Structural Engineering and Mechanics, Vol. 43, No. 2 (2012) 239-251
DOI: http://dx.doi.org/10.12989/sem.2012.43.2.239

Can finite element and closed-form solutions for laterally loaded piles be identical?

Vishwas A. Sawant*[1] and Sanjay Kumar Shukla[2a]

[1]*Department of Civil Engineering, Indian Institute of Technology, Roorkee, 247667, Uttarakhand, India*
[2]*Discipline of Civil Engineering, School of Engineering, Edith Cowan University, Perth, WA 6027, Australia*

(Received January 6, 2012, Revised May 19, 2012, Accepted June 1, 2012)

Abstract. The analysis of laterally loaded piles is generally carried out by idealizing the soil mass as Winkler springs, which is a crude approximation; however this approach gives reasonable results for many practical applications. For more precise analysis, the three- dimensional finite element analysis (FEA) is one of the best alternatives. The FEA uses the modulus of elasticity E_s of soil, which can be determined in the laboratory by conducting suitable laboratory tests on undisturbed soil samples. Because of the different concepts and idealizations in these two approaches, the results are expected to vary significantly. In order to investigate this fact in detail, three-dimensional finite element analyses were carried out using different combinations of soil and pile characteristics. The FE results related to the pile deflections are compared with the closed-form solutions in which the modulus of subgrade reaction k_s is evaluated using the well-known k_s-E_s relationship. In view of the observed discrepancy between the FE results and the closed-form solutions, an improved relationship between the modulus of subgrade reaction and the elastic constants is proposed, so that the solutions from the closed-form equations and the FEA can be closer to each other.

Keywords: closed-form solution; finite element analysis; lateral load; modulus of elasticity; pile; subgrade modulus

1. Introduction

Piles are constructed to support lateral loads in several field applications. In general, the laterally loaded piles are categorized as short or long with respect to their length as well as the stress-deformation characteristics of both the piles and the surrounding soils (Terzaghi *et al*. 1996). Due to the difficulties in realistic idealization of the behaviour of the pile and the soil, the design of the laterally loaded piles has been a complex problem for practicing engineers and researchers. Several analytical and numerical approaches have been developed to analyze the response of laterally loaded piles in the past (Reese and Matlock 1956, Matlock and Reese 1960, Davisson and Gill 1963, Matlock 1970, Poulos 1971, Reese and Welch 1975, Randolph 1981, Pise 1982, Norris 1986, Budhu and Davies 1988, Prakash and Kumar 1996, Ashour *et al*. 1998, Fan and Long 2005, Harikumar *et al*. 2005, Houston *et al*. 2005, Karthigeyan *et al*. 2006, 2007, Dewaikar *et al*. 2011,

*Corresponding author, Assistant Professor, E-mail: sawntfce@iitr.ernet.in
[a]Associate Professor, E-mail: s.shukla@ecu.edu.au

Basu *et al.* 2009, Zhang 2009). Osman and Randolph (2011) presented a closed-form analytical solution for the consolidation of the soil around a laterally loaded pile under plane strain conditions by assuming elastic deformation of soil. Recently Chae *et al*. (2004) described the results of several numerical studies performed on laterally loaded short rigid piles and pier foundation located near slope with a three-dimensional finite element model and prototype tests. The finite element (FE) modeling approach provides a more precise tool that is capable of modeling soil continuity, pile-soil interface behaviour, and three-dimensional (3D) boundary conditions (Desai and Abel 1976, Randolph 1981). Additionally, the realistic determination of modulus of elasticity E_s of soil is possible in the laboratory by conducting triaxial tests on undisturbed soil specimens collected from the site (Bowles 1997, Das 1999). However, the FE modeling approach is more rigorous in its analytical methodology than any other existing methods, and therefore, this method primarily remains a research tool. The elastic subgrade reaction approach treats laterally loaded pile in the soil medium as a beam supported on a system of mutually independent elastic springs, called the Winkler's springs (Winkler 1867, Reese and Matlock 1956, Selvadurai 1979). A closed-form analysis based on the modulus of subgrade reaction (Reese and Matlock 1956, Poulos and Davies 1980, Bowles 1997) is quite convenient for computing the pile deflection under small working loads because of simplicity in the analysis in terms of mathematical steps; this has been the primary reason for the wide use of this approach in routine practice. The subgrade reaction approach, although approximate, is a powerful technique to model the response of single piles subjected to lateral loads. The error in the computed bending moments based on the subgrade reaction approach is no more than a few percent when compared to the theory of elasticity solutions (Vesic 1961).

Due to the different idealizations made in the finite element analysis and the subgrade reaction approach, the results related to pile deflections and bending moments are likely to vary. The key parameter in the subgrade reaction approach is the modulus of subgrade reaction (k_s), which is often evaluated by conducting a plate load test that involves an application of a horizontal load to a plate supported on the vertical soil wall (Terzaghi 1955, Teng 1962, Selvadurai 1979, Bowles 1997, Das 1999). It is experienced that the variation of k_s along the pile length cannot be determined experimentally in an economic manner. Therefore, the value of k_s is generally estimated using its relationship with elastic constants (E_s and μ) given in the literature (Vesic 1961, Sevaduarai 1979, Bowels 1998). This may be an important cause for difference between the closed-form and the FE solutions. An attempt is made here to investigate the difference in the results related to deflections and bending moments for laterally loaded piles obtained from the 3D finite element and the closed-form analyses considering various pile geometries and soil characteristics. Further attempt is made to propose an improved relationship between the modulus of subgrade reaction (k_s) and the elastic constants (E_s and μ_s) so that the closed-form solution can be conveniently used for practical applications without significantly deviating from the results based on rigorous finite element analysis.

