Structural Engineering and Mechanics, Vol. 43, No. 4 (2012) 439-448
DOI: http://dx.doi.org/10.12989/sem.2012.43.4.439

Finite element generalized tooth contact analysis of double circular arc helical gears

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(Received May 1, 2010, Revised June 13, 2012, Accepted August 6, 2012)

Abstract. This paper investigates the load sharing of double circular arc helical gears considering the influence of assembly errors. Based on a load sharing formulae, a three-dimensional finite element tooth contact analysis (TCA) is implemented with commercial software package ANSYS. The finite element grid for the double circular arc gear contact model is automatically generated by using the APDL (ANSYS Parameter Design Language) embedded in ANSYS. The realistic rotation of gears is achieved by using a coupling degree-of-freedom method. Numerical simulations are carried out to exemplify the proposed approach. The distribution of contact stress and bending stress under specific loading conditions are computed and compared with those obtained from Hertz contact theory and empirical formulae to demonstrate the efficiency of the proposed load sharing calculation formulae and TCA approach.

Keywords: double circular arc; helical gears; load share; tooth contact analysis; finite element method

1. Introduction

Double circular arc gears have many excellent attributes including high load carrying capacity, good closely running capacity, long life etc. They get more and more recognition and have wide application in heavy industry fields such as oil, chemical industry, mining and so on. Due to their unique convex-concave tooth contact profile and multi-point meshing characteristics, double circular arc gears have higher contact strength than involute gears. However, the meshing status of double circular arc gears is quite complex. It involves multi-teeth, multi-point contact and repeating alternative loading. In order to improve the running of double circular arc gear drives, it is essential to investigate gear-tooth contact inherent in meshing. Correct determinations of load sharing among gear teeth, stress distribution on tooth surface and synthetic mesh stiffness are important fundamental works to improve the static and dynamic characteristics and carrying capacity of gear drives.

In the past decades, numerical approaches including Finite Element Method (FEM) have been applied in the stress analysis of various types of gear drives and proved to be effective gear computer-aided design tools. Zhang and Fang (1999) proposed a tooth contact model for helical

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gears. This model takes into account the effect of tooth profile modifications, gear manufacturing errors and tooth surface deformation on gear mesh quality. Tseng and Tsay (2004, 2006) studied the transmission errors and bearing contacts of curvilinear-tooth gears under different assembly conditions by using tooth contact analysis and surface topology method. They demonstrated via numerical examples that the transmission error of the curvilinear-tooth gear pair is small and the bearing contacts are located in the middle region of the curvilinear-tooth surface even under axial misalignments. Falah and Elkholv (2008) applied a slicing method to estimate the load share of single enveloping worm gear drives by calculating the instantaneous tooth meshing stiffness and stresses. They investigated the effect of transmission error on loading and stresses and determined tooth composite deflection caused by bending, shearing, contact deformation as well as initial separation due to profile mismatch. Simon (2007) presented an approach to analyze the load distribution in mismatched spiral bevel gears by assuming point contact over a surface along a potential contact line. Gear teeth deflection under bending and shearing, contact deformations, gear body bending and torsion, the deflections of supporting shafts were obtained.

