

A curvature method for beam-column with different materials and arbitrary cross-section shapes

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Abstract. This paper presents a curvature method for analysis of beam-columns with different materials and arbitrary cross-section shapes and subjected to combined biaxial moments and axial load. Both material and geometric nonlinearities (the p-delta effect in this case) were incorporated. The proposed method considers biaxial curvatures and uniform normal strains of discrete cross-sections of beam-columns as basic unknowns, and seeks for a solution of the column deflection curve that satisfies force equilibrium conditions. A piecewise representation of the beam-column deflection curve is constructed based on the curvatures and angles of rotation of the segmented cross-sections. The resulting bending moments were evaluated based on the deformed column shape and the axial load. The moment curvature relationship and the beam-column deflection calculation are presented in matrix form and the Newton-Raphson method is employed to ensure fast and stable convergence. Comparison with results of analytic solutions and eccentric compression tests of wood beam-columns implies that this method is reliable and effective for beam-columns subjected to eccentric compression load, lateral bracings and complex boundary conditions.

Keywords: beam columns; curvature; biaxial moment; lateral bracing; stability

1. Introduction

Structural beam-column members are subjected to biaxial bending moments and axial load. Solution of such loaded members should be approached via three-dimensional models, in which the equilibrium conditions need to be satisfied based on the deformed column shape to consider the amplified moments due to axial load (as called p-delta effect). When lateral bracings are provided to prevent buckling, the bracing forces must be taken into account as well.

Many models have been proposed based on linear elastic material assumptions (Zhang *et al.* 1993, Aristizabal-Ochoa 2004, Areiza-Hurtado *et al.* 2005, Saha and Banu 2007) and the beam-column governing differential equations (Al-Noury and Chen 1982, Iwai *et al.* 1986). Numerical methods have been also constructed to consider the material and geometric nonlinearities (Wang and Hsu 1992, Song and Lam 2009, 2010). However, for beam-columns with nonrectangular cross-sections or made with different materials, such as steel bars and concrete, issues may arise regarding the analysis efficiency and accuracy.

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Much effort has been asserted onto modeling the moment curvature relationship of cross-sections consisted of different materials and or in arbitrary cross-section shapes (Zak 1993, De Vivo and Rosati 1998, Rodriguez and Aristizabal-Ochoa 1999, Consolazio *et al.* 2004). These studies provided a good starting point, from where the cross-sectional moment curvature relationship can be incorporated into the analysis of structural behaviour of complex beam-columns.

This paper presents a curvature method for analysis of beam-columns of different material and arbitrary cross-section shapes. The focus is on the incorporation of the cross-sectional moment-curvature relationship into the structural analysis of complex beam-columns. A piecewise representation of the beam-column deflection curve is constructed based on the curvatures and angles of rotation of the segmented cross-sections. The resulting bending moments were evaluated based on the deformed column shape and the axial load. The moment curvature relationship and the beam-column deflection calculation are presented in matrix form and the Newton-Raphson method is employed to ensure fast and stable convergence. Comparison with results of analytic solutions and eccentric compression tests of wood beam-columns implies that this method is reliable and effective for beam-columns subjected to eccentric compression load, lateral bracings and complex boundary conditions.

2. Basic assumptions

Formulation of the moment curvature relationship, the beam-column deflection and the resulting bending moments at the discrete cross-sections are based on following assumptions:

- Plane section remains plane after deformation,
- Stress-strain relationship is known and independent of loading rate,
- Deflection is small compared to member geometry,
- Column deflection can be approximated by second order Taylor's series,
- Shear stress is negligible, and
- Torque and torsional failures are negligible.

The last assumption (ignoring torsional buckling) is generally valid for close-form cross-sections of a relatively small height-to-width ratio. This covers most of the reinforced concrete members and wood members made of sawn lumber; therefore, it is considered to be sufficient for the numerical analysis problems presented in this study.