2. Finite element formulation

Fig. 1 shows a schematic diagram of laterally loaded pile of length L, subjected to a lateral load H at the pile top along X-direction. Z axis is vertical coinciding with the pile axis, and X-Y defines a horizontal plane. For the sake of convenience in three-dimensional finite element (3D FE) mesh generation, a square pile of width D has been used. The pile is completely embedded in the soil.

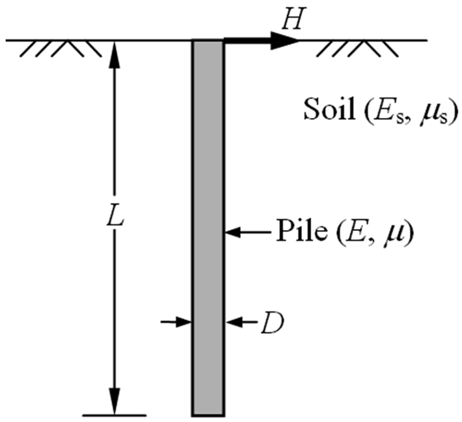


Fig. 1 Schematic diagram of a laterally loaded pile

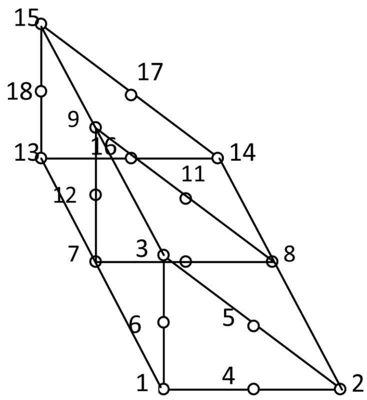
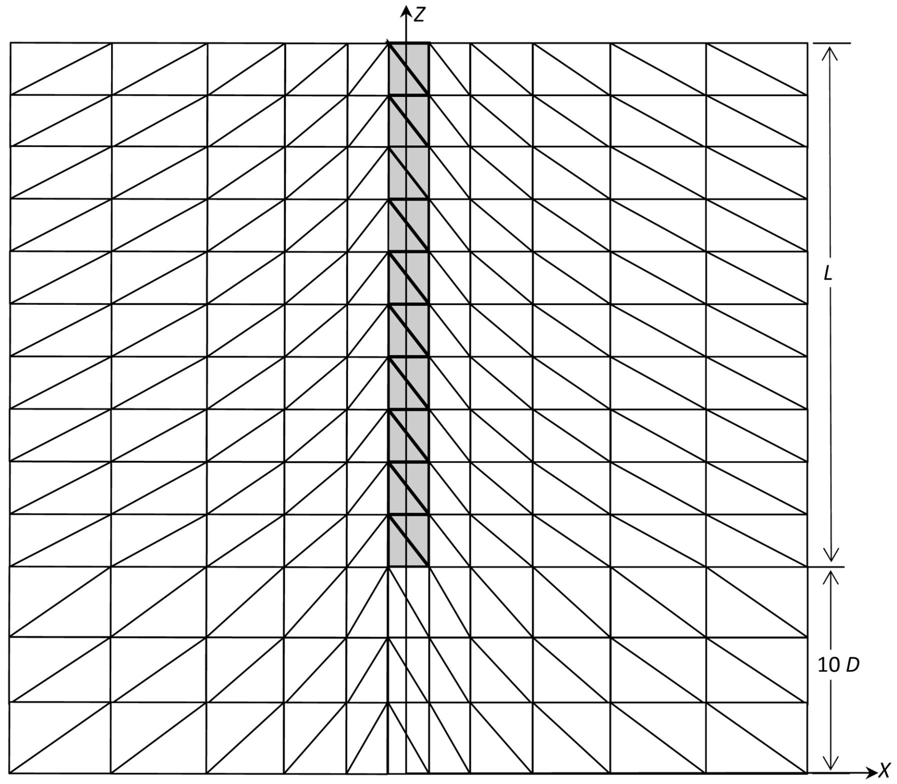