Compared to the extensive works on the numerical analysis of spur and helical gears in the literature, little efforts have been put on circular-arc gears. Pioneering works can be found in Litvin and Lu (1993, 1995). They investigated the generation, geometry, meshing and contact of double circular-arc helical gears. They determined contact paths on the gear tooth surfaces and transmission errors caused by gear misalignment. They also analyzed the impact of gear misalignment on transmission errors and developed a modified geometry for the gears to absorb transmission errors caused by misalignment. By using imaginary rack cutters and double envelope concept, Tsai *et al.* (2008) developed a mathematical model for planetary gear mechanism with double circular-arc teeth in a second envelope. Contact stress on the planet gear and ring gear, and planet gear and sun gear were determined to investigate the effect of various assembly errors. By delicately choosing design elements to reduce the tensile stresses in the round corner of gear teeth, Medvedev *et al.* (2008) increased the loading capacity of circular-arc teeth of conic gears without a significant increase in dimensions. Yang (2009) presented an imaginary rack cutter with stepped triple circular-arc tooth profile stemmed from the standard Chinese double circular arc tooth profile and involute tooth profile. He then investigated the contact stress of a stepped triple circular-arc tooth via finite element analysis. By using theory of gearing, Wu *et al.* (2009) developed a mathematical model for a circular-arc curvilinear tooth gear drive in which the gear and pinion tooth surfaces were generated by two complemented circular-arc rack cutters with curvilinear tooth-traces. They investigated the kinematical errors of circular-arc curvilinear tooth gear drives under different assembly errors by tooth contact analysis. Based on theory of gearing and the gear generation mechanism, Bair *et al.* (2007, 2009) developed mathematical models to depict tooth profile of elliptical gears manufactured by circular-arc shaper cutters. A geometric relation was accordingly developed to prevent tooth undercutting and pointed teeth on the elliptical gears. These works provide information for understanding the meshing mechanism of double circular-arc gears.

Regarding to the calculation of tooth root stress on double circular-arc gears, the usual way is to put concentrated normal-direction force on the contact point. This method is applicable to such situations as the loading point is far away from tooth root and tooth contact zone is small. However, the contact area of double circular arc gears takes on elliptical or cetacean shape and is much bigger than that of other gear drive types. The single-point-loading method is obviously lack of accuracy. Lu *et al.* (1995) investigated tooth surface contact and stresses for double circular-arc helical gear drives by using finite element method and Hertz contact theory. Load share between neighboring pairs of teeth is based on the analysis of position errors resulted from surface mismatch and elastic deformation of teeth. They assumed that the loading on the contact surface is distributed on a half ellipsoid. However, this method has to search for nodes in the elliptical contact zone and still puts concentrated forces on nodal points. In this paper, taking advantage of the powerful contact-problem solving capability of commercial FEA package ANSYS, the distribution of contact stress and bending stress in double circular-arc gear drives is directly calculated for the purpose of improving computational accuracy and efficiency.

2. Determination of load share along contact trace

The flexibility-matrix-format method based on mathematical programming is the common method used in many applications (Rahgozar *et al.* 2011) including the calculation of load sharing among gear teeth. It examines the reciprocal influence among points and teeth on the same contact line. Because double circular arc gears have many meshing points and the contact zone is irregular, it will be quite difficult to implement computation if simultaneously consider load sharing in and among the contact traces. By using the principle of angular displacement compensation and based on the loading sharing formulation proposed by Lu *et al.* (1995), this paper presents a feasible and efficient approach for contact analysis of double circular arc gears and determination of load sharing among teeth as well as among contact traces.

2.1 Gear-drive transmission error analysis

Gear-drive transmission error stemming from machining errors, assembly errors and backlash are called tooth surface mismatch error $\Delta\phi_2^{(m)}$, which can be calculated as follows (Lu *et al.* 1995)

$$\Delta\phi_2^{(m)} = (\phi_2 - \phi_2^{(0)}) - \frac{N_1}{N_2}(\phi_1 - \phi_1^{(0)}) \tag{1}$$

where N_1 and N_2 are the tooth number of the pinion and the gear, respectively; ϕ_1 and ϕ_2 are the rotation angles of the pinion and the gear, respectively; $\phi_1^{(0)}$ and $\phi_2^{(0)}$ are the initial rotation angles. The rotation angles can be obtained by solving the following gear-tooth surface meshing equations

$$\begin{cases} \mathbf{r}^{(1)}(u_1, \theta_1, \phi_1) = \mathbf{r}^{(2)}(u_2, \theta_2, \phi_2) \\ \mathbf{n}^{(1)}(u_1, \theta_1, \phi_1) = \mathbf{n}^{(2)}(u_2, \theta_2, \phi_2) \end{cases} \tag{2}$$

where $\mathbf{r}$ is the tooth surface position vector, $\mathbf{n}$ denotes the tooth surface normal vector, u and θ are the tooth surface coordinates.

