$

Combining Eqs. (22), (23), (25) and (26), we can find the relationship between $\mathbf{F}_1$ and $\mathbf{F}_2$

$$F_1 = K_1 D_1 D_2^{-1} K_2^{-1} F_2 \tag{28}$$

Similarly, combining Eqs. (22), (24), (25) and (27), we can find the relationship between $\mathbf{F}_3$ and $\mathbf{F}_2$

$$F_3 = K_3 D_3 D_2^{-1} K_2^{-1} F_2 \tag{29}$$

In the Global Co-ordinate System calculating the force and moment balance equations and written in matrix form

$$A_1 F_1 + A_2 F_2 + A_3 F_3 = F \tag{30}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_1 \cos\beta_1 & l_1 \sin\beta_1 \sin\theta_1 & 1 & 0 & 0 \\ l_1 \cos\beta_1 & 0 & -l_1 \sin\beta_1 \cos\theta_1 & 0 & 1 & 0 \\ -l_1 \sin\beta_1 \sin\theta_1 & l_1 \sin\beta_1 \cos\theta_1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_2 \cos\beta_2 & l_2 \sin\beta_2 \sin\theta_2 & 1 & 0 & 0 \\ l_2 \cos\beta_2 & 0 & -l_2 \sin\beta_2 \cos\theta_2 & 0 & 1 & 0 \\ -l_2 \sin\beta_2 \sin\theta_2 & l_2 \sin\beta_2 \cos\theta_2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_3 \cos\beta_3 & l_3 \sin\beta_3 \sin\theta_3 & 1 & 0 & 0 \\ l_3 \cos\beta_3 & 0 & -l_3 \sin\beta_3 \cos\theta_3 & 0 & 1 & 0 \\ -l_3 \sin\beta_3 \sin\theta_3 & l_3 \sin\beta_3 \cos\theta_3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining Eqs. (28)-(30) we have the following equation

$$(A_1 K_1 D_1 D_2^{-1} K_2^{-1} + A_2 + A_3 K_3 D_3 D_2^{-1} K_2^{-1}) F_2 = F \tag{31}$$

So we can calculate $\mathbf{F}_2$

$$F_2 = E_3^{-1} F \tag{32}$$

where

$$E_3 = A_1K_1D_1D_2^{-1}K_2^{-1} + A_2 + A_3K_3D_3D_2^{-1}K_2^{-1} \tag{33}$$

Combining Eqs. (23), (26) and (32) we can find the relationship between the force of wheel center and the displacement of the wheel center as the following equation

$$d_0 = D_2^{-1}K_2^{-1}E_3^{-1}F \tag{34}$$

We can write the Eq. (34) in the following form

$$F = E_3K_2D_2d_0 \tag{35}$$

So the equivalent stiffness of the wheel center ($K_0^3$) is

$$K_0^3 = E_3K_2D_2 = A_1K_1D_1 + A_2K_2D_2 + A_3K_3D_3 \tag{36}$$

Comparing the Eqs. (21) and (36), we will find the following relationship:

$$K_0^3 = K_0^2 + A_3K_3D_3 \tag{37}$$

### *2.4 Multi-bushings and multi-linkages*

Using the mathematical induction we can calculate the stiffness of the wheel center with $N$ bushings and linkages. We assume that we already calculate the equivalent stiffness of the wheel center with $N$-1($N > 2$) bushings and linkages is $K_0^{N-1}$, so we can know the $\mathbf{E}_{N\text{-}1}$. There is another bushing and linkage connecting the wheel center with the wheel carrier, the length of the rod 0-N is $\mathbf{l}_N$ and the stiffness of the bushing is $\mathbf{K}_N$.