Fig. 2 18 noded triangular prism element

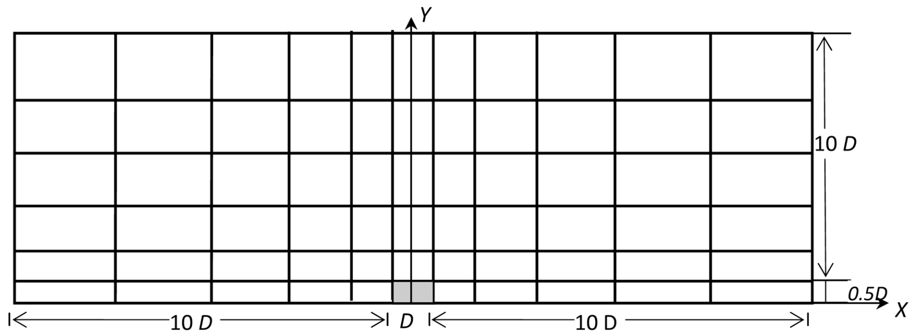
The pile and soil system is idealized as an assemblage of 18 node triangular prism continuum elements (Fig. 2) with linearly varying strain across the element. These elements are suitable for modeling the response of a system dominated by bending deformations. Each node of the element has three translational degrees of freedom, u, v and w, in the X, Y and Z coordinate directions, respectively. A typical finite element mesh developed in the study is shown in Fig. 3. Taking an advantage of the symmetry, only half of the actual domain was modelled, thus reducing the computational effort. The mesh size selected for the finite element solution has been optimized for both accuracy and computational economy based on the analyses of several meshes with different numbers of elements and mesh sizes.

The relations used in the formulation are outlined below. The shape functions which describe the relation between the displacements at any point within the element are defined as follows

$$u = \sum_{i=1}^{1} N_i u_i; \quad v = \sum_{i=1}^{18} N_i v_i; \quad w = \sum_{i=1}^{18} N_i w_i \tag{1}$$



(a) Elevation



(b) Plan

Fig. 3 Finite element discretisation of the pile-soil system

where

$$N_i = M_j f_k(\eta); \quad j = 1,6; \quad k = 1,3 \quad \text{and} \quad i = 6(k-1)+j$$
$$f_1(\eta) = -0.5\,\eta(1-\eta); \quad f_2(\eta) = 1-\eta^2; \quad f_3(\eta) = 0.5\,\eta(1+\eta)$$
$$M_i = L_1(2L_i - 1); \quad i = 1,3$$
$$M_k = 4L_iL_j; \quad k = 4,5,6; \quad i = 1,2,3; \quad j = 2,3,1 \tag{2}$$

In Eq. (2), the functions $f_1(\eta)$ to $f_3(\eta)$ define the variables in Y direction. M_1 to M_6 are the components of the shape function defined in a triangular XZ plane. Any point P is defined in XZ plane with the set of natural coordinates (L_1, L_2, L_3) as: $L_1 = A_1/A_t$; $L_2 = A_2/A_t$; $L_3 = A_3/A_t$, where A_1, A_2, and A_3 are the areas of the three subtriangles, subtended by the point P and A_t is total area of triangle.

The relation between strains and nodal displacements is expressed as

$$\{\varepsilon\}_e = [B]\{\delta\}_e \tag{3}$$

where $\{\varepsilon\}_e$ is the strain vector, $\{\delta\}_e$ is the vector of nodal displacements, and $[B]$ is the strain displacement transformation matrix. The stress-strain relation is given by

$$\{\sigma\}_e = [D]\{\varepsilon\}_e \tag{4}$$

where $\{\sigma\}_e$ is the stress vector, and $[D]$ is the constitutive relation matrix.

The stiffness matrix of an element $[K]_e$ is expressed as

$$[K]_e = \int_V [B]^T[D][B]dv \tag{5}$$

The lateral force H, acting on the pile top, is considered as a uniformly distributed force. The intensity of the uniformly distributed force is, $q = H/A$, where A is the area of pile-top. Equivalent nodal force vector, $\{Q\}_e$, is then expressed as

$$\{Q\}_e = \int_V q[N]^T dA \tag{6}$$

where $[N]$ represents matrix of shape functions (Zienkiewicz 1977).

The element stiffness matrix $[K]_e$ and the nodal force vector $\{Q\}_e$, are evaluated analytically. The 3D finite element program based on the formulation explained above is developed in FORTRAN-90, in which, the element stiffness matrix $[K]_e$ for each element is assembled in global stiffness matrix in the skyline storage form. Similarly, the nodal load vectors are assembled into the global load vector. The system of simultaneous equations is solved for the unknown nodal displacements using active column solver. Bending moments are computed using well-known moment curvature relationship, $M = EIu^{ZZ}$, where E is the Young's modulus of elasticity of the pile material, I is the moment of inertia of the pile cross-section, and u^{ZZ} is second order derivative with respect to z.

For verifying the accuracy of the program, a standard problem of the cantilever beam subjected to a load at its free end is considered. The analytical result is obtained by integration of moment curvature relationship. The displacement pattern was computed using the developed code, and it was compared with the analytical results. The deflection of the beam obtained from the finite element and the analytical approaches along the beam length is compared in the Fig. 4. It is observed that deflection increases with the distance from the fixed end of the beam. A close agreement between the two solutions is noticed with a maximum error of 3%. Thus, the accuracy of the developed program is validated.