2.1 Moment curvature relationship of a cross-section under eccentric load

General procedures to formulate the moment curvature relationship of a cross-section under eccentric loading are briefly introduced here for completeness of the proposed curvature method. Consider a rectangular cross-section for simplicity. The moment curvature relationship can be constructed based on the force equilibrium. Assuming that the cross-section is deformed by curvatures ϕ_x, ϕ_y around principle axes x and y and a uniform strain ε_0 due to the axial displacement and lateral deflection, strain ε_i at an arbitrary point can be uniquely determined by its coordinates x, y as

$$\varepsilon = y\phi_x - x\phi_y + \varepsilon_0 \quad (1)$$

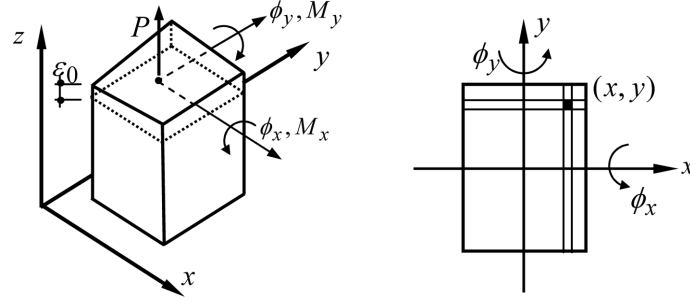


Fig. 1 Biaxial cross-section deformation under external load effects

The sign convention is shown in Fig. 1. Knowing the stress-strain relationship, the normal stress σ can be calculated from ε as

$$\sigma = E\varepsilon \quad (2)$$

where E is the secant modulus evaluated based on the strain ε . The resultant forces M_x , M_y and P can then be calculated via integration of the stresses over the entire cross-section area as

$$\begin{Bmatrix} M_x \\ M_y \\ P \end{Bmatrix} = \mathbf{k} \begin{Bmatrix} \phi_x \\ \phi_y \\ \varepsilon_0 \end{Bmatrix} = \begin{bmatrix} \int_A E y^2 dx dy & -\int_A E x y dx dy & \int_A E y dx dy \\ -\int_A E x y dx dy & \int_A E x^2 dx dy & -\int_A E x dx dy \\ \int_A E y dx dy & -\int_A E x dx dy & \int_A E dx dy \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \\ \varepsilon_0 \end{Bmatrix} \quad (3)$$

where \mathbf{k} denotes the stiffness matrix. The tangent stiffness matrix \mathbf{k}_{tan} can be obtained by differentiating the resultant forces with respect to the curvatures and uniform strain as

$$\mathbf{k}_{\text{tan}} = \begin{bmatrix} \int_A E_{\text{tan}} y^2 dx dy & -\int_A E_{\text{tan}} x y dx dy & \int_A E_{\text{tan}} y dx dy \\ -\int_A E_{\text{tan}} x y dx dy & \int_A E_{\text{tan}} x^2 dx dy & -\int_A E_{\text{tan}} x dx dy \\ \int_A E_{\text{tan}} y dx dy & -\int_A E_{\text{tan}} x dx dy & \int_A E_{\text{tan}} dx dy \end{bmatrix} \quad (4)$$

where E_{tan} is the tangent modulus evaluated based on the strain value ε . Considering a cross-section consisting of n materials, the stiffness matrix \mathbf{k} corresponding to the area of each type of material can be evaluated following Eqs. (3) and (4), and the total stiffness matrix \mathbf{K} can be assembled based on the principle of superposition as

$$\mathbf{K} = \sum_{j=1}^n \mathbf{k}_j \quad (5)$$

The total tangent stiffness matrix \mathbf{K}_{tan} can be assembled similarly. These matrices can be used to calculate the curvatures and uniform strains caused by certain external bending moments and axial loads, and vice versa. The solution can be facilitated by using Newton-Raphson method. For

example, the cross-sectional deformation obtained at m th iteration can be updated as

$$\begin{Bmatrix} \phi_x \\ \phi_y \\ \varepsilon_0 \end{Bmatrix}^{m+1} = \begin{Bmatrix} \phi_x \\ \phi_y \\ \varepsilon_0 \end{Bmatrix}^m + \mathbf{K}_{\text{tan},m}^{-1} \begin{Bmatrix} \Delta M_x \\ \Delta M_y \\ \Delta P \end{Bmatrix}^m \quad (6)$$

where $\{\phi_x \ \phi_y \ \varepsilon_0\}^m$ is the deformation of m th iteration; $\{\Delta M_x \ \Delta M_y \ \Delta P\}^m$ is the out-of-balance forces evaluated at m th iteration; and $\mathbf{K}_{\text{tan},m}^{-1}$ is the inverse matrix of the total tangent stiffness matrix evaluated at m th iteration, and so on.

