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Influence of pressure-dependency of the yield criterion and temperature on residual stresses and strains in a thin disk

S. Alexandrov[1,2a], Y.R. Jeng*[1] and E. Lyamina[2]

[1]*Department of Mechanical Engineering and Advanced Institute for Manufacturing with High-tech Innovations, National Chung Cheng University, Chia-Yi 62102, Taiwan*
[2]*A.Yu Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow 119526, Russia*

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Abstract. Existing plane stress solutions for thin plates and disks have shown several qualitative features which are difficult to handle with the use of commercial numerical codes (non-existence of solutions, singular solutions, rapid growth of the plastic zone with a loading parameter). In order to understand the effect of temperature and pressure-dependency of the yield criterion on some of such features as well as on the distribution of residual stresses and strains, a semi-analytic solution for a thin hollow disk fixed to a rigid container and subject to thermal loading and subsequent unloading is derived. The material model is elastic-perfectly/plastic. The Drucker-Prager pressure-dependent yield criterion and the equation of incompressibity for plastic strains are adopted. The distribution of residual stresses and strains is illustrated for a wide range of the parameter which controls pressure-dependency of the yield criterion.

Keywords: thin disk; plane stress conditions; pressure-dependent yield criterion; thermal loading; residual stress and strain; semi-analytic solution

1. Introduction

Thin plates and disks with holes and embedded inclusions have many structural applications. A significant amount of analytical and numerical research for various material models has been carried out in the area of stress and strain analysis of such structures (Hsu and Forman 1975, Guven 1992, Gamer 1992, Lippmann 1992, Mack and Bengeri 1994, Ball 1995, Poussard *et al*. 1995, Debski and Zyczkowski 2002, Alexandrova and Alexandrov 2004a,b, Gupta *et al*. 2005, You *et al*. 2007, Jang and Kim 2008, Deepak *et al*. 2009, Alexandrov *et al*. 2010, Chakherlou and Yaghoobi 2010, Masri *et al*. 2010, Alexandrov *et al*. 2011a among others). An excellent review of previous works devoted to the problem of enlargement of a circular hole in thin plates has been given in Masri *et al*. (2010). Solutions at large strains for isotropic and anisotropic plates with a hole subject to mechanical

*Corresponding author, Professor, E-mail: imeyrj@ccu.edu.tw
[a]Professor, E-mail: sergei_alexandrov@spartak.ru

loading have been provided in Durban and Birman (1982), Cohen *et al.* (2009). In these works, deformation theories of plasticity have been adopted. The assumptions made regarding yield criterion, strain hardening and unloading have a significant effect on the predicted response and residual stress and strain fields (Ball 1995). Knowledge of the distribution of residual stresses around holes is necessary for accurate fatigue life prediction. Residual stresses determined by numerical simulations must be verified by experimental measurements or analytical solutions (Dutta and Rasty 2010). Therefore, even though closed form solutions involve more assumptions than numerical solutions, the former are necessary for studying qualitative effects and verifying numerical codes. Typical qualitative effects under plane stress conditions are the singularity of the velocity field and non-existence of the solution under certain conditions (Debski and Zyczkowski 2002, Alexandrova and Alexandrov 2004a,b, Alexandrova *et al*. 2004, Alexandrov *et al*. 2010, Alexandrov *et al*. 2011a). These features of boundary value problems can cause difficulties with their treatment by means of standard commercial numerical codes. In particular, some specific difficulties with numerical solution for plane stress problems have been mentioned in Kleiber and Kowalczyk (1996).

In the present paper, the effect of temperature and pressure-dependency of the yield criterion on the elastic/plastic solution for thin hollow disks loaded by thermal expansion assuming plane stress conditions is investigated including the stage of unloading. Various aspects of thermal loading of such disks have been considered in Lippmann (1992), Mack (1993), Bengeri and Mack (1994), Mack and Bengeri (1994), Alexandrov and Alexandrova (2001). In particular, an efficient analytic method has been developed and applied in Alexandrov and Alexandrova (2001). It has been shown that the size of the plastic zone is very sensitive to the increase in temperature. This method of solution has been extended to rotating disks in Alexandrova and Alexandrov (2004a), Alexandrova *et al*. (2004). As in the case of thermal loading, it has been shown that the size of the plastic zone is very sensitive to the angular velocity of the disk. Moreover, it has been demonstrated that no plane stress solution may exist under certain conditions. Plastic yielding of many metallic materials reveals dependency on the hydrostatic stress, though the equation of incompressibity is valid with a high accuracy (Yoshida *et al*. 1971, Spitzig *et al*. 1976, Spitzig 1979, Kao *et al.* 1990). It is therefore of interest to study the effect of this material property on the development of plastic zones and the distribution of residual stresses and strains in thin plates. Previous results on this subject include analytic solutions for a hollow disk subject to pressure over its inner radius (Alexandrov *et al*. 2011a) and a rotating hollow disk (Alexandrov *et al*. 2010). In these papers, the Drucker-Prager yield criterion (Drucker and Prager 1952) has been adopted. This criterion is confirmed by experimental data for several materials (Wilson 2002, Liu 2006). Therefore, the Drucker-Prager yield criterion is also used in the present paper. Strain hardening is neglected and the assumption of plastic incompressibility is accepted (i.e., the associated flow rule is not satisfied). A detailed description of the Drucker-Prager model including hardening and the associated flow rule is provided in Wilson (2002). This model has been used in Durban and Fleck (1997) to describe spherical cavity expansion at large strains. It is assumed that the disk is fixed to a rigid container such that its outer radius is motionless during the process of loading and unloading. At the initial instant the disk has no stress. Thermal expansion caused by a rise of temperature leads to an elastic strain distribution in the disk. Once the temperature has attained a certain magnitude, a plastic zone begins to develop. Subsequent unloading is assumed to be pure elastic. This assumption is verified for the specific set of parameters considered in the numerical example.

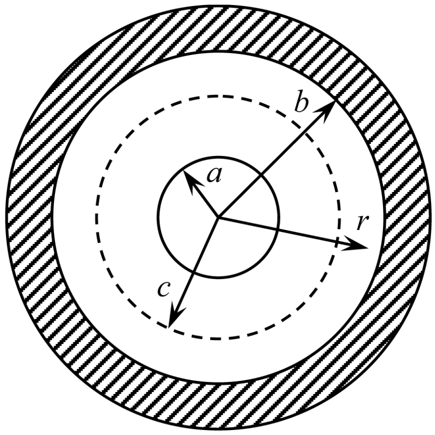

Fig. 1 Illustration of the boundary value problem

2. Statement of the problem and elastic solution

Consider a thin disk of radius b with a central circular hole of radius a fixed to a rigid container of radius b as shown in Fig. 1. The disk has no stress at the initial temperature. Thermal expansion caused by a rise of temperature and the constraints imposed on the disk affect the zero-stress state. Strains are supposed to be small. The state of stress is two-dimensional $(\sigma_z = 0)$ in a cylindrical coordinate system $r\theta z$ with its z-axis coinciding with the axis of symmetry of the disk. At the stage of loading, the rise of temperature above the reference state, T, is a monotonically increasing function of the time, t. At the stage of unloading the magnitude of T drops to its initial value. The boundary conditions are

$$u = 0 \quad \text{at} \quad r = b \tag{1}$$

and

$$\sigma_r = 0 \quad \text{at} \quad r = a \tag{2}$$

where u is the radial displacement and σ_r is the radial stress (σ_θ will stand for the circumferential stress).