It was also attempted to verify the accuracy with field results available in literature. For this purpose, data presented by Prakash (1962) on experimental study on behaviour of lateral load pile embedded in sand was considered. The test pile was a hollow circular tube with diameter of 1.6″ and length 24″. However, for the sake of convenience, the hollow circular section of pile is converted into an equivalent square section of 1.6″. For the pile, Young's modulus and Poisson's ratio were taken as 43520 psi and 0.2, respectively. An equivalent modulus of elasticity of sand was defined. As detailed information pertaining to stress-strain behaviour was not available, this

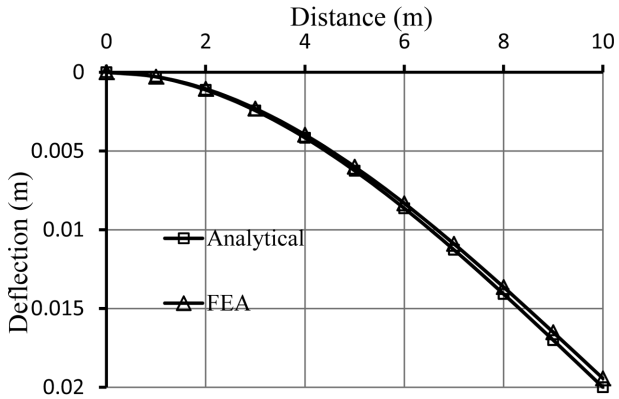
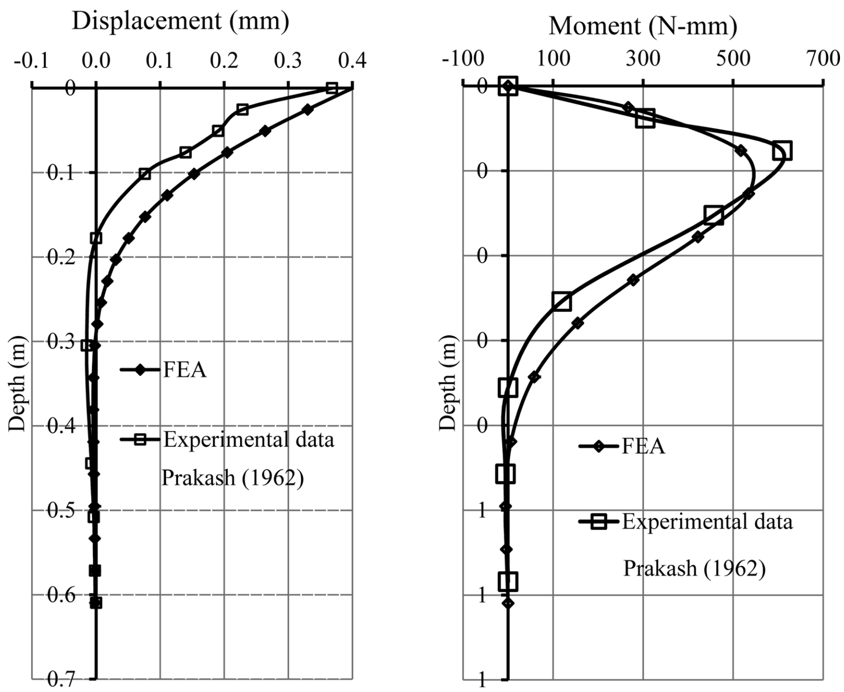


Fig. 4 Deflection of a cantilever beam with FEA and analytical solution



(a) (b)

Fig. 5 Comparison of pile response with experimental data from Prakash (1962). (a) Variation in displacement along depth, (b) Variation in bending moment along depth

modulus was approximately computed from the relation $E = J\gamma z$ as given by Terzaghi and Peck (1967) where γ is the unit weight of sand, z is the depth from surface and J is the dimensionless parameter whose value is taken as 350. The unit weight of sand was taken as 18.9 kN/m^3. Poisson's ratio for sand was considered as 0.25. Lateral load of magnitude 12.26 N was applied at the top of the pile. Variations in pile displacement and bending moment along depth are compared in Fig. 5. It is observed that the results obtained from the finite element analysis (FEA) are in fair agreement with the experimental results.

3. Closed-form solution

In the past, many researchers have analyzed laterally loaded pile as a vertical beam supported by linear soil springs. The governing differential equation for the horizontal displacement u of the pile along depth z can be given as (Hetenyi 1946, Reese and Matlock 1956, Poulos and Davis 1980)

$$EI\frac{d^4u}{dz^4} + (k_sD)u = 0 \tag{7}$$

Vesic (1961) proposed simplified and generalized relationships between the modulus of subgrade reaction k_s and the elastic constants (E_s and μ_s) as given below (Selvadurai 1979, Bowles 1997)

$$k_s = \frac{E_s}{D(1-\mu_s^2)} \tag{8a}$$

$$k_s = 0.65\frac{E_s}{D(1-\mu_s^2)}\sqrt[12]{\frac{E_sD^4}{EI}} \tag{8b}$$

For constant value of k_s along the pile length, solution for the differential equation (Eq. (7)) is given as (Hetenyi 1946)

$$u = e^{\lambda z}(C_1\cos\lambda z + C_2\sin\lambda z) + e^{-\lambda z}(C_3\cos\lambda z + C_4\sin\lambda z)\,; \quad \lambda = (k_sD/4EI)^{0.25} \tag{9}$$