Another source of transmission error comes from elastic deformation when loading is exerted on the gear drive. This includes the sink distortion at the tooth-contact point and tooth bending deformation. The elastic deformation can be obtained by applying normal force on the center of the contact ellipse and solving with FEM. Suppose $\mathbf{d}_1$ and $\mathbf{d}_2$ are the deformation vectors of the pinion and the gear at the contact point, respectively; $\mathbf{n}_1$ and $\mathbf{n}_2$ are the corresponding unit normal vectors at the contact point in the global coordinate system, then the total displacement u along the normal direction in a pair of engaged gears can be expressed as

$$u = |\mathbf{d}_1 \cdot \mathbf{n}_1| + |\mathbf{d}_2 \cdot \mathbf{n}_2| \tag{3}$$

The distortion is related to the angle displacement error as

$$\Delta\phi_2^{(l)} = \frac{N_1 u}{N_2 \cdot (\mathbf{r}_1 \times \mathbf{n}_1) \cdot \mathbf{k}_1} \tag{4}$$

The normal force exerted on contact point j can be calculated by the following equation (Lu *et al.* 1995)

$$F_i^{(j)} = \frac{T_j}{(\mathbf{r}_1^{(j)} \times \mathbf{n}_1^{(j)}) \cdot \mathbf{k}_1} \tag{5}$$

where $F_i^{(j)}$ denotes the normal force at the j-th contact point in the i-th gear with $i = 1$ standing for the pinion and $i = 2$ standing for the gear. T_j is the torque exerted on the j-th contact point, $\mathbf{k}_1$ is the axial unit vector of the pinion. These vectors are all expressed in the global coordinate system. The initial displacement u_{j0} can be computed by assuming that the torque T is the only load exerted on the gear drive at the beginning. The force at the j-th contact point in the gear may obtained directly by the force and counterforce relation.

2.2 Calculation of load sharing

The theory of angular displacement compensation states: (a) under loading, in order to ensure contact continuity of two gear teeth in meshing process, the total angular displacement at any point on the contact line should keep to be the same at any meshing instant. (b) the summation of torque applied to the teeth engaged in meshing of the pinion (or the gear) equals to the import torque (or export torque).

Suppose there currently have N teeth engaged in meshing simultaneously, and there have s contact traces. When the rotation angle ϕ_1 of the pinion is given, under the torque T, load share on the s contact traces should be determined

$$(\Delta\phi_2^{(m)} - \Delta\phi_2^{(l)})_1 = (\Delta\phi_2^{(m)} - \Delta\phi_2^{(l)})_2 = \ldots = (\Delta\phi_2^{(m)} - \Delta\phi_2^{(l)})_s \tag{6}$$

where $(\Delta\phi_2^{(m)})_j$ can be obtained from Eq. (1) and $(\Delta\phi_2^{(l)})_j$ from Eq. (4). Assume that the load share coefficient of the j-th contact trace is c_j, then the torque allotted to the j-th contact trace is $T_j = c_j \cdot T$. If the total displacement u_j at the contact point is linear to the applied torque T, the real displacement of the j-th contact point is

$$u_j = c_j \cdot u_{j0}, \quad j = 1, 2, \ldots, s \tag{7}$$

Combining (4), (6) and (7) gives (Lu *et al.* 1995)

$$\left(\Delta\phi_2^{(m)} - \frac{N_1 c u_0}{N_2 \cdot (\mathbf{r}_1 \times \mathbf{n}_1) \cdot \mathbf{k}_1}\right)_1 = \left(\Delta\phi_2^{(m)} - \frac{N_1 c u_0}{N_2 \cdot (\mathbf{r}_1 \times \mathbf{n}_1) \cdot \mathbf{k}_1}\right)_2 = \ldots = \left(\Delta\phi_2^{(m)} - \frac{N_1 c u_0}{N_2 \cdot (\mathbf{r}_1 \times \mathbf{n}_1) \cdot \mathbf{k}_1}\right)_s \tag{8}$$