Elastic strains are related to stresses and temperature by the classical Duhamel-Neumann law. The yield criterion is taken in the form proposed in Drucker and Prager (1952)

$$\alpha\sigma + \sigma_{eq} = \sigma_0 \tag{3}$$

where σ is the first invariant of the stress tensor (hydrostatic stress), σ_{eq} is the second invariant of the stress tensor (equivalent stress), α and σ_0 are material constants. The stress invariants are defined by

$$\sigma = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \quad \sigma_{eq} = \sqrt{\frac{3}{2}(s_1^2 + s_2^2 + s_3^2)} \tag{4}$$

where σ_1, σ_2 and σ_3 are the principal stresses. Also, $s_1 = \sigma_1 - \sigma$, $s_2 = \sigma_2 - \sigma$ and $s_3 = \sigma_3 - \sigma$. Obviously, the yield criterion given by Eq. (3) reduces to the von Mises yield criterion for $\alpha = 0$.

In this case σ_0 is the yield stress in tension. The plastic potential is taken in the form of $\sigma_{eq} = \sigma_0$. Then, the flow rule gives

$$\xi_1^p = \lambda s_1, \quad \xi_2^p = \lambda s_2, \quad \xi_3^p = \lambda s_3 \tag{5}$$

where ξ_1^p, ξ_2^p and ξ_3^p are the plastic portions of the principal strain rates and λ is a non-negative multiplier. Thus the material is plastically incompressible even though the yield criterion is pressure-dependent. This assumption is in agreement with experiment for some metallic materials (Spitzig *et al*. 1976, Spitzig 1979, Kao *et al*. 1990). Finally, the total strain tensor is the sum of its elastic and plastic portions. In terms of the principal strains

$$\varepsilon_1 = \varepsilon_1^e + \varepsilon_1^p, \quad \varepsilon_2 = \varepsilon_2^e + \varepsilon_2^p, \quad \varepsilon_3 = \varepsilon_3^e + \varepsilon_3^p \tag{6}$$

For axisymmetric deformation under plane stress conditions, the equations of linear thermoelasticity have the general solution in the form

$$\frac{\sigma_r}{\sigma_0} = \frac{A}{r^2} + B, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{A}{r^2} + B \tag{7}$$

$$u = -\frac{\sigma_0}{E}\left[(1+\nu)\frac{A}{r} - (1-\nu)Br\right] + \gamma T r \tag{8}$$

where E is the Young's modulus, ν is the Poisson's ratio, γ is the thermal coefficient of linear expansion, and A and B are arbitrary functions of T.

At the beginning of the process the entire disk is elastic. Therefore, the boundary conditions given by Eqs. (1) and (2) lead to

$$A = \frac{Eb^2a^2\gamma T}{\sigma_0[(1+\nu)a^2 + (1-\nu)b^2]} \quad \text{and} \quad B = -\frac{Eb^2\gamma T}{\sigma_0[(1+\nu)a^2 + (1-\nu)b^2]} \tag{9}$$

Combining Eqs. (7), (8) and (9) results in

$$\frac{\sigma_r}{\sigma_0} = \frac{\tau b^2(a^2 - r^2)}{[b^2(1-\nu) + a^2(1+\nu)]r^2}, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{\tau b^2(a^2 + r^2)}{[b^2(1-\nu) + a^2(1+\nu)]r^2}$$

$$\varepsilon_r = \frac{k\tau a^2(1+\nu)(b^2 + r^2)}{[b^2(1-\nu) + a^2(1+\nu)]r^2}, \quad \varepsilon_\theta = \frac{k\tau a^2(1+\nu)(r^2 - b^2)}{[b^2(1-\nu) + a^2(1+\nu)]r^2}$$

$$\varepsilon_z = \frac{k\tau(1+\nu)(b^2 + a^2)}{[b^2(1-\nu) + a^2(1+\nu)]}, \quad \frac{u}{a} = \left[\frac{k\tau a(1+\nu)(r^2 - b^2)}{[b^2(1-\nu) + a^2(1+\nu)]r^2}\right]$$

$$\tau = \frac{\gamma T}{k}, \quad k = \frac{\sigma_0}{E} \tag{10}$$

This solution is valid up to the value of τ at which the plastic zone begins to develop. This value of τ may be determined from Eq. (10) and the yield criterion (3).

3. Elastic/plastic solution for stress at loading

With no loss of generality, it is possible to assume that $\sigma_1 \equiv \sigma_r$, $\sigma_2 \equiv \sigma_\theta$ and $\sigma_3 \equiv \sigma_z = 0$. The last equation leads to $s_z = -\sigma$. Then, taking into account the identity $s_r + s_\theta + s_z = 0$ the value of s_θ can be expressed as $s_\theta = \sigma - s_r$. Substituting these values of s_z and s_θ into Eq. (4) results in $\sigma_{eq}^2 = 3(s_r^2 + \sigma^2 - \sigma s_r)$. Using this expression for σ_{eq} the yield criterion (3) can be rewritten in the form

$$3s_r^2 + (3-\alpha^2)\sigma^2 - 3\sigma s_r + 2\alpha\sigma\sigma_0 = \sigma_0^2 \tag{11}$$

Substituting the stress solution given by Eqs. (10) into Eq. (11) shows that the plastic zone starts to develop at $r = a$. The corresponding value of τ will be denoted by τ_e. It is determined from the following quadratic equation

$$(\alpha^2-9)\tau_e^2 + 3\alpha\tau_e\left[1-\nu+\frac{a^2}{b^2}(1+\nu)\right] + \frac{9}{4}\left[1-\nu+\frac{a^2}{b^2}(1+\nu)\right]^2 = 0 \tag{12}$$

The dependence of τ_e on a/b and α at $\nu = 0.3$ is illustrated in Fig. 2. Curve 1 in this figure corresponds to $\alpha = 0$, curve 2 to $\alpha = 0.1$ curve 3 to $\alpha = 0.2$, and curve 4 to $\alpha = 0.4$.