Setting the displacement of the point 0 is $d_0$: $d_0 = (d_{0x}\ \ d_{0y}\ \ d_{0z}\ \ \theta_{0x}\ \ \theta_{0y}\ \ \theta_{0z})^T$

Using the formula (6), we can calculate the displacement of the bushing $N$ and the bushing 2

$$d_N = (d_{Nx}\quad d_{Ny}\quad d_{Nz}\quad \theta_{Nx}\quad \theta_{Ny}\quad \theta_{Nz})^T = D_Nd_0 \tag{38}$$

$$d_2 = (d_{2x}\quad d_{2y}\quad d_{2z}\quad \theta_{2x}\quad \theta_{2y}\quad \theta_{2z})^T = D_2d_0 \tag{39}$$

where

$$D_N = \begin{bmatrix} 1 & 0 & 0 & 0 & l_N\cos\beta_N & -l_N\sin\beta_N\sin\theta_N \\ 0 & 1 & 0 & -l_N\cos\beta_N & 0 & l_N\sin\beta_N\cos\theta_N \\ 0 & 0 & 1 & l_N\sin\beta_N\sin\theta_N & -l_N\sin\beta_N\cos\theta_N & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & l_2\cos\beta_2 & -l_2\sin\beta_2\sin\theta_2 \\ 0 & 1 & 0 & -l_2\cos\beta_2 & 0 & l_2\sin\beta_2\cos\theta_2 \\ 0 & 0 & 1 & l_2\sin\beta_2\sin\theta_2 & -l_2\sin\beta_2\cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Using the formula (1), we can calculate the force of the bushing $N$ and bushing 2

$$F_N = (F_{Nx} \quad F_{Ny} \quad F_{Nz} \quad M_{Nx} \quad M_{Ny} \quad M_{Nz})^T = K_N d_N \tag{40}$$

$$F_2 = (F_{2x} \quad F_{2y} \quad F_{2z} \quad M_{2x} \quad M_{2y} \quad M_{2z})^T = K_2 d_2 \tag{41}$$

Combining Eqs. (38)-(41), we can find the relationship between $\mathbf{F}_{\mathrm{N}}$ and $\mathbf{F}_2$

$$F_N = K_N D_N D_2^{-1} K_2^{-1} F_2 \tag{42}$$

In the Global Co-ordinate System calculating the force and moment balance equations and written in matrix form

$$E_{N-1}F_2 + A_N F_N = F \tag{43}$$

where

$$A_N = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -l_N\cos\beta_N & l_N\sin\beta_N\sin\theta_N & 1 & 0 & 0 \\ l_N\cos\beta_N & 0 & -l_N\sin\beta_N\cos\theta_N & 0 & 1 & 0 \\ -l_N\sin\beta_N\sin\theta_N & l_N\sin\beta_N\cos\theta_N & 0 & 0 & 0 & 1 \end{bmatrix}$$

where, $\theta_N$ is the angle between the rod 0-$N$ projection in 0-$xy$ plane and the positive direction of $x$-axis and $\beta_N$ is the angle between the rod 0-$N$ and the positive direction of $z$-axis.

Combining Eqs. (42) and (43) we have the following equation

$$(E_{N-1} + A_N K_N D_N D_2^{-1} K_2^{-1}) F_2 = F \tag{44}$$

So we can calculate $\mathbf{F}_2$

$$F_2 = E_N^{-1} F \tag{45}$$

where

$$E_N = E_{N-1} + A_N K_N D_N D_2^{-1} K_2^{-1} \tag{46}$$

Combining Eqs. (39), (41) and (45) we can find the relationship between the force of wheel center and the displacement of the wheel center as the following equation

$$d_0 = D_2^{-1} K_2^{-1} E_N^{-1} F \tag{47}$$

So the equivalent stiffness of the wheel center ($K_0^N$) is

$$K_0^N = E_N K_2 D_2 = K_0^{N-1} + A_N K_N D_N \tag{48}$$

where, ($K_0^N$) is the equivalent stiffness of the wheel center and **N** is the number of the rod.

## 3. Numerical examples

We apply the above method to calculate the equivalent stiffness of the wheel center. There are three bushings and three linkages. The geometric parameters of the linkages are elaborated in Table 1. The stiffness of the bushings are elaborated in Table 2.

Taking the above data into the formula (36), we can calculate the equivalent stiffness of the wheel center ($K_0^3$)

Table 1 The geometric parameters of the linkages

| Bar number | $\theta_N$ (°) | $\beta_N$ (°) | The length of the bar (**l**$_N$) (m) |
|---|---|---|---|
| 0-1 | 0 | 0 | 0.3 |
| 0-2 | 30 | 30 | 0.5 |
| 0-3 | 60 | 60 | 0.7 |