The angular displacement compensation theory (b) should be satisfied, hence

$$\sum_{i=1}^{s} c_i = 1 , \quad c_i \geq 0 \tag{9}$$

Eqs. (8) and (9) consist of s linear equations and can be solved for the load share coefficients c_j. A negative load share coefficient ($c_j < 0$) indicates that the point is not under contact and the corresponding equation in the linear system should be omitted. Then the dimension of the linear system equation reduces to $s-1$. Keep solving the linear system equation until all the load share coefficients c_j are not negative. The load sharing for each rotation angle can be obtained by circularly solving the linear system following the above procedure in a whole meshing cycle.

3. Contact analysis with ANSYS

After the rotation angle corresponding to the maximum load share coefficient is obtained, the tooth contact analysis (TCA) model can be built up in ANSYS. For the sake of computational efficiency, an approximate three-tooth model is taken for each gear in the TCA model. Finite element mesh is automatically generated in ANSYS by APDL language (ANSYS Parameter Design Language). Dense mesh is locally put at the tooth root and contact zone. The tooth finite element model is shown in Fig. 1.

Convex and concave portion of a single tooth may simultaneously engage in contact. Hence two contact pairs should be set in advance, as shown in Fig. 2. Because the curvature radius of concave surface is bigger than that of the convex surface, the concave surfaces of the two teeth are set to be the target surfaces TARGE1 and TARGE2, respectively. The convex surfaces are set to be the contact surfaces CONTA1 and CONTA2. Hexahedron iso-parametric elements are chosen to model the gear teeth. Total nodes are 23452 and total elements are 20882 with 1120 contact elements.

In past gear tooth contact analyses, the boundary conditions were usually set to be fully restricting the driven gear tooth and restricting the degrees-of-freedom (DOF) of the two boundary surfaces of the drive gear tooth except the rotational DOF with regard to the gear axle. Tangential force was applied as an equivalent torque to the inner ring of the drive gear. Another way for contact analysis