2.2 Formulation of beam-column deflection and bending moments

The curvature method calculates the beam-column lateral deflection exclusively by the curvatures of discrete cross-sections with consideration of boundary conditions. The process is explained with the aid of a discretized beam-column member as shown in Fig. 2.

Assuming that the curvature is linear within the segment length, the relative deflection of node i with respect to node $i-1$ can be calculated from the angles of rotation at node $i-1$ and the average curvature as

$$\delta_{i,x} = \theta_{i-1,y}\Delta + \frac{1}{4}(\phi_{i-1,y} + \phi_{i,y})\Delta^2, \quad \delta_{i,y} = -\theta_{i-1,x}\Delta - \frac{1}{4}(\phi_{i-1,x} + \phi_{i,x})\Delta^2 \quad (7)$$

where Δ is the segment length, assumed to be constant throughout the column length for simplicity.

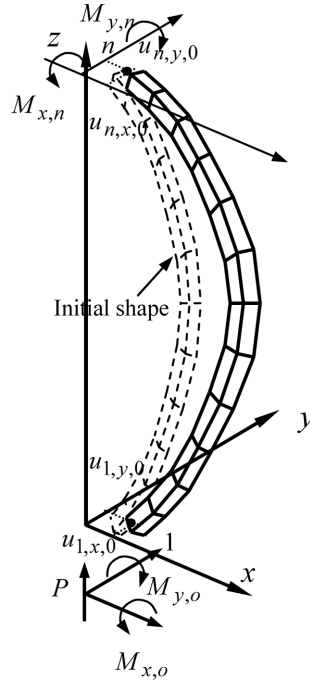


Fig. 2 Discretization of a beam-column with initial deflection

The sign convention is shown in Figs. 1 and 2. The angles of rotation at node i can be calculated from the curvatures up to node i as

$$\theta_{i,x} = \theta_{0,x} + \frac{1}{2} \sum_{j=0}^{i-1} (\phi_{j,x} + \phi_{j+1,x}) \Delta, \quad \theta_{i,y} = \theta_{0,y} + \frac{1}{2} \sum_{j=0}^{i-1} (\phi_{j,y} + \phi_{j+1,y}) \Delta \quad (8)$$

where $\theta_{0,x}$ and $\theta_{0,y}$ are the angles of rotation at node 0. The total deflection at node i can then be calculated from the relative deflections up to node i with respect to the deflection of node 0 as

$$\begin{aligned} u_{i,x} &= u_{0,x} + i\Delta\theta_{0,y} + \frac{\Delta^2}{2} \sum_{j=0}^{i-1} (\phi_{j,y} + \phi_{j+1,y})(i-j-0.5) \\ u_{i,y} &= u_{0,y} - i\Delta\theta_{0,x} - \frac{\Delta^2}{2} \sum_{j=0}^{i-1} (\phi_{j,x} + \phi_{j+1,x})(i-j-0.5) \end{aligned} \quad (9)$$

where $u_{0,x}$ and $u_{0,y}$ are the deflections of node 0. Eq. (9) can be used to calculate the deflection $u_{n,x}$ and $u_{n,y}$ of node n with i substituted by n . Note that the deflections at the boundary nodes 0 and n equal zero, $\theta_{0,x}$ and $\theta_{0,y}$ can then be solved from Eq. (9) as

$$\begin{aligned} \theta_{0,x} &= -\frac{\Delta}{2n} \sum_{j=0}^{n-1} (\phi_{j,x} + \phi_{j+1,x})(n-j-0.5) \\ \theta_{0,y} &= -\frac{\Delta}{2n} \sum_{j=0}^{n-1} (\phi_{j,y} + \phi_{j+1,y})(n-j-0.5) \end{aligned} \quad (10)$$