Table 2 The stiffness of the bushings

| Bushing number (N) | The stiffness of the bushing $N$ (**K**$_N$) (N/m & N m/rad) |
|---|---|
| 1 | [628, -475, -3, 27330, -32550, -90046; -475, 15273, 347, 271759, -63999, 261819; -3, 347, 614, 154320, 295744, 36154; 27330, 271759, 154320, 184432089, 44877463, 14545450; -32550, -63999, 295744, 44877463, 160408083, -6070126; -90046, 261819, 36154, 14545450, -6070126, 596847763] |
| 2 | [9.0891e7, 0, 0, 0, -1.0162e7, 0; 0, 9.0891e7, 0, 1.0162e7, 0, 0; 0, 0, 22.7228e7, 0, 0, 0; 0, 1.0162e7, 0, 0.1137e7, 0, 0; -1.0162e7, 0, 0, 0, 0.1137e7, 0; 0, 0, 0, 0, 0, 0.0001e7] |
| 3 | [9.0891e7, 0, 0, 0, -1.0162e7, 0; 0, 9.0891e7, 0, 1.0162e7, 0, 0; 0, 0, 22.7228e7, 0, 0, 0; 0, 1.0162e7, 0, 0.1137e7, 0, 0; -1.0162e7, 0, 0, 0, 0.1137e7, 0; 0, 0, 0, 0, 0, 0.0001e7] |

$$K_0^3 = \begin{pmatrix} 181782628 & -475 & -3 & 2.74725e{+}4 & 5.0812446e{+}7 & 5.8989104e{+}7 \\ -475 & 181797273 & 347 & -5.05776304e{+}7 & -6.41415e{+}4 & 4.7490168e{+}7 \\ -3 & 347 & 454456614 & 1.47852416e{+}8 & -1.177754e{+}8 & 36154 \\ 2.74725e{+}4 & -5.05776304e{+}7 & 1.47852416e{+}8 & 2.649868e{+}8 & 2.596082e{+}6 & 1.58375e{+}6 \\ 5.0812446e{+}7 & -6.41415e{+}4 & -1.177754e{+}8 & 2.596082e{+}6 & 2.0645275e{+}8 & 8.918401e{+}6 \\ 5.8989104e{+}7 & 4.7490168e{+}7 & 36154 & 1.58375e{+}6 & 8.918401e{+}6 & 635932893 \end{pmatrix}$$

We can find that the equivalent stiffness matrix of the wheel center is a symmetric and positive matrix. Find the eigenvalues of the equivalent stiffness of the wheel center ($K_3$)

$$\lambda_3 = \begin{pmatrix} 6.49056e8 \\ 5.66335e8 \\ 2.66423e8 \\ 0.97482e8 \\ 1.94016e8 \\ 1.521e8 \end{pmatrix}$$

Comparing the eigenvalues of the stiffness of the bushings and the equivalent stiffness of the wheel center we find that the eigenvalues of the equivalent stiffness of the wheel center is much bigger than the eigenvalues of the stiffness of the bushings. As we know that the more the rod for a statically indeterminate structure, the more rigid of the structure. So the calculating result consistent with the real situation.

## 4. Conclusions

The paper gives a method to use the stiffness of bushings to calculate the equivalent stiffness of the wheel center. Comparing the $\mathbf{A}_i$ and $\mathbf{D}_i$ we can have the following equation

$$A_i = D_i^T \tag{49}$$

From the formula (48) we can find that the equivalent stiffness of the wheel center is a symmetric matrix which meets the conditions of the stiffness matrix of a structure. We find the relationship between the stiffness of the wheel center and the stiffness of the bushings. So using this method we can easily calculate the equivalent stiffness of the wheel center. We also can use this method to optimize the suspension system. We can optimize the stiffness of the suspension through changing the stiffness and the location of the bushings. So this method is very useful for the design of the suspension.