It has been shown in Alexandrov *et al.* (2011a) that the yield criterion given by Eq. (11) is satisfied by the following substitution

$$\frac{s_r}{\sigma_0} = \beta_0 + \beta_1\sin\psi + \beta_2\cos\psi \quad \text{and} \quad \frac{\sigma}{\sigma_0} = 2\beta_0 + 2\beta_2\cos\psi \tag{13}$$

where ψ is a new unknown function and

$$\beta_0 = \frac{2\alpha}{4\alpha^2-9}, \quad \beta_1 = \sqrt{\frac{3}{9-4\alpha^2}}, \quad \beta_2 = \frac{3}{9-4\alpha^2} \tag{14}$$

Using Eq. (13) and the condition $\sigma_z = 0$ it is possible to find that

$$\frac{\sigma_r}{\sigma_0} = 3\beta_0 + \beta_1\sin\psi + 3\beta_2\cos\psi \quad \text{and} \quad \frac{\sigma_\theta}{\sigma_0} = 3\beta_0 - \beta_1\sin\psi + 3\beta_2\cos\psi \tag{15}$$

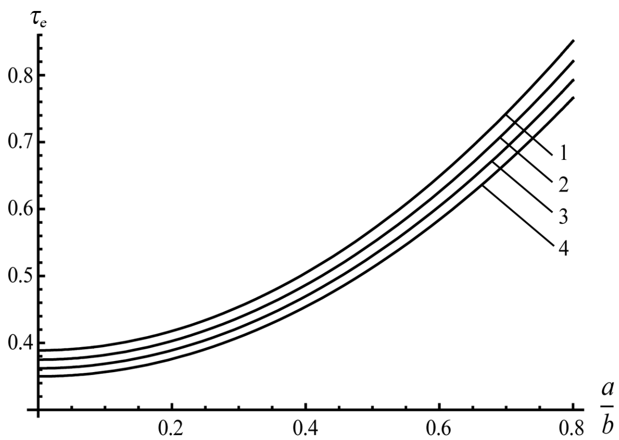

Fig. 2 Variation of τ_e with a/b at $\nu = 0.3$

Since the plastic zone starts to develop from the edge $r = a$, the boundary condition (2) should be reformulated in terms of ψ for the elastic/plastic stage of the process of loading. In particular, using the expression for σ_r in Eq. (15) this boundary condition can be rewritten in the form

$$3\beta_0 + \beta_1 \sin\psi_a + 3\beta_2 \cos\psi_a = 0 \tag{16}$$

where ψ_a is the value of ψ at $r = a$. The constraints imposed on the disk suggest that $\sigma_r > \sigma_\theta$. Then, it immediately follows from Eq. (15) that $\sin\psi > 0$. This inequality allows one to find the unique solution to Eq. (16). Note that ψ_a is independent on τ because β_0, β_1 and β_2 are constants, as follows from Eq. (14). The only non-trivial equilibrium equation has the form

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{17}$$

Substituting Eq. (15) into Eq. (17) leads to

$$(\beta_1 \cos\psi - 3\beta_2 \sin\psi)\frac{\partial\psi}{\partial r} + \frac{2\beta_1 \sin\psi}{r} = 0 \tag{18}$$

This equation can be immediately integrated with the use of the boundary condition $\psi = \psi_a$ at $r = a$ to give

$$\frac{r}{a} = \sqrt{\frac{\sin\psi_a}{\sin\psi}}\exp\left[\frac{3\beta_2}{2\beta_1}(\psi - \psi_a)\right] \tag{19}$$

Assume that $\psi = \psi_c$ at the elastic/plastic boundary $r = c$. Then, Eq. (19) gives

$$\frac{c}{a} = \sqrt{\frac{\sin\psi_a}{\sin\psi_c}}\exp\left[\frac{3\beta_2}{2\beta_1}(\psi_c - \psi_a)\right] \tag{20}$$

Since $c \geq a$, it follows from this equation that $\psi_a \leq \psi_c$. Therefore, when $\tau \geq \tau_e$ the stress distribution given by Eq. (15) is vaild in the range $\psi_a \leq \psi \leq \psi_c$ (or $a \leq r \leq c$). The general stress solution in the form given by Eq. (7) is valid in the elastic zone $c \leq r \leq b$. Eq. (8) is valid in this zone as well. Then, the boundary condition (1) gives

$$\frac{A}{b^2} = \frac{(1-\nu)B + \tau}{1+\nu} \tag{21}$$

Moreover, the radial and circumferential stresses should be continuous across the elastic/plastic boundary. Then, it follows from Eqs. (7), (15), and (21) that

$$(1-\nu)B + \tau = \frac{c^2}{b^2}(1+\nu)\beta_1 \sin\psi_c, \quad B = 3(\beta_0 + \beta_2 \cos\psi_c) \tag{22}$$

Eliminating B and replacing c/b by means of Eq. (20) result in

$$\tau = \frac{a^2}{b^2}\exp\left[\frac{3\beta_2}{\beta_1}(\psi_c - \psi_a)\right](1+\nu)\beta_1 \sin\psi_a - 3(1-\nu)(\beta_0 + \beta_2 \cos\psi_c) \tag{23}$$

The entire disk becomes plastic when $b = c$. Therefore, the maximum possible value of ψ_c which will be denoted by ψ_b is determined from Eq. (20) in the following implicit form

$$\frac{b}{a} = \sqrt{\frac{\sin\psi_a}{\sin\psi_b}}\exp\left[\frac{3\beta_2}{2\beta_1}(\psi_b - \psi_a)\right] \tag{24}$$

The corresponding value of τ which will be denoted by τ_b is obtained from Eq. (23) where ψ_c should be replaced with ψ_b. The variation of the parameter $\Delta\tau = (\tau_b - \tau_e)/\tau_e$ with a/b and α at $\nu = 0.3$ is illustrated in Fig. 3. The radius of the elastic/plastic boundary as a function of τ is determined from Eqs. (20) and (23) in parametric form. The variation of this radius with τ for several values of α at $\nu = 0.3$ is depicted in Figs. 4-6 ($a/b = 1/2$ in Fig. 4, $a/b = 1/5$ in Fig. 5, and $a/b = 1/10$ in Fig. 6). The right ends of all the curves correspond to $\psi = \psi_b$. In Figs. 3-6, curve 1 corresponds to $\alpha = 0$, curve 2 to $\alpha = 0.1$ curve 3 to $\alpha = 0.2$, and curve 4 to $\alpha = 0.4$. The radial distribution of the stresses in the plastic zone is immediately obtained from Eqs. (15) and (19) in parametric form. In order to find the radial distribution of the stresses in the elastic zone, it is sufficient to eliminate A and B in Eq. (7) by means of Eqs. (21) and (22).