Constants C_1 to C_4 are obtained using the four boundary conditions with respect to shear force and bending moment at the pile top and tip. For the pile subjected to a lateral load H at the top, the constants are given as

$$C_1 = \frac{\Psi - \sin\theta e^{\theta}}{e^{2\theta}-2e^{\theta}-1+2\cos\theta}\left(\frac{H}{2EI\lambda^3}\right) \qquad C_2 = \frac{(\cos\theta-1)e^{\theta}}{e^{2\theta}-2e^{\theta}-1+2\cos\theta}\left(\frac{H}{2EI\lambda^3}\right)$$

$$C_3 = \frac{\Psi^2+\Psi - \sin\theta e^{\theta}}{e^{2\theta}-2e^{\theta}-1+2\cos\theta}\left(\frac{H}{2EI\lambda^3}\right) \qquad C_4 = C_2; \quad \theta = 2\lambda L \text{ and } \Psi = e^{\theta}-1 \tag{10}$$

Pile displacement u, can be expressed in the non-dimensional form $\bar{u} = u(2EI\lambda^3)/H$ as below

$$\bar{u} = \frac{[(\Psi-\sin\theta e^{\theta})e^{\lambda z} + (\Psi^2+\Psi-\sin\theta e^{\theta})e^{-\lambda z}]\cos\lambda z + (\cos\theta-1)e^{\theta}(e^{\lambda z}+e^{-\lambda z})\sin\lambda z}{e^{2\theta}-2e^{\theta}-1+2\cos\theta} \tag{11}$$

Pile top displacement (at $z = 0$) can be computed directly from the following expression.

$$u_{top} = \frac{H}{2EI\lambda^3}\frac{e^{2\theta}-1-2\sin\theta e^{\theta}}{e^{2\theta}-2e^{\theta}-1+2\cos\theta} \tag{12}$$

Bending moment M, can be expressed in the non-dimensional form $\bar{M} = M\lambda/H$ as given below

$$\bar{M} = \frac{[-(\Psi-\sin\theta e^{\theta})e^{\lambda z} + (\Psi^2+\Psi-\sin\theta e^{\theta})e^{-\lambda z}]\sin\lambda z + (\cos\theta-1)e^{\theta}(e^{\lambda z}+e^{-\lambda z})\cos\lambda z}{e^{2\theta}-2e^{\theta}-1+2\cos\theta} \tag{13}$$

4. Results and discussions

In order to investigate the difference in the pile response with respect to deflections and bending moments obtained from the 3D finite element and the closed-form approaches, a parametric study is carried out considering various pile geometries and soil characteristics. The material properties are reported in Table 1. The width of pile is taken as 0.6 m, which is kept constant throughout the study. The pile length and modulus of elasticity of soil were varied to examine their effects on the pile response. The ratio of pile length to its width (L/D) was varied from 10 to 50. The modulus of elasticity E_s of soil is varied from 4000 kPa to 14000 kPa.

Variations in the pile top displacement with E_s and L/D ratio are presented in Fig. 6. It can be noted that the pile top displacement decreases with an increase in E_s. Further, it is observed that the pile top displacement decreases with an increase in L/D ratio; the reductions become very small beyond L/D greater than 15, and the displacements become almost constant for L/D greater than 40.

Table 1 Geometrical and material properties

	Modulus of elasticity (kPa)	Poisson's ratio
Pile	2.0×10^7	0.30
Soil	4000, 6000, 8000 10000, 12000, 14000	0.45
L/D ratio	10, 15, 20, 25, 30, 40, 45, 50	

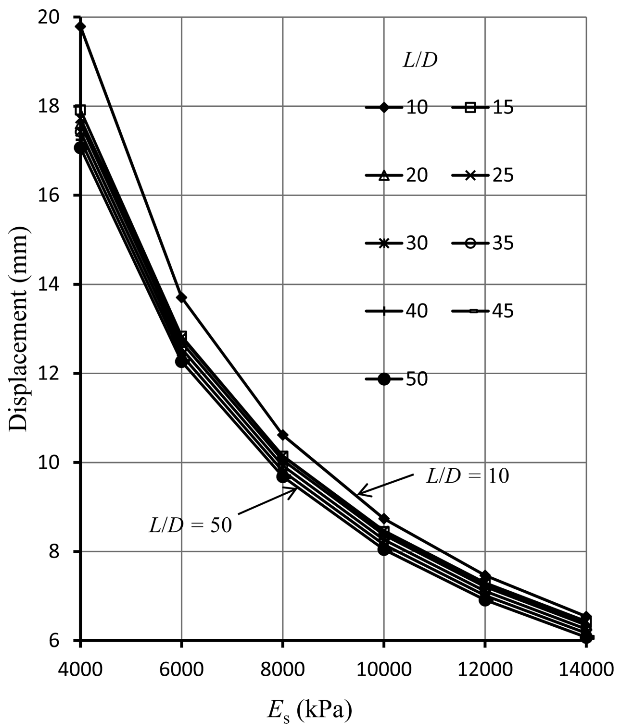


Fig. 6 Variation in pile top displacement with soil modulus for different L/D ratios