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## Appendix I

The length of rod is **l**. $\theta$ is the angle between the rod 0-1 projection in 0-*xy* plane and the positive direction of *x*-axis and $\beta$ is the angle between the rod 0-1 and the positive direction of *z*-axis as Fig. I. When there is a displacement of the point 0, what is the displacement of the point 1?

ssuming the displacement vector at the point 1 is $d_0$

$$d_0 = (d_x \quad d_y \quad d_z \quad \theta_x \quad \theta_y \quad \theta_z)^T$$

Without loss of generality, we assume that the point 0 is origin point of the coordinate system, so we can know the coordinate at the point 1 is

$$a = (l \sin\beta \cos\theta \quad l \sin\beta \sin\theta \quad l \cos\beta) \tag{I1}$$

When there is a rotation angle ($\theta_z$) around the *z*-axis at the point 0, the coordinate at the point 1 is

$$a_1 = (l \sin\beta \cos(\theta+\theta_z) \quad l \sin\beta \sin(\theta+\theta_z) \quad l \cos\beta) \tag{I2}$$

Then when there is a rotation angle ($\theta_x$) around the *x*-axis at the point 0, the coordinate at the point 1 is

$$a_2 = (l \sin\beta \cos(\theta+\theta_z) \quad l' \cos(\gamma+\theta_x) \quad l' \sin(\gamma+\theta_x)) \tag{I3}$$

where $\gamma$ is the angle between the rod 0-1 projection in the 0-*yz* plane and the *y*-axis positive direction; where

$$l' \cos\gamma = l \sin\beta \sin(\theta+\theta_z)$$

$$l' \sin\gamma = l \cos\beta$$

Finally when there is a rotation angle ($\theta_y$) around the *y*-axis at the point 0, the coordinate at the point 1 is

$$a_3 = (l'' \sin(\eta+\theta_y) \quad l' \cos(\gamma+\theta_x) \quad l'' \cos(\eta+\theta_y)) \tag{I4}$$

where $\eta$ is the angle between the rod 0-1 projection in the 0-*zx* plane and the *z*-axis positive direction; where

$$l'' \cos\eta = l(\cos\beta \cos\theta_x + \sin\beta \sin(\theta+\theta_z) \sin\theta_x)$$

$$l'' \sin\eta = l \sin\beta \cos(\theta+\theta_z)$$

At this time the translation vector at the point 1 is

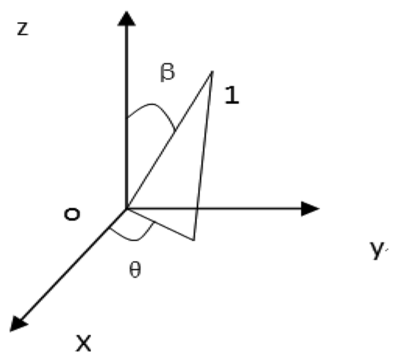


Fig. I The rod in the 3-dimensional space

$$\vec{d} = a_3 - a \tag{I5}$$

Simplifying the formula (I5), we can have the translation vector:

$$\begin{pmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \end{pmatrix} = \begin{pmatrix} l\sin\beta\cos(\theta+\theta_z)\cos\theta_y + l(\cos\beta\cos\theta_x + \sin\beta\sin(\theta+\theta_z)\sin\theta_x)\sin\theta_y - l\sin\beta\cos\theta \\ l\sin\beta\sin(\theta+\theta_z)\cos\theta_x - l\cos\beta\sin\theta_x - l\sin\beta\sin\theta \\ l(\cos\beta\cos\theta_x + \sin\beta\sin(\theta+\theta_z)\sin\theta_x)\cos\theta_y - l\sin\beta\cos(\theta+\theta_z)\sin\theta_y - l\cos\beta \end{pmatrix} \tag{I6}$$

And the rotation vector is

$$\begin{pmatrix} \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \end{pmatrix} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix} \tag{I7}$$

So we can have the displacement vector is

$$\begin{pmatrix} d_{1x} \\ d_{1y} \\ d_{1z} \\ \theta_{1x} \\ \theta_{1y} \\ \theta_{1z} \end{pmatrix} = \begin{pmatrix} l\sin\beta\cos(\theta+\theta_z)\cos\theta_y + l(\cos\beta\cos\theta_x + \sin\beta\sin(\theta+\theta_z)\sin\theta_x)\sin\theta_y - l\sin\beta\cos\theta \\ l\sin\beta\sin(\theta+\theta_z)\cos\theta_x - l\cos\beta\sin\theta_x - l\sin\beta\sin\theta \\ l(\cos\beta\cos\theta_x + \sin\beta\sin(\theta+\theta_z)\sin\theta_x)\cos\theta_y - l\sin\beta\cos(\theta+\theta_z)\sin\theta_y - l\cos\beta \\ \theta_x \\ \theta_y \\ \theta_z \end{pmatrix} \tag{I8}$$