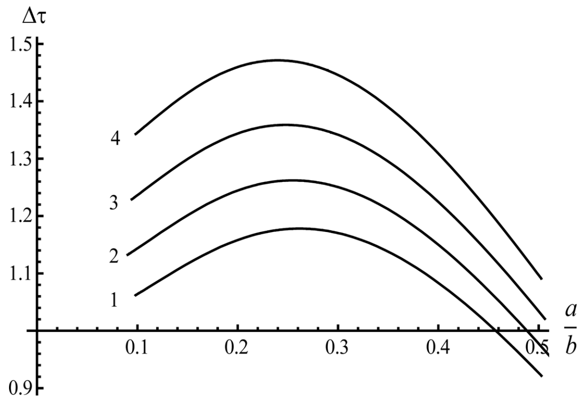

Fig. 3 Variation of $\Delta\tau$ with a/b at $\nu = 0.3$

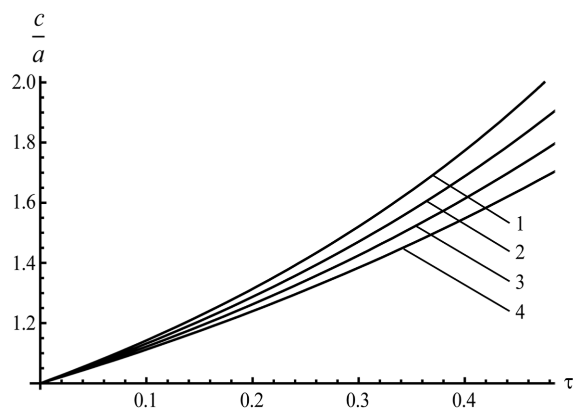

Fig. 4 Variation of the radius of the elastic/plastic boundary with τ at $a/b = 1/2$ and $\nu = 0.3$

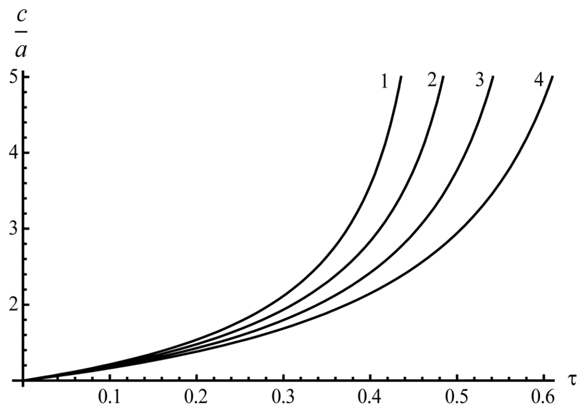

Fig. 5 Variation of the radius of the elastic/plastic boundary with τ at $a/b = 1/5$ and $\nu = 0.3$

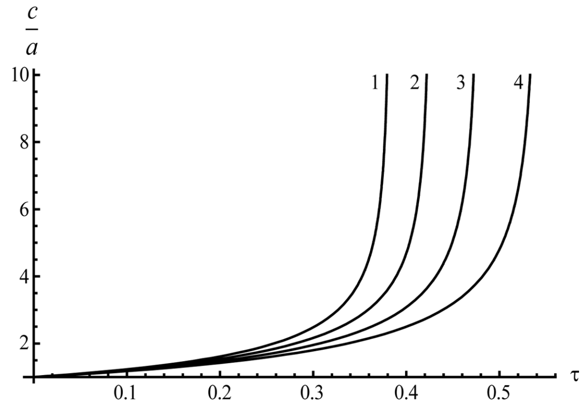

Fig. 6 Variation of the radius of the elastic/plastic boundary with τ at $a/b = 1/10$ and $\nu = 0.3$

4. Elastic/plastic solution for strains at loading

Since $\sigma_z = 0$, the elastic portions of the principal strains, $\varepsilon_r^e, \varepsilon_\theta^e$ and ε_z^e, are related to the stress components and temperature by the Duhamel-Neumann law as

$$\frac{\varepsilon_r^e}{k} = \left(\frac{\sigma_r}{\sigma_0} - \nu\frac{\sigma_\theta}{\sigma_0}\right) + \tau, \quad \frac{\varepsilon_\theta^e}{k} = \left(\frac{\sigma_\theta}{\sigma_0} - \nu\frac{\sigma_r}{\sigma_0}\right) + \tau, \quad \frac{\varepsilon_z^e}{k} = -\nu\left(\frac{\sigma_r}{\sigma_0} - \nu\frac{\sigma_\theta}{\sigma_0}\right) + \tau \tag{25}$$

In the case under consideration, the compatibility condition for the total strain components is

$$r\frac{d\varepsilon_\theta}{dr} = \varepsilon_r - \varepsilon_\theta \tag{26}$$

The corresponding condition for the total principal strain rate components ξ_r and ξ_θ is

$$r\frac{d\xi_\theta}{dr} = \xi_r - \xi_\theta \tag{27}$$

The third total principal strain rate component will be denoted by ξ_z. Introduce the quantities

$$\zeta_r = \xi_r\frac{dt}{d\tau}, \quad \zeta_\theta = \xi_\theta\frac{dt}{d\tau}, \quad \zeta_z = \xi_z\frac{dt}{d\tau}$$

$$\zeta_r^p = \xi_r^p\frac{dt}{d\tau}, \quad \zeta_\theta^p = \xi_\theta^p\frac{dt}{d\tau}, \quad \zeta_z^p = \xi_z^p\frac{dt}{d\tau}$$

$$\zeta_r^{e,p} = \xi_r^{e,p}\frac{dt}{d\tau}, \quad \zeta_\theta^{e,p} = \xi_\theta^{e,p}\frac{dt}{d\tau}, \quad \zeta_z^{e,p} = \xi_z^{e,p}\frac{dt}{d\tau} \tag{28}$$

where $\xi_r^{e,p}$, $\xi_\theta^{e,p}$, and $\xi_z^{e,p}$ are the elastic portions of the total principal strain rates in the plastic zone. Substituting Eq. (28) into Eq. (5) and eliminating λ lead to

$$\frac{\zeta_r^p}{\zeta_\theta^p} = \frac{s_r}{s_\theta}, \quad \frac{\zeta_z^p}{\zeta_\theta^p} = \frac{s_z}{s_\theta} \tag{29}$$