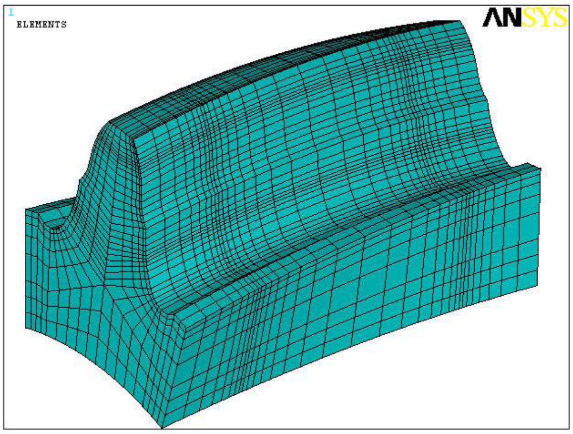


Fig. 1 Mesh of a tooth model

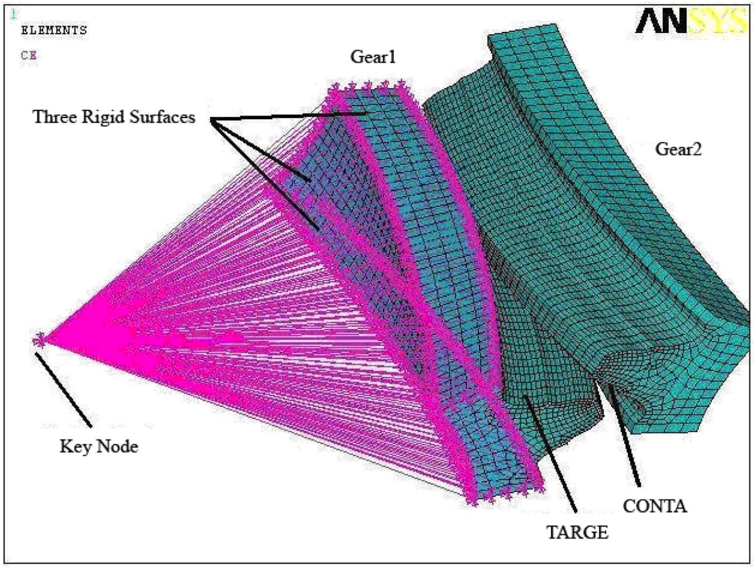


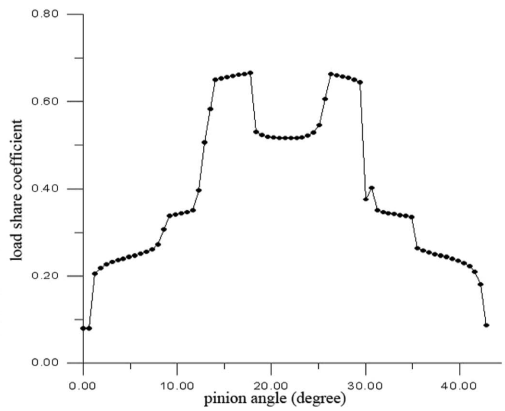
Fig. 2 Gear tooth contact model

is by transforming stress boundary conditions and kinematic conditions into multiple point constraints for nodal displacements (Liu *et al*. 2010). The commercial CAE software package ANSYS allows mixed element types to be involved in one single model and its point elements have rotational DOF. Hence, a new model which has the capability of representing the real meshing situation can be implemented as follows: first define the boundary surfaces of gear teeth as rigid surfaces, then couple the translational DOF of the main control point with the rigid surfaces and restrict the rotational DOF of the main control point except that around the gear axle. The import torque can be put on the main control point. Thin lines in Fig. 2 denote this relation of DOF coupling.

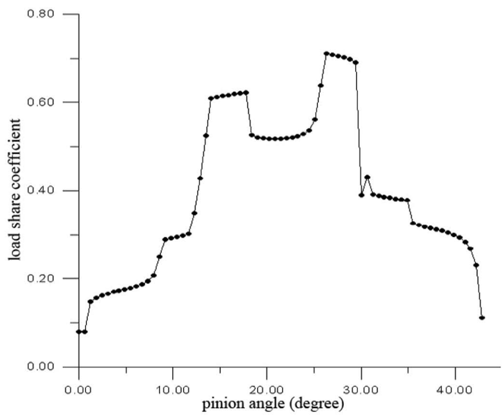
Considering that the bending deformation of gear tooth is much larger than that from contact extrusion, the normal contact stiffness factor is set to be a small number of 0.1. Besides, the minimum automatic time step is set to be 0.02 second for the purpose of ensuring computational stability and convergence.

4. Numerical examples

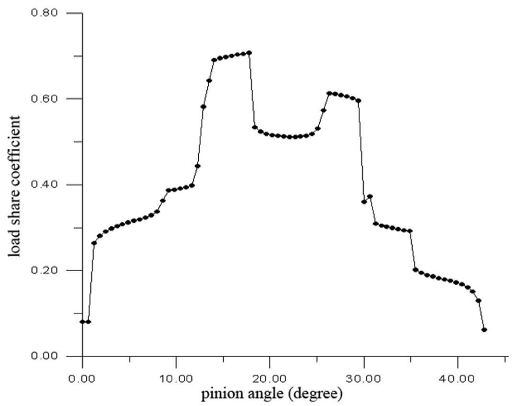
A pair of double circular arc gears with the following parameters are selected as an example to demonstrate the proposed TCA approach: normal modulus $m_n = 2.25$, tooth number $z_1 = z_2 = 21$, helical angle $\beta = 28.955^{\circ}$, tooth face width b = 26.0 mm; The elastic properties of the gear material are Young's modulus $E = 2.06e5$ MPa, Poisson's ratio $\nu = 0.3$. The import torque varies from 50 Nm to 125 Nm. Assume that the maximum stress is smaller than the yield strength of the material so that the proposed elastic contact computational method is applicable. The calculated theoretical contact ratio of the gear is 1.78. Each gear has 3~4 contact traces at the same time. According to toothing quality 7 in accordance with China Standard JB12759-91, the crossing error