Therefore, the total deflection of node i can be expressed exclusively by the cross-sectional curvatures as

$$\begin{aligned} u_{i,x} &= \frac{\Delta^2}{2} \left[\sum_{j=0}^{i-1} (\phi_{j,y} + \phi_{j+1,y})(j+0.5) \left(\frac{i}{n} - 1 \right) + \sum_{j=i}^{n-1} (\phi_{j,y} + \phi_{j+1,y}) \left(\frac{i}{n} (j+0.5) - i \right) \right] \\ u_{i,y} &= -\frac{\Delta^2}{2} \left[\sum_{j=0}^{i-1} (\phi_{j,x} + \phi_{j+1,x})(j+0.5) \left(\frac{i}{n} - 1 \right) + \sum_{j=i}^{n-1} (\phi_{j,x} + \phi_{j+1,x}) \left(\frac{i}{n} (j+0.5) - i \right) \right] \end{aligned} \quad (11)$$

Eq. (11) can be rearranged as $\mathbf{u} = \mathbf{K}_u \boldsymbol{\phi}$, where $\mathbf{u} = \{u_{i,x} \ u_{i,y}\}^T$, $\boldsymbol{\phi} = \{\phi_{i,x} \ \phi_{i,y}\}^T$, \mathbf{K}_u is the stiffness matrix and can be calculated by

$$\mathbf{K}_u = \begin{bmatrix} \mathbf{k}_{u,0,0} & \cdot & \cdot & \cdot & \mathbf{k}_{u,0,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{k}_{u,i,j} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{k}_{u,n,0} & \cdot & \cdot & \cdot & \mathbf{k}_{u,n,n} \end{bmatrix} \quad (12)$$

where $\mathbf{k}_{u,i,j}$ represents the biaxial deflection at node i due to the curvatures at node j and can be calculated according to Eq. (11) as

$$\mathbf{k}_{u,i,j} = \frac{\Delta^2}{2} \begin{bmatrix} 0 & a_{i,j} + b_{i,j} \\ -a_{i,j} - b_{i,j} & 0 \end{bmatrix}, \quad i, j = 0, 1, \dots, n \quad (13)$$

where $a_{i,j}$ and $b_{i,j}$ are the coefficients and determined by

$$a_{i,j} = \begin{cases} (j+0.5)(i/n-1) & 0 \leq j \leq i-1 \\ 0.5i/n + ji/n - i & i < j \leq n-1 \\ 0 & n \leq j \end{cases}, \quad b_{i,j} = \begin{cases} 0 & j = 0 \\ (j-0.5)(i/n-1) & 1 \leq j \leq i \\ ji/n - 0.5i/n - i & i < j \leq n \end{cases}$$

Assuming that the beam-column is subjected to a axial load P , the resulting bending moments at node i with consideration of the p-delta effect can be calculated by

$$M_{i,x} = -Pu_{i,y}, \quad M_{i,y} = Pu_{i,x} \quad (14)$$

Similarly, the moments can be expressed in matrix form as $\mathbf{M} = \mathbf{K}_G \boldsymbol{\phi}$, where \mathbf{K}_G takes similar form as \mathbf{K}_u except substituting $\mathbf{k}_{u,i,j}$ by $\mathbf{k}_{G,i,j}$, which can be expressed as

$$\mathbf{k}_{G,i,j} = \frac{P\Delta^2}{2} \begin{bmatrix} a_{i,j} + b_{i,j} & 0 \\ 0 & a_{i,j} + b_{i,j} \end{bmatrix} \quad (15)$$

where $a_{i,j}$ and $b_{i,j}$ are defined in Eq. (13). As \mathbf{K}_G is independent of $\boldsymbol{\phi}$, the tangent stiffness $\mathbf{K}_{G,\tan}$, defined by the first order derivative of \mathbf{M} with respect to $\boldsymbol{\phi}$, takes the same form as \mathbf{K}_G .