Also, Eq. (27) becomes

$$r\frac{d\zeta_\theta}{dr} = \zeta_r - \zeta_\theta \tag{30}$$

4.1 Solution in the plastic zone, a ≤ r ≤ c

Since the radial and circumferential stresses in this zone are given by Eq. (15), the distribution of the elastic portion of the total strain tensor in the plastic zone can be immediately found from Eq. (25) in the form

$$k^{-1}\varepsilon_r^{e,p} = 3\beta_0(1-\nu) + \beta_1(1+\nu)\sin\psi + 3\beta_2(1-\nu)\cos\psi + \tau$$

$$k^{-1}\varepsilon_\theta^{e,p} = 3\beta_0(1-\nu) - \beta_1(1+\nu)\sin\psi + 3\beta_2(1-\nu)\cos\psi + \tau$$

$$k^{-1}\varepsilon_z^{e,p} = -6\nu(\beta_0 + \beta_2\cos\psi) + \tau \tag{31}$$

Since ψ_a is independent of τ, it follows from Eq. (19) that $\partial\psi/\partial\tau = 0$. Then, differentiating Eq. (31) with respect to τ gives with the use of Eq. (28)

$$\zeta_r^{e,p} = k, \quad \zeta_\theta^{e,p} = k, \quad \zeta_z^{e,p} = k \tag{32}$$

Differentiating Eq. (6) with respect to τ gives with the use of Eqs. (28) and (32)

$$\zeta_r = k + \zeta_r^p, \quad \zeta_\theta = k + \zeta_\theta^p, \quad \zeta_z = k + \zeta_z^p \tag{33}$$

Using Eqs. (13) and (15) it is possible to find that

$$s_\theta = \sigma_\theta - \sigma = \beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi \tag{34}$$

Then, it follows from Eqs. (13) and (29) that

$$\zeta_r^p(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi) = \zeta_\theta^p(\beta_0 + \beta_1 \sin\psi + \beta_2 \cos\psi) \tag{35}$$

Substituting Eq. (33) into Eq. (30) results in

$$r\frac{d\zeta_\theta^p}{dr} = \zeta_r^p - \zeta_\theta^p \tag{36}$$

Eliminating ζ_r^p in this equation by means of Eq. (35) and replacing differentiation with respect to r with differentiation with respect to ψ with the use of Eq. (18) give

$$\frac{d\zeta_\theta^p}{\zeta_\theta^p} = \frac{(3\beta_2 \sin\psi - \beta_1 \cos\psi)}{(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi)} d\psi \tag{37}$$

Even though this equation can be integrated in elementary functions, it is more convenient to represent its solution in the two following equivalent forms

$$\zeta_\theta^p = \zeta_a \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \tag{38}$$

and

$$\zeta_\theta^p = \zeta_c \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \tag{39}$$

where ζ_a and ζ_c are the values of ζ_r^p at $\psi = \psi_a$ (or $r = a$) and $\psi = \psi_c$ (or $r = c$), respectively. It is obvious that ζ_a and ζ_c depend on τ. Moreover, it follows from Eqs. (38) and (39) that

$$\zeta_a = \zeta_c \exp\left[\int_{\psi_c}^{\psi_a} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \tag{40}$$

The incompressibility equation for the plastic portion of the strain rate tensor results in $\zeta_r^p + \zeta_\theta^p + \zeta_z^p = 0$. Using this equation, Eq. (35) and Eq. (38) it is possible to find that

$$\zeta_r^p = \zeta_a \frac{(\beta_0 + \beta_1 \sin\psi + \beta_2 \cos\psi)}{(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi)} \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right]$$

$$\zeta_z^p = -\frac{2\zeta_a(\beta_0 + \beta_2 \cos\psi)}{(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi)} \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \tag{41}$$

It follows from the definition for $\zeta_r^p, \zeta_\theta^p$ and ζ_z^p given in Eq. (28) that

$$\varepsilon_r^p = \int_{\tau_p}^{\tau} \zeta_r^p d\tau, \quad \varepsilon_\theta^p = \int_{\tau_p}^{\tau} \zeta_\theta^p d\tau, \quad \varepsilon_z^p = \int_{\tau_p}^{\tau} \zeta_z^p d\tau \tag{42}$$

where τ_p is the value of τ at which the plastic strains appear at the point where the strain components are calculated. It is obvious that the value of τ_p depends on the position of the point (i.e., on ψ) and is determined by the condition $\psi = \psi_c$. Therefore, it is convenient to rewrite Eq. (42) in the following form

$$\varepsilon_r^p = \int_{\psi}^{\psi_c} \zeta_r^p \frac{d\tau}{d\psi_c} d\mu, \quad \varepsilon_\theta^p = \int_{\psi}^{\psi_c} \zeta_\theta^p \frac{d\tau}{d\psi_c} d\mu, \quad \varepsilon_z^p = \int_{\psi}^{\psi_c} \zeta_z^p \frac{d\tau}{d\psi_c} d\mu \tag{43}$$

Here $\zeta_r^p, \zeta_\theta^p$, ζ_z^p and $d\tau/d\psi_c$ are understood as functions of μ which is a dummy variable. Since ψ and ψ_a are independent of τ, ζ_a is the only quantity in Eqs. (38) and (41) which depends on τ. Therefore, substituting Eqs.(38) and (41) into Eq. (42) and taking into account Eq. (43) give

$$\varepsilon_\theta^p = \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \int_{\psi}^{\psi_c} \zeta_a \frac{d\tau}{d\psi_c} d\mu$$

$$\varepsilon_r^p = \frac{(\beta_0 + \beta_1 \sin\psi + \beta_2 \cos\psi)}{(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi)} \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \int_{\psi}^{\psi_c} \zeta_a \frac{d\tau}{d\psi_c} d\mu$$

$$\varepsilon_z^p = -\frac{2(\beta_0 + \beta_2 \cos\psi)}{(\beta_0 - \beta_1 \sin\psi + \beta_2 \cos\psi)} \exp\left[\int_{\psi_a}^{\psi} \frac{(3\beta_2 \sin x - \beta_1 \cos x)}{(\beta_0 - \beta_1 \sin x + \beta_2 \cos x)} dx\right] \int_{\psi}^{\psi_c} \zeta_a \frac{d\tau}{d\psi_c} d\mu \tag{44}$$

The total strain components in the plastic zone can be calculated by substituting Eqs. (31) and (44) into Eq. (6). However, in order to find the function $\zeta_a(\psi_c)$ involved in the integrands in Eq. (44) it is necessary to find the solution in the elastic zone.