(a) Without assembly error



(b) With crossing angle error $\gamma = +0.02^\circ$



(c) With crossing angle error $\gamma = -0.02^\circ$

Fig. 3 Load sharing curves for single tooth with import torque of 50 Nm

of the two gear axles is chosen to be $\gamma = \pm 0.02^{\mathrm{o}}$ in the analysis. The computed load sharing coefficients for a single tooth under an import torque of 50 Nm and different crossing angle errors are shown in Fig. 3. Because the total contact traces change between 3~4, the load share coefficient is always less than 1 under the current import torque.

Without assembly error, the maximum load sharing coefficient of a single tooth happens at $\phi_1 = 27.2^{\mathrm{o}}$ with a value of 66.19%, among which the convex portion shares 34.19% and the concave portion shares 32.0%. The load sharing curve is almost symmetric with regard to the central section of the tooth width. When the crossing angle error is $\gamma = +0.02^{\mathrm{o}}$, the maximum single tooth load share is 71.03%, among which the convex portion is 39.19% and the concave portion is 31.84%; the maximum load sharing coefficient happens at $\phi_1 = 28.2^{\circ}$. When the crossing angle error is $\gamma = -0.02^{\mathrm{o}}$, the maximum load sharing coefficient of a single tooth happens at $\phi_1 = 17.56^{\mathrm{o}}$ with a value of 70.78%, among which the convex portion shares 35.57% and the concave portion shares 35.21%; The computational results demonstrate that the maximum load sharing coefficient becomes larger and the occurrence of the peak value of the load share will shift backward or forward when

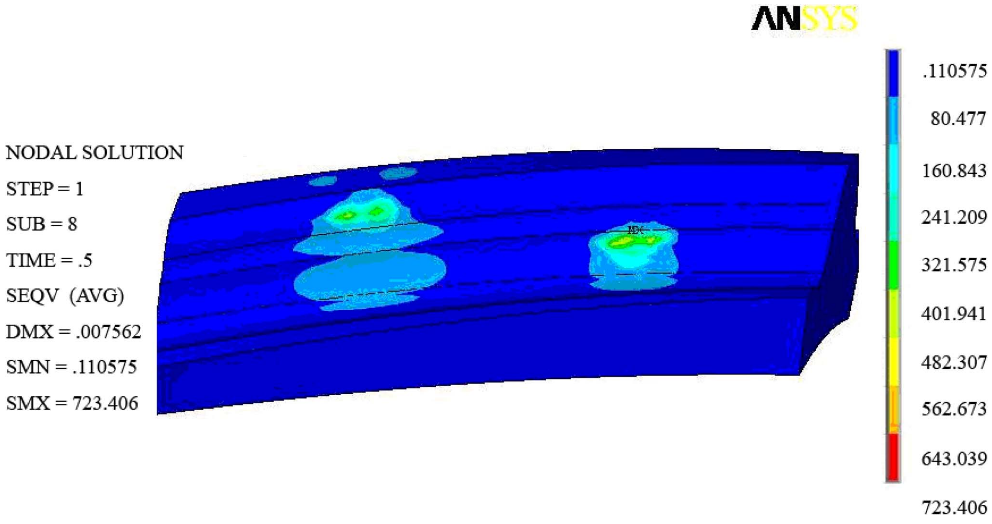


Fig. 4 Contact stress of convex and concave contact points (MPa)