At any combination of cross-section curvatures and uniform strains, the out-of-balance load of the beam-column can be evaluated as the difference between the external loads and the internal stress resultants as

$$\boldsymbol{\psi} = \mathbf{K} \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix} - \mathbf{K}_G \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix} - P \begin{Bmatrix} -u_{i,y,0} \\ u_{i,x,0} \\ 1 \end{Bmatrix} = (\mathbf{K} - \mathbf{K}_G) \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix} - \boldsymbol{\psi}_0, \quad i = 0, 1, \dots, n \quad (16)$$

where $\boldsymbol{\psi}$ is the out-of-balance load vector including the moments and the axial load, $\boldsymbol{\psi}_0$ is the axial load and the moments due to initial deflection. \mathbf{K}_G is the stiffness matrix calculated by Eq. (15) and needs to be expanded to consider the effect of uniform strain ε_0 . Note that ε_0 does not affect the deflection and the moments under constant axial load; therefore, $\mathbf{k}_{G,i,j}$ and $\mathbf{k}_{G,\tan,i,j}$ can be modified as

$$\mathbf{k}_{G,i,j} = \mathbf{k}_{G,\tan,i,j} = \frac{P\Delta^2}{2} \begin{bmatrix} a_{i,j} + b_{i,j} & 0 & 0 \\ 0 & a_{i,j} + b_{i,j} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (17)$$

Employing Newton-Raphson method, $\boldsymbol{\psi}$ can be reduced stepwisely by updating the deformation as

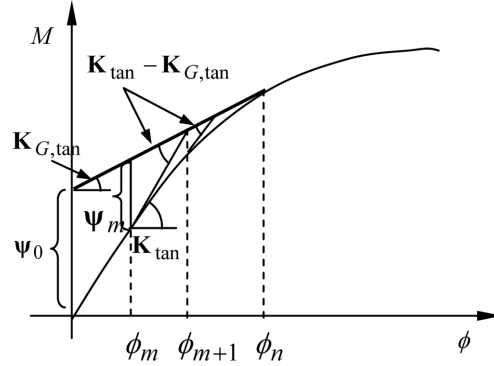


Fig. 3 Nonlinear solution process using Newton-Raphson method

$$\begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix}^{m+1} = \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix}^m + (\mathbf{K}_{\tan} - \mathbf{K}_{G,\tan})_m^{-1} \psi_m, \quad i = 0, 1, \dots, n \quad (18)$$

where $(\mathbf{K}_{\tan} - \mathbf{K}_{G,\tan})_m$ is the effective tangent stiffness matrix evaluated at m th iteration, ψ_m is the out-of-balance load vector evaluated at m th iteration by Eq. (16). The nonlinear solution process for a beam-column under compression load is shown schematically in Fig. 3.

2.3 Incorporation of lateral bracing forces

Provision of lateral bracings can induce lateral bracing forces and additional bending moments to the beam-columns. The lateral bracing forces can be determined from the bracing stiffness and column deflection at the bracing nodes. Consider a beam-column braced at node i_0 as in Fig. 4, the lateral deflection at node i_0 and the resulting bracing force F_{br} can be calculated by

$$u_{i_0} = \frac{\Delta^2}{2} \left[\sum_{j=0}^{i_0-1} (\phi_j + \phi_{j+1})(j+0.5) \left(\frac{i_0}{n} - 1 \right) + \sum_{j=i_0}^{n-1} (\phi_j + \phi_{j+1}) \left(\frac{i_0}{n} (j+0.5) - i_0 \right) \right] \quad (19)$$

$$F_{br} = k_{br} u_{i_0}$$

where k_{br} is the bracing stiffness. The bending moment at node i due to the bracing force can be determined from static analysis as

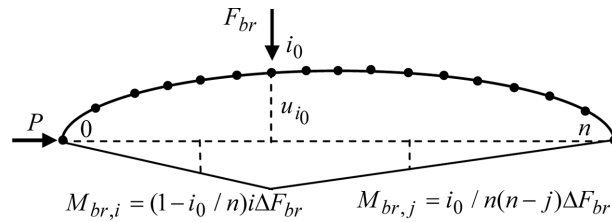


Fig. 4 Bending moments due to lateral bracing force

$$M_{br,i} = \begin{cases} (i - i_0/n)k_{br}u_{i_0}i\Delta & 0 \leq i < i_0 \\ i_0/nk_{br}u_{i_0}(n-i)\Delta & i_0 \leq i \leq n \end{cases} \quad (20)$$