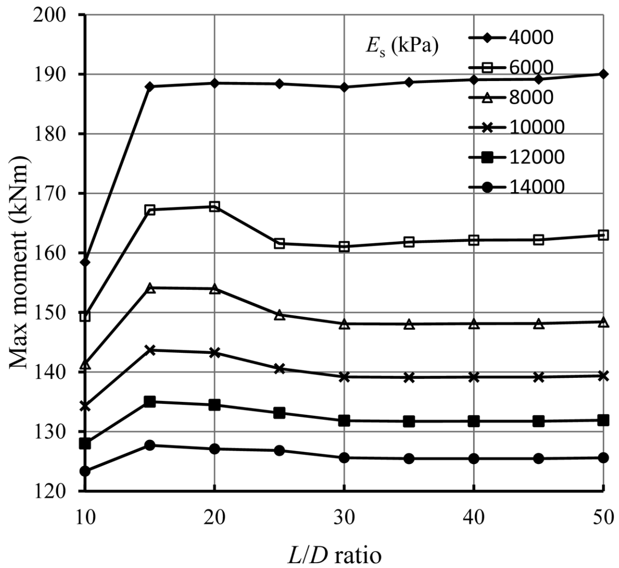


Fig. 7 Variation in maximum moment with L/D ratio for different soil moduli

Variations in the maximum bending moments in the pile for different combinations of E_s and L/D ratio are depicted in Fig. 7. It can be noticed that the maximum bending moment in the pile decreases with an increase in E_s. With respect to the effect of L/D ratio, an increase in the moment is observed up to L/D ratio of 25, and thereafter values attain almost a constant value with a further increase in the L/D ratio.

For specific L/D ratios and soil parameters (E_s and μ_s), the deflections and bending moments are obtained from the finite element analysis. For the given soil parameters (E_s and μ_s), k_s value is determined using Eq. (8), and the pile deflections are computed from the closed-form solution given by Eq. (11). Variations in the pile top displacements obtained from the FE and closed-form approaches are presented in Fig. 8 for L/D = 10 to 50 in non-dimensional form. As expected, the pile top displacement decreases with an increase in E_s. Pile top displacements calculated using Eq. (11) are significantly greater than those obtained from the FE analysis. In order to get closer match, closed form solutions are also obtained as plotted in these figures with an improved form of Eq. (8a) as

$$k_s^* = k \times k_s = \frac{kE_s}{D(1-\mu_s^2)} \tag{14a}$$

$$k_s^* = k \times k_s = k \times 0.65 \frac{E_s}{D(1-\mu_s^2)} \sqrt[12]{\frac{E_s D^4}{EI}} \tag{14b}$$

where k is a factor, which may be called, subgrade modification factor. The value of k is obtained by trial-and-error, so that the solutions from the closed-form equation and the FE analysis will almost be identical.

During the trial-and-error computation, the factor k was observed to vary from 1.38 to 1.45 (Table 2), the higher value was observed for a high L/D ratio. In Fig. 8, it is clearly observed that

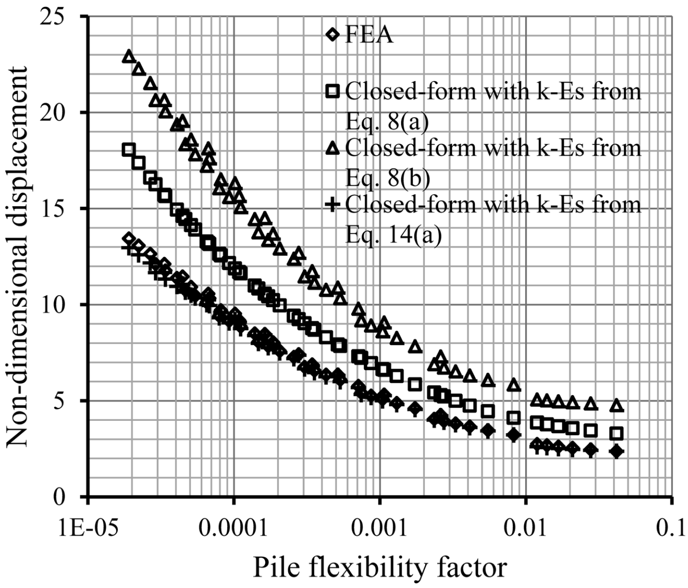


Fig. 8 Comparison of FEA and closed-form results

Table 2 Factor k for different L/D ratio

L/D ratio	10	15	20	25	30	35	40	45	50
Factor k	1.45	1.38	1.38	1.38	1.4	1.4	1.42	1.45	1.45

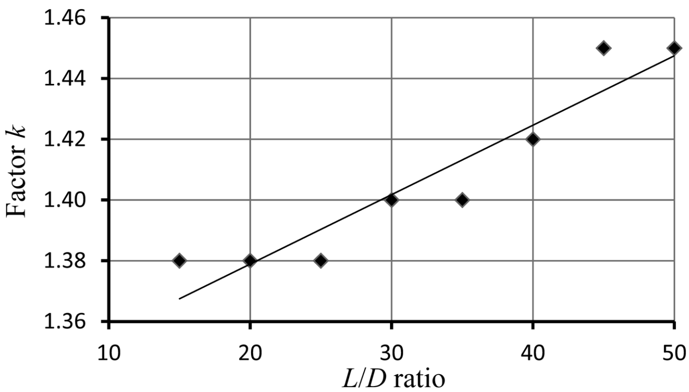


Fig. 9 Variation in factor k with L/D ratio

with the subgrade modification factor k, the pile top displacement obtained from the closed-form equation (Eq. (11)) is in good agreement with the FEM results. For the L/D ratios greater than 10 (long flexible piles), the best fit to the scatter points yielded the following expression for factor k.

$$k = 1.33321 + 0.00229L/D \tag{15}$$

Similarly the modification factor k was in the range of 3.16 to 3.68 for k-E_s relationship described by Eq. (8b).