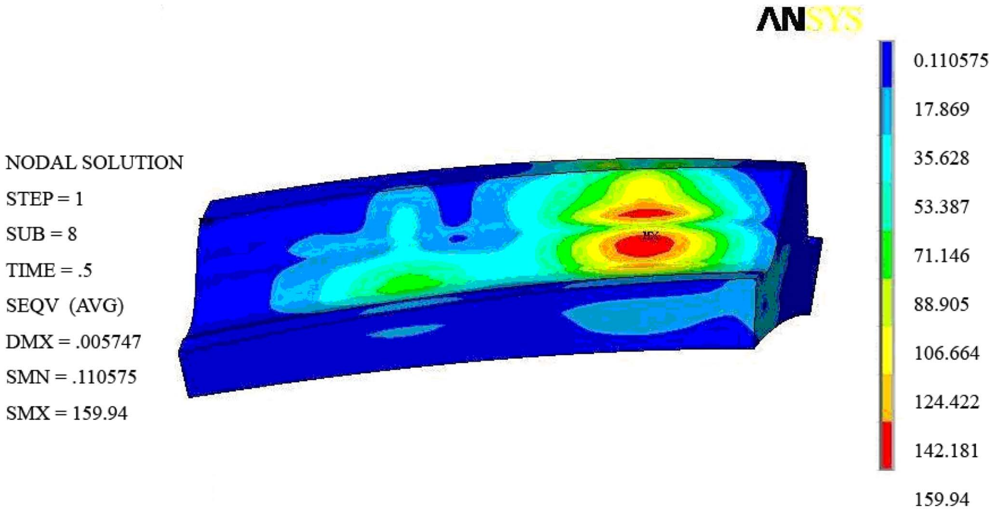


Fig. 5 Bending von-Mises stress for no error case (MPa)

assembly angle error presents.

The TCA is carried out according to the location corresponding to the maximum load share of a single tooth. Fig. 4 shows the contact stress distribution with an import torque of 50 Nm and no assembly error. The maximum contact stress is 723 MPa. Fig. 5 shows the corresponding bending stress distribution on the tooth root compressed surface. Along the tooth-lengthwise direction, there are two stress peak values, which correspond to the effects of the loaded convex and concave surfaces on the tooth root. The maximum bending stress is 159.9 MPa. It can be seen from Fig. 5 that the tooth root bending stress from loaded convex surface is obviously larger than that from the loaded concave surface.

Table 1 Maximum stress of peak load of convex tooth for all cases

Import torque (Nm)	50	75	100	125
Contact stress (MPa)	723	979	1279	1459
Hertz contact stress (Chen 2004) (MPa)	697	917	1138	1286
Bending stress (MPa)	160	220	286	332
Bending stress (Xu 1984) (MPa)	146	207	265	321
Bending stress from concentrated-load-model (MPa)	143	188	243	287

Table 1 shows the maximum contact and bending stresses under different loading conditions on the convex tooth surface. Form Table 1 it can been seen that the contact stress obtained from the proposed TCA approach is very close to the Hertz contact stress (Xu 1984). On the other hand, the bending stress from the proposed TCA method is a bit larger than that from the empirical formula (Chen 2004). But the discrepancy is small. The calculation results also show that the bending stress from a concentrated-force-model is always smaller than those from the TCA model. The difference becomes larger with the increase of loading. This shows that the surface-load-share mode of double circular arc gears has significant influence on tooth root bending stress and the TCA model can take into account this effect.

5. Conclusions

This paper presents a feasible and efficient approach for contact analysis of double circular arc gears and load sharing determination among teeth as well as among contact traces. Three-dimensional FE TCA is implemented with commercial CAE software package ANSYS. The contact status and contact stress distribution are obtained and compared with that from Hertz contact theory. The comparison demonstrates the efficiency and accuracy of the proposed TCA model. Because the contact zone of double circular arc gears is large, load is distributed on the contact surface. Hence, the bending stress obtained from the proposed contact analysis approach is more accurate than that from a concentrated loading approach.

Acknowledgements

This work is supported by the Science and Technology Innovation Project of Shaanxi Province (grant No. 2011KTCG01-06), and Innovation Fund for PHD of Xi'an Shiyou University (grant No. Z07061).

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