Plug in Eq. (19), the biaxial bending moments can be expressed by the curvatures and uniform strains as

$$\{M_{br,i,x} \ M_{br,i,y} \ 0\}^T = \mathbf{K}_{G,br} \{\phi_{i,x} \ \phi_{i,y} \ \varepsilon_{i,0}\}^T, \quad i = 0, 1, \dots, n \quad (21)$$

where $\mathbf{K}_{G,br}$ is the bracing stiffness matrix and can be assembled similarly as in Eq. (12) with $\mathbf{k}_{G,br,i,j}$ calculated by

$$\mathbf{k}_{G,br,i,j} = w_i \begin{bmatrix} k_{br,y}(a_{i_0,j} + b_{i_0,j}) & 0 & 0 \\ 0 & k_{br,x}(a_{i_0,j} + b_{i_0,j}) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (22)$$

$$w_i = \frac{\Delta^3}{2} \begin{cases} (i - i_0/n)i & 0 \leq i < i_0 \\ i_0/n(n-i) & i_0 \leq i \leq n \end{cases}$$

where $k_{br,x}, k_{br,y}$ are the bracing stiffness in x and y axes, respectively; and $a_{i_0,j}$ and $b_{i_0,j}$ are defined in Eq. (13) with $i = i_0$. As can be seen from Eq. (22), $\mathbf{K}_{G,br}$ is also independent of the deformation and therefore the tangent stiffness matrix $\mathbf{K}_{G,br,tan}$ equals $\mathbf{K}_{G,br}$. For a beam-column with multiple lateral bracings, the bending moments and the stiffness and tangent stiffness matrices can be assembled similarly for each individual lateral bracing and summed up by superposition as the beam-column is statically determinate.

2.4 Incorporation of boundary conditions

Beam-columns can have various boundary conditions, such as pinned, fixed and pinned with rotational and/or translational springs. Note that a rotational spring with pin end can simulate the pinned or fix end support by setting the spring stiffness k_{rot} to zero or infinite large. Consider a beam-column with rotational springs at both ends. The bending moments at the intermediate nodes induced by the springs are shown in Fig. 5. The angles of rotation at node 0 and n are calculated by Eqs. (8) and (10). The resulting moment distribution is determined by static analysis. The bending moment at intermediate nodes can also be expressed in matrix form by

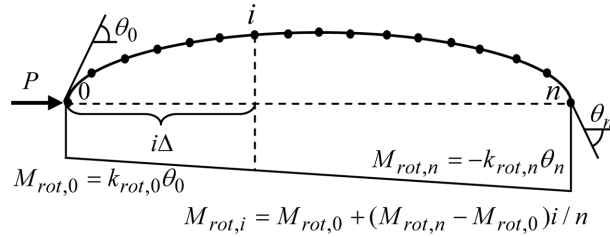


Fig. 5 Bending moments due to rotational springs at beam-column ends

$$\begin{Bmatrix} M_{rot,i,x} \\ M_{rot,i,y} \\ 0 \end{Bmatrix} = \mathbf{K}_{G,rot} \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix} = -\frac{\Delta}{2n} (\mathbf{K}_{G,rot,n} - \mathbf{K}_{G,rot,0}) \begin{Bmatrix} \phi_{i,x} \\ \phi_{i,y} \\ \varepsilon_{i,0} \end{Bmatrix}, \quad i = 0, \dots, n \quad (23)$$

where $\mathbf{K}_{G,rot,0}$ and $\mathbf{K}_{G,rot,n}$ can be assembled by Eq. (12) with $\mathbf{k}_{G,rot,0,i,j}$ and $\mathbf{k}_{G,rot,n,i,j}$ calculated by

$$\mathbf{k}_{G,rot,i_0,i,j} = w_i \begin{bmatrix} k_{r,x}^{i_0}(a_{i_0,j} + b_{i_0,j}) & 0 & 0 \\ 0 & k_{r,y}^{i_0}(a_{i_0,j} + b_{i_0,j}) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad j = 0, \dots, n \quad (24)$$

where $i_0 = 0$ or n , $a_{i_0,j}$ and $b_{i_0,j}$ can be calculated by

$$a_{i_0,j} = n - i_0 - j - 0.5, \quad 0 \leq j \leq n-1, \quad \text{otherwise } 0$$

$$b_{i_0,j} = n - i_0 - j - 0.5, \quad 1 \leq j \leq n, \quad \text{otherwise } 0$$

$$w_i = \begin{cases} (1 - i/n) & i_0 = 0 \\ i/n & i_0 = n \end{cases}$$

where $k_{r,x}^{i_0}, k_{r,y}^{i_0}$, $i_0 = 0$ or n are the stiffness values of the rotational springs at nodes 0 and n , respectively. Stiffness matrix $\mathbf{K}_{G,rot}$ is independent of the deformation if constant spring stiffness is assumed, therefore, the tangent stiffness matrix $\mathbf{K}_{G,rot,tan}$ equals $\mathbf{K}_{G,rot}$.