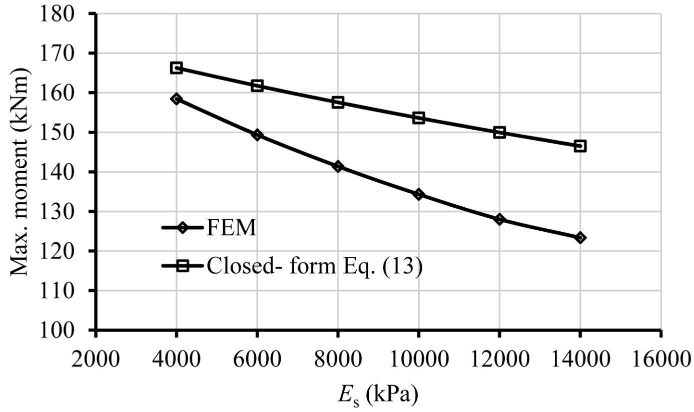


Fig. 10 Comparison of maximum moments from FEA and closed-form solution

A typical comparison of maximum moments in the pile obtained from FEA and those from closed-form solution using the relationship for k-E_s given in Eq. (14), is depicted in Fig. 10. It is observed that the maximum moments calculated using closed-form solutions are on higher side. Overall maximum bending moments were on average 17% higher than those computed from FEA. It may be attributed to one-dimensional load sharing mechanism in closed-form solution where effect of side shear is neglected. Depth of occurrence of maximum moment was observed from 5*D* to 3*D* for FEA as well as for closed-form solution. A higher value was observed for lower soil modulus which reduced with an increase in soil modulus. Both approaches indicated nearly the same depth, except marginally higher depth (0.5*D*) in few cases with FEA predictions.

It should be noted that one can achieve the accuracy of the finite element approach into the closed-form solutions using the subgrade modification factor as suggested above in Eq. (14); however, this new approach requires to be verified by experiments, which are expected in future.

Present approach is suggested to minimize the difference between closed form solution and finite element analysis. Though it is suggested as an alternative approach to complex finite element analysis, some of the limitations of the closed form solutions will remain as it is. For instance, the closed form solution could be valid for one-dimensional structures with symmetric geometry, symmetric loading, unidirectional lateral loading, symmetric axial loading, and only displacements from which the bending moments are computed. It is also required that applied load should be small so as to have linear stress-strain relationship for soil medium. If one wants to analyse other factors such as effect of realistic nonlinear models, stress and strain variations, and cracking leading to softening and multi-dimensional geometries, then finite element analysis is the only alternative.

5. Conclusions

In the present study, a three dimensional finite element formulation is presented for a laterally loaded pile in order to estimate pile deflections and bending moments along the pile length, considering linear elastic characteristics of soil and pile materials. The results obtained are compared with the closed-form solutions based on the subgrade reaction approach, which is commonly used for designing laterally loaded piles in the day-to-day practice. A significant variation in the two approaches is observed. This discrepancy appears mainly because of different assumptions and idealisations in the two approaches. This study shows that an inclusion of a subgrade modification factor *k* lying in the range of 1.38 to 1.45 in the simplified Vesic's equation (Eq. (8a)) and 3.16 to 3.68 in the generalized Vesic's equation (Eq. (8ba)) makes the results obtained from the closed-form solution (Eq. (11) and (13)) almost identical to the results from finite element analysis. For the *L*/*D* ratios greater than 10 (long flexible piles), the best fit to the scatter points yielded a simple expression (Eq. (15)) for factor *k* for routine use in the simplified Vesic's equation (Eq. (8a)) by the designers. The use of this new subgrade reaction approach of the closed-from solution is simple to apply, and additionally the designers can save significant time avoiding the complex finite element formulation and required computation time.

Acknowledgements

The principal author V.A. Sawant wish to express his sincere thanks to the Australian Government, Department of Education, Employment and Workplace Relations (DEEWR) for financial support through the Endeavour Award scheme.