4.2 Solution in the elastic zone, $c \le r \le b$

The general solution for the displacement given in Eq. (8) as well as Eq. (25) are valid in the elastic zone. Therefore, the total principal strains in this zone are

$$k^{-1}\varepsilon_r = [3(1-\nu)(\beta_0 + \beta_2 \cos\psi_c) + \tau]\left(1 + \frac{b^2}{r^2}\right)$$

$$k^{-1}\varepsilon_\theta = [3(1-\nu)(\beta_0 + \beta_2 \cos\psi_c) + \tau]\left(1 - \frac{b^2}{r^2}\right)$$

$$k^{-1}\varepsilon_z = -6\nu(\beta_0 + \beta_2 \cos\psi_c) + \tau \tag{45}$$

Here Eqs. (21) and (22) have been taken into account to exclude A and B. Differentiating the expression for ε_θ given in Eq. (45) with respect to τ leads to

$$\zeta_\theta = k\left[-3(1-\nu)\beta_2 \sin\psi_c \frac{d\psi_c}{d\tau} + 1\right]\left(1 - \frac{b^2}{r^2}\right) \tag{46}$$

4.3 Complete solution

Since ζ_θ is proportional to the radial velocity, this quantity must be continuous across the elastic/plastic boundary where $\psi = \psi_c$ and $r = c$. Therefore, it follows from Eqs. (33), (39) and (46) that

$$k^{-1}\zeta_c = -3(1-\nu)\beta_2 \sin\psi_c \frac{d\psi_c}{d\tau}\left(1-\frac{b^2}{c^2}\right)-\frac{b^2}{c^2} \tag{47}$$

Differentiating Eq. (23) with respect to ψ_c results in

$$\frac{d\psi_c}{d\tau} = \frac{1}{3\beta_2}\left\{\frac{a^2}{b^2}(1+\nu)\sin\psi_a \exp\left[\frac{3\beta_2}{\beta_1}(\psi_c-\psi_a)\right]+(1-\nu)\sin\psi_c\right\}^{-1} \tag{48}$$

Eliminating the derivative $d\psi_c/d\tau$ by means of Eq.(48) and the ratio c/b by means of Eq. (20) in Eq.(47) gives ζ_c as a function of ψ_c. Then, Eq.(40) determines ζ_a as a function of ψ_c. Having this function and Eq. (48) integration in Eq. (44) can be performed numerically to find the distribution of the plastic strains in the plastic zone. Since the elastic strains in this zone are given in Eq. (31), the total strains can be found with no difficulty. The solution obtained is written in terms of ψ_c and ψ (in the plastic zone). In order to rewrite it in terms of τ and r, it is sufficient to replace ψ_c with τ by means of Eq.(23), and ψ with r by means of Eq. (19).

5. Residual stresses and strains

Assume that the temperature drops down from its current magnitude to $T = 0$. Then, if unloading is pure elastic, the solution given in Eq. (10) where τ should be replaced with $-\tau$ is valid for the increments of stresses and strains. Thus

$$\frac{\Delta\sigma_r}{\sigma_0} = -\frac{\tau b^2(a^2-r^2)}{[b^2(1-\nu)+a^2(1+\nu)]r^2},\quad \frac{\Delta\sigma_\theta}{\sigma_0} = \frac{\tau b^2(a^2+r^2)}{[b^2(1-\nu)+a^2(1+\nu)]r^2}$$

$$\Delta\varepsilon_r = -\frac{k\tau a^2(1+\nu)(b^2+r^2)}{[b^2(1-\nu)+a^2(1+\nu)]r^2},\quad \Delta\varepsilon_\theta = -\frac{k\tau a^2(1+\nu)(r^2-b^2)}{[b^2(1-\nu)+a^2(1+\nu)]r^2}$$

$$\Delta\varepsilon_z = -\frac{k\tau(1+\nu)(b^2+r^2)}{[b^2(1-\nu)+a^2(1+\nu)]} \tag{49}$$

Denote the solution at the end of the loading stage in the range $a \le r \le b$ by $\sigma_r^l, \sigma_\theta^l, \varepsilon_r^l, \varepsilon_\theta^l$, and ε_z^l. Then, the residual stresses and strains are given by

$$\sigma_r^{res} = \sigma_r^l + \Delta\sigma_r,\quad \sigma_\theta^{res} = \sigma_\theta^l + \Delta\sigma_\theta$$

$$\varepsilon_r^{res} = \varepsilon_r^l + \Delta\varepsilon_r,\quad \varepsilon_\theta^{res} = \varepsilon_\theta^l + \Delta\varepsilon_\theta,\quad \varepsilon_z^{res} = \varepsilon_z^l + \Delta\varepsilon_z \tag{50}$$

The validity of this solution is restricted by the yield criterion (3). Using Eq. (11) this restriction can be written in the form

$$3(s_r^{res})^2 + (3-\alpha^2)(\sigma^{res})^2 - 3\sigma^{res}s_r^{res} + 2\alpha\sigma_0 + \sigma_0^2 \le 0$$
$$\sigma^{res} = (\sigma_r^{res} + \sigma_\theta^{res})/3, \quad s_r^{res} = \sigma_r^{res} - \sigma^{res} \tag{51}$$

Having the solution at the end of the loading stage found in the previous section and the solution given in Eq. (49) the distribution of the residual stresses and strains can be found from Eq. (50) with no difficulty.

6. Numerical example

The integrals involved in Eq. (44) have been evaluated numerically. The accuracy of the numerical solution has been controlled by substituting the strain components found into Eq. (26). In all calculations $\nu = 0.3$, $k = 10^{-3}$ and $a/b = 1/2$. The solution at the end of the loading stage has been illustrated in Alexandrov *et al.* (2011b). In order to reveal the effect of pressure-dependency of the yield criterion on the distribution of residual stresses and strains, the maximum temperature for the stage of loading has been taken independently of the value of α as $\tau_{\max} = (\tau_e + \tau_b)/2$ where the values of τ_e and τ_b correspond to $\alpha = 0$ (pressure-independent plasticity). Under these conditions, the solution is illustrated in Figs. 7 to 11. The dashed curves correspond to $\alpha = 0$, curves 1 to $\alpha = 0.1$, curves 2 to $\alpha = 0.2$, and curves 3 to $\alpha = 0.3$. The inequality (51) has been verified numerically for all cases considered. The distribution of the residual radial and circumferential stresses along the radius at several values of α is depicted in Figs. 7 and 8, respectively. The effect of α-value on the distribution of these stresses is rather significant and is obvious from these diagrams. The effect on the radial stress is more pronounced in the elastic zone. It is explained by the fact that the value of this stress at $r = a$ is fixed by the boundary condition (2). On the other hand, the effect on the circumferential stress is negligible in the elastic zone but is significant in the plastic zone. Note that the residual radial stress is positive at $r = b$. Therefore, the solution obtained is not valid for the disk just inserted into but not fixed to the container. For the disk inserted into