2.5 Verification with analytical solutions

The curvature method was first verified by Euler's critical load for a slender beam-column under compression load. The beam-column was assumed to have an effective length of 5000 mm and $100 \times 100 \text{ mm}^2$ in cross-sectional dimension. The initial modulus of elasticity was assumed to be $1.0 \times 10^4 \text{ N/mm}^2$. The beam-column was assumed to be either simply supported or pin-and-fix-end

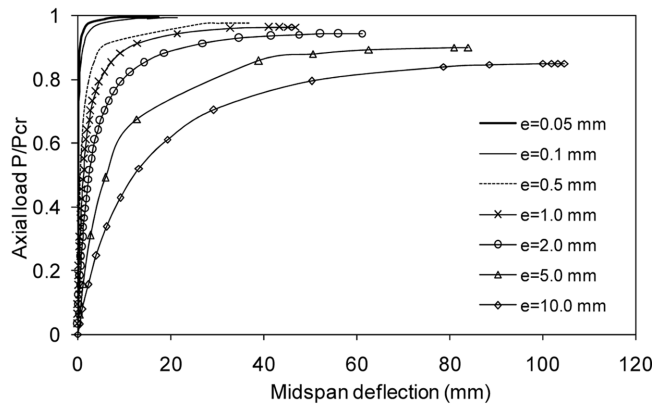


Fig. 6 Simply-supported slender beam-column under compression load

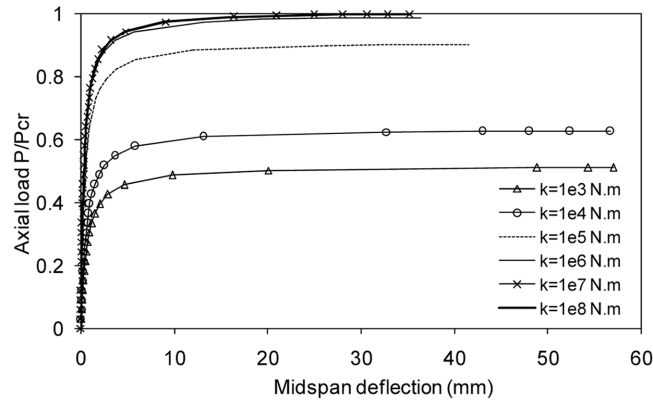


Fig. 7 Simply-supported slender beam-column with rotational spring at one end

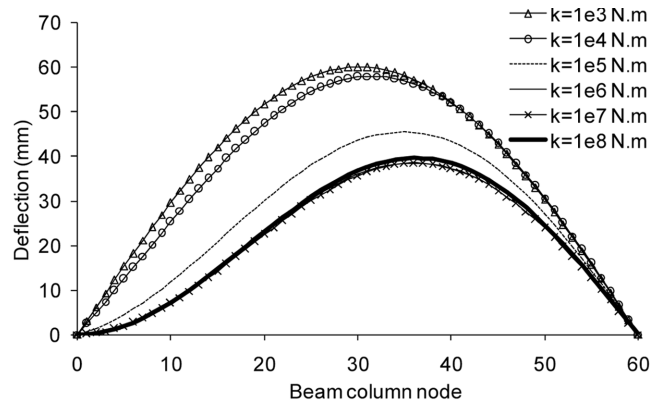


Fig. 8 Lateral deflection of simply supported slender beam-column with rotational spring at left end

supported. Various load eccentricities were considered to initiate the lateral deflection and buckling failure. A rotational spring with large stiffness was used in the calculation to simulate the fix-end support. The results are shown in Figs. 6 and 7, where good agreement can be found.

Fig. 8 shows the column deflection curves obtained using the curvature method with various rotational spring stiffness values. It can be seen that the column deflection curve were simulated reasonably, including the transition from pin support to fix end support.