References

Ashour, M., Norris, G.M. and Pilling, P. (1998), "Laterally loading of a pile in layered soil using the strain wedge model", *J. Geotech. Geoenviron. Eng.*, **124**(4), 303-315.
Basu, D., Salgado, R. and Prezz, M. (2009), "A continuum-based model for analysis of laterally loaded piles in layered soils", *Geotechnique*, **59**(2), 127-140.
Bowles, J.E. (1997), *Foundation Analysis and Design*, 6th edition, McGraw-Hill International Press.
Budhu, M. and Davies, T.G. (1988), "Analysis of laterally loaded piles in soft clays", *J. Geotech. Eng.*, **114**(1), 21-39.
Chae, K.S., Ugai, K. and Wakai, A. (2004), "Lateral resistance of short single piles and pile groups located near slopes", *Int. J. Geomech.*, **4**(2), 93-103.
Das, B.M. (1999), *Principles of Foundation Engineering*, 4th Edition, PWS Publishing, Pacific Grove, CA.
Davisson, M.T. and Gill, H.L. (1963), "Laterally loaded piles in a layered soil system", *J. Soil Mech. Found. Div.*, **89**(3), 63-94.
Desai, C.S. and Appel, G.C. (1976), "3-D analysis of laterally loaded structures", *Proceedings of the 2nd International Conference on Numerical Methods in Geomechanics*, Blacksburg, 405-418.
Dewaikar, D.M., Chore, H.S., Goel, M.D. and Mutgi, P.R. (2011), "Lateral resistance of long piles in cohesive soils using *p-y* curves", *J. Struct. Eng.*, **38**(3), 222-227.
Fan, C.C. and Long, J.H. (2005), "Assessment of existing methods for predicting soil response of laterally loaded piles in sand", *Comput. Geotech.*, **32**(4), 274-289.
Harikumar, A., Dodagoudar, G.R. and Quadri, S.S. (2005), "Response analysis of laterally loaded piles using nonlinear *p-y* curves", *Proc. Indian Geotechnical Conference-2005*, 55-58.
Hetenyi, M. (1946), *Beams on Elastic Foundation*, University of Michigan Press, Ann Arbor.
Houston, W.N., Walsh, K.D., Harraz, A.M. and Houston, S.L. (2005), "Moment/rotation effects on laterally loaded drilled shaft group response", *Int. J. Geomech.*, **5**(4), 304-310.
Karthigeyan, S., Ramakrishna, V.V.G.S.T. and Rajagopal, K. (2006), "Influence of vertical load on the lateral response of piles in sand", *Comput. Geotech.*, **33**, 121-131
Karthigeyan, S., Ramakrishna, V.V.G.S.T. and Rajagopal, K. (2007), "Numerical investigation of the effect of vertical load on the lateral response of piles", *J. Geotech. Geoenviron. Eng., ASCE*, **133**(5), 512-521.
Matlock, H. (1970), "Correlations for design of laterally loaded piles in soft clay", *Proc., 2nd Offshore*

Technology Conf., Houston, 577-594.
Matlock, H. and Reese, L.C. (1960), "Generalized solutions for laterally loaded piles", *J. Soil Mech. Found.*, **86**(5), 63-91.
Norris, G.M. (1986), "Theoretically based BEF laterally loaded pile analysis", *Proceedings 3rd International Conference on Numerical Methods in Offshore Piling*, Nantes, France.
Osman, A. and Randolph, M. (2011), "An analytical solution for the consolidation around a laterally loaded pile", *International Journal of Geomechanics*, March.
Pise, P.J. (1982), "Laterally loaded piles in a two-layer soil system", *J. Geotech. Engrg. Div*., **108**(9), 1177-1181.
Poulos, H.G. (1971), "Behavior of laterally loaded piles-I: Single piles", *J. Soil Mech. Found. Div.,* **97**(5), 711-731.
Poulos, H.G. and Davis, E.H. (1980), *Pile Foundation Analysis and Design*, John Wiley & Sons, New York.
Prakash, S. (1962), "Behaviour of pile groups subjected to lateral loads", Ph.D. Thesis, University of Illinois, Urbana.
Prakash, S. and Kumar, S. (1996), "Nonlinear lateral pile deflection prediction in sands", *J. Geotech. Eng.*, **122**(2), 130-138
Randolph, M.F. (1981), "The response of flexible piles to lateral loading", *Geotechnique*, **31**(2), 247-259.
Reese, L.C. and Matlock, H. (1956), "Non-dimensional solutions for laterally loaded piles with soil modulus assumed proportional to depth", *Proceedings of the 8th Texas Conf. on Soil Mechanics and Foundation Engineering,* Austin, Texas.
Reese, L.C. and Welch, R.C. (1975), "Lateral loading of deep foundations in stiff clay", *J. Geotech. Eng. Div*., **101**(7), 633-649.
Selvadurai, A.P.S. (1979), *Elastic Analysis of Soil-Foundation Interactions*, Elsevier Scientific Publishing, Netherlands.
Teng, W.C. (1962), *Foundation Design*, Prentice Hall, Englewood Cliffs, NJ.
Terzaghi, K. (1955), "Evaluation of coefficient of subgrade reaction", *Geotechnique*, **5**(4), 297-326.
Terzaghi, K. and Peck, R.B. (1967), *Soil Mechanics in Engineering Practice*, Second edition, John Wiley & Sons, New York.
Vesic, A.B. (1961), "Beams on elastic subgrade and Winkler's hypothesis", *Proceedings of the 5th Int. Conf. on Soil Mechanic and Foundation Engineering*, Paris.
Winkler, E. (1867), *Die Lehre von der Elasticitaet und Festigkeit*, Prag, Dominicus.
Zhang, L. (2009), "Nonlinear analysis of laterally loaded rigid piles in cohesionless soil", *Comput. Geotech.,* **36**, 718-724.
Zienkiewicz, O.C. (1977), *Finite Element Method*, 3rd Edition, Tata McGraw Hill.