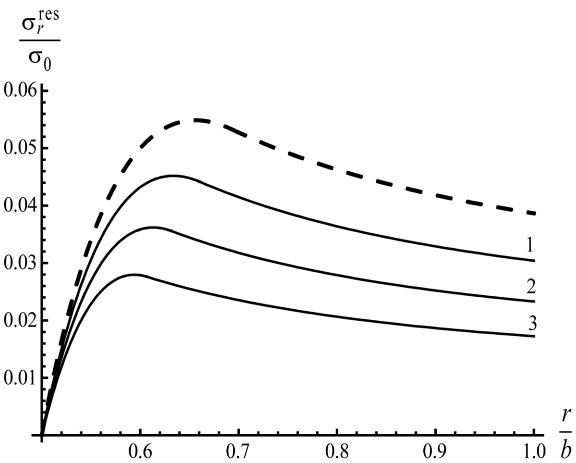

Fig. 7 Distribution of the residual radial stress along the radius

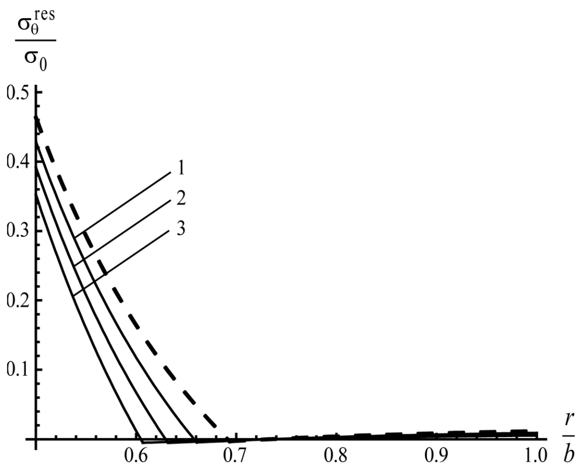

Fig. 8 Distribution of the residual circumferential stress along the radius

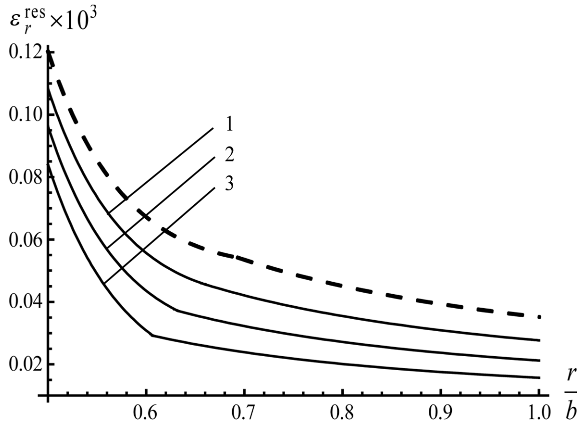

Fig. 9 Distribution of the residual radial strain along the radius

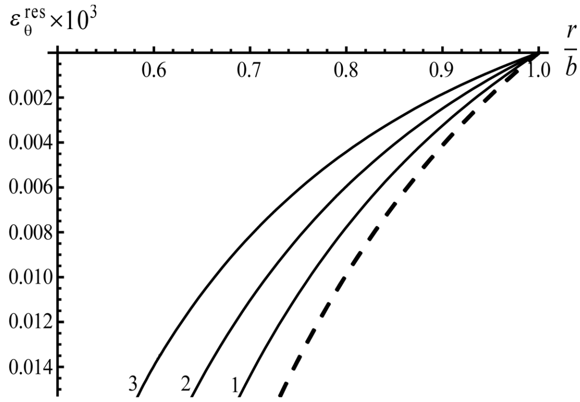

Fig. 10 Distribution of the residual circumferential strain along the radius

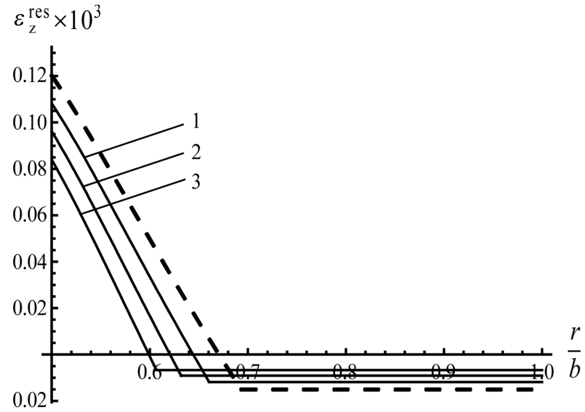

Fig. 11 Distribution of the residual axial strain along the radius

the container, the description of the process of unloading should be divided into two steps. At the beginning of the process, the solution given in (49) is in general valid but τ should be replaced with some value τ_1 which is determined by the condition that $\sigma_r^{res} = 0$ at $r = b$. The solution for the increment of stresses for the second step follows from Eq. (7) where A and B are determined from the boundary conditions $\Delta\sigma_r = 0$ at $r = a$ and $r = b$.

The variation of the residual principal strains with the radius is shown in Figs. 9 to 11. The effect of α-value on the distribution of these strains is significant except for the circumferential strain in the vicinity of the outer radius of the disk where the value of this strain is fixed by the boundary condition (1). Moreover, it is necessary to mention that the magnitude of this strain is much smaller than that of ε_r^{res} and ε_z^{res}.

7. Conclusions

A new semi-analytic solution for a thin hollow disk made of plastically pressure-dependent material subject to thermal loading and subsequent unloading has been found. A numerical treatment is only necessary to evaluate ordinary integrals.

The significant influence of α-value and the ratio a/b on the relative increase in temperature from its magnitude corresponding to the initiation of plastic deformation to the value at which the entire disk becomes plastic has been revealed (Fig. 3). It is interesting to mention that the dependence of $\Delta\tau$ on a/b at a given value of α is not monotonic and this function attains a maximum at some value of a/b. Another important qualitative effect following from the solution is a sharp increase in the radius of the elastic/plastic boundary for sufficiently large values of τ and small values of a/b (Fig. 6). The effect of pressure-dependency of the yield criterion on the distribution of residual stresses and strains is in general significant as can be seen from Figs. 7 to 11. For this reason and since some metallic materials reveal pressure-dependency of the yield criterion (Yoshida *et al*. 1971, Spitzig *et al*. 1976, Spitzig 1979, Kao *et al*. 1990) it is of importance to take into account this material property in fatigue life predictions.

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References

Alexandrov, S. and Alexandrova, N. (2001), "Thermal effects on the development of plastic zones in thin axisymmetric plates", *J. Strain Anal. Eng. Design*., **36**(2), 169-176.

Alexandrov, S.E., Lomakin, E.V. and Jeng, Y.R. (2010), "Effect of the pressure dependency of the yield condition on the stress distibution in a rotating disk", *Doklady-Physics*, **55**(12), 606-608.

Alexandrov, S., Jeng, Y.-R. and Lomakin, E. (2011a), "Effect of pressure-dependency of the yield criterion on the development of plastic zones and the distribution of residual stresses in thin annular dsks" *Trans. ASME J. Appl. Mech*., **78**(3), 031012.

Alexandrov, S., Jeng, Y.R. and Lyamina, E. (2011b), "Influence of pressure-dependence of yield criterion and temperature on the development of plastic zones in thin plates", *Proceedings of 2011 World Congress Advances in Structural Engineering and Mechanics*, ASEM'11, Seuol, September.

Alexandrova, N. and Alexandrov, S. (2004a), "Elastic-plastic stress distribution in a rotating annular disk", *Mech. Base. Des. Struct. Mach.,* **32**(1), 1-15.

Alexandrova, N. and Alexandrov, S. (2004b), "Elastic-plastic stress distribution in a plastically anisotropic rotating disk", *Trans. ASME J. Appl. Mech.,* **71**(3), 427-429.

Alexandrova, N., Alexandrov, S. and Vila Real, P. (2004), "Displacement field and strain distribution in a rotating annular disk", *Mech. Base. Des. Struct. Mach.,* **32**(4), 441-457.

Ball, D.L. (1995), "Elastic-plastic stress analysis of cold expanded fastener holes", *Fat. Fract. Eng. Mater. Struct*., **18**, 47-63.

Bengeri, M. and Mack, W. (1994), "The influence of the temperature dependence of the yield stress on the stress distribution in a thermally assembled elastic-plastic shrink fit", *Acta Mech*., **103**, 243-257.

Chakherlou, T.N. and Yaghoobi, A. (2010), "Numerical simulation of residual stress relaxation around a cold-expanded fastener hole under longitudinal cyclic loading using different kinematic hardening models", *Fat.*

Fract. Eng. Mater. Struct., **33**, 740-751.

Cohen, T., Masri, R. and Durban, D. (2009), "Analysis of circular hole expansion with generalized yield criteria", *Int. J. Solids Struct.*, **46**, 3643-3650.

Debski, R. and Zyczkowski, M. (2002), "On decohesive carrying capacity of variable-thickness annular perfectly plastic disks", *Z. Angew. Math. Mech (ZAMM)*, **82**(10), 655-669.

Deepak, D., Gupta, V.K. and Dham, A.K. (2009), "Impact of stress exponent on steady state creep in a rotating composite disc", *J. Strain Anal. Eng. Des.*, **44**, 127-135.

Drucker, D.C. and Prager, W. (1952), "Soil mechanics and plastic analysis for limit design", *Q. Appl. Math.*, **10**, 157-165.

Durban, D. and Fleck, N.A. (1997), "Spherical cavity expansion in a Drucker-Prager solid", *Trans. ASME J. Appl. Mech.*, **64**, 743-750.

Durban, D. and Birman, V. (1982), "On the elastic-plastic stress concentration at a circular hole in an anisotropic sheet", *Acta Mech.*, **43**, 73-84.

Dutta, N. and Rasty, J. (2010), "Prediction of elastic-plastic boundary around cold-expanded holes using elastic strain measurement", *Trans. ASME J. Eng. Mater. Technol.*, **132**, 031009.

Gamer, U. (1992), "A concise treatment of the shrink fit with elastic-plastic hub", *Int. J. Solids Struct.*, **29**, 2463-2469.

Gupta, V.K., Singh, S.B., Chandrawat, H.N. and Ray, S. (2005), "Modeling of creep behavior of a rotating disc in the presence of both composition and thermal gradients", *Trans. ASME J. Eng. Mater. Technol.*, **127**, 97-105.

Guven, U. (1992), "Elastic-plastic annular disk with variable thickness subjected to external pressure", *Acta Mech.*, **92**, 29-34.

Hsu, Y.C. and Forman, R.G. (1975), "Elastic-plastic analysis of an infinite sheet having a circular hole under pressure", *Trans. ASME, J. Appl. Mech.*, **42**, 347-352.

Jang, J.S. and Kim, D.W. (2008), "Re-cold expansion process simulation to impart the residual stresses around fastener holes in 6061 A-T6 aluminium alloy", *Proc. IMechE Part B, J. Eng. Manufact.*, **222**, 1325-1332.

Kao, A.S., Kuhn, H.A., Spitzig, W.A. and Richmond, O. (1990), "Influence of superimposed hydrostatic pressure on bending fracture and formability of a low carbon steel containing globular sulfides", *ASME J. Eng. Mater. Technol.*, **112**, 26-30.

Kleiber, M. and Kowalczyk, P. (1996), "Sensitivity analysis in plane stress elasto – plasticity and elasto – viscoplasticity", *Comput. Meth. Appl. Mech. Eng.*, **137**, 395-409.

Lippmann, H. (1992), "The effect of a temperature cycle on the stress distribution in a shrink fit", *Int. J. Plasticity*, **8**, 567-582.

Liu, P.S. (2006), "Mechanical behaviors of porous metals under biaxial tensile loads", *Mater. Sci. Eng.*, **A422**, 176-183.

Mack, W. (1993), "Thermal assembly of an elastic-plastic hub and a solid shaft", *Arch. Appl. Mech.*, **63**, 42-50.

Mack, W. and Bengeri, M. (1994), "Thermal assembly of an elastic-plastic shrink fit with solid inclusion", *Int. J. Mech. Sci.*, **36**, 699-705.

Masri, R., Cohen, T. and Durban, D. (2010), "Enlargement of a circular hole in a thin plastic sheet: Taylor-Bethe controversy in retrospect Q.", *J. Mech. Appl. Math.*, **63**(4), 589-616.

Poussard C., Pavier, M.Y. and Smith, D.J. (1995), "Analytical and finite element predictions of residual stresses in cold worked fastener holes", *J. Strain Anal. Eng. Des.*, **30**, 291-304.

Spitzig, W.A. (1979), "Effect of hydrostatic pressure on plastic-flow properties of iron single crystals", *Acta Metall.*, **27**, 523-534.

Spitzig, W.A., Sober, R.J. and Richmond, O. (1976), "The effect of hydrostatic pressure on the deformation behavior of maraging and HY-80 steels and its implications for plasticity theory", *Metallurg. Trans.*, **7A**, 1703-1710.

Wilson, C.D. (2002), "A critical reexamination of classical metal plasticity", *ASME J. Appl. Mech.*, **69**(1), 63-68.

Yoshida, S., Oguchi, A. and Nobuki, M. (1971), "Influence of high hydrostatic pressure on the flow stress of copper polycrystals", *Trans. Jpn. Inst. Met.*, **12**, 238-242.

You, L.H., You, X.Y., Zhang, J.J. and Li, J. (2007), "On rotating circular disks with varying material properties", *Z. Angew. Math. Phys.*, **58**, 1068-1084.