2.6 Verification with experimental results

Biaxial eccentric compression tests of wood beam-columns were conducted to provide input parameters and verification for the proposed method. The raw materials included Spruce Pine Fir (SPF) dimension lumber and 10d common nails (76.2 mm in length and 3.76 mm in diameter). The SPF lumber was of two machine stress rated (MSR) grades (graded in conformance with the National Lumber Grades Authority Special Product Standard 2 of Canada, SPS 2) and two cross-sectional sizes. All specimens made with dimension lumber were kept in a conditioning room at 65% relative humidity and 20°C before testing, until the equilibrium moisture content was reached.

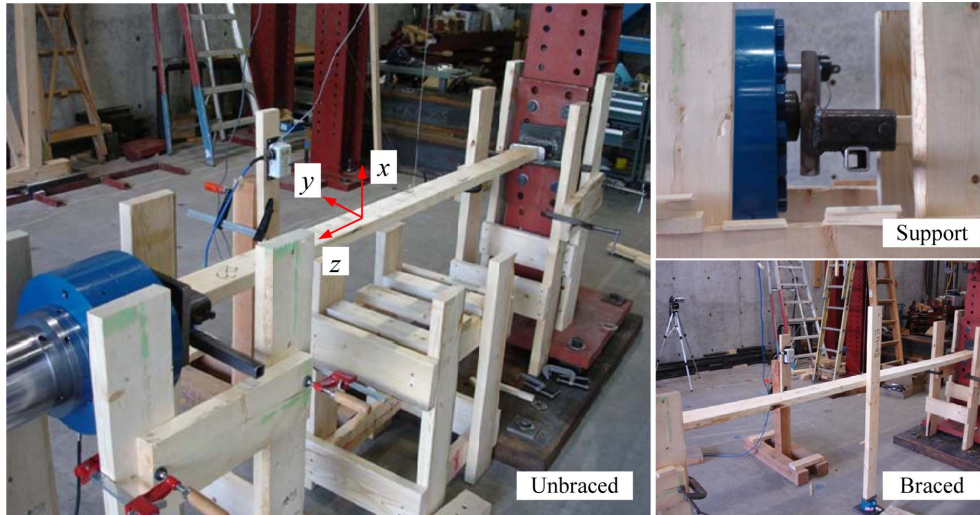


Fig. 9 Eccentric compression tests of wood beam-columns with and without lateral bracing

Table 1 Beam-column specimens of the biaxial eccentric compression tests (unit: mm)

Group	Lumber grade*	Cross section	Length	Eccentricities		Braced	Replication
				x axis	y axis		
1	MSR 1650f-1.5E	38 × 89	2134	0	20	NO	30
2	MSR 1650f-1.5E	38 × 89	2134	20	20	NO	30
3	MSR 1650f-1.5E	38 × 89	2134	20	50	NO	30
4	MSR 1650f-1.5E	38 × 89	3048	20	20	NO	30
5	MSR 1650f-1.5E	38 × 139	2134	20	20	NO	30
6	MSR 2400f-2.0E	38 × 89	3048	20	20	NO	30
7	MSR 2400f-2.0E	38 × 139	3048	20	20	NO	30
8	MSR 1650f-1.5E	38 × 89	3048	0	20	YES	15
9	MSR 1650f-1.5E	38 × 89	3048	20	20	YES	15
10	MSR 2400f-2.0E	38 × 89	3048	20	20	YES	15
11	MSR 2400f-2.0E	38 × 139	3048	20	50	YES	15

Note: MSR1650f-1.5E refers to design values of bending strength 1650 psi (11.38 MPa) and MOE 1.5×10^6 psi (10342 MPa), MSR2400f-2.0E refers to design values of bending strength 2400 psi (16.55 MPa) and MOE of 2.0×10^6 psi (13789 MPa).

The wood beam-columns were tested by a compression load with a biaxial load eccentricity. Seven groups of beam-columns were tested without bracing, and four groups were braced at midspan and in the weak axis (x axis, defined in Fig. 9). The lateral bracing members were made of SPF dimension lumber ($39 \times 89 \text{ mm}^2$ in cross section size and 610 mm in length) which was graded as MSR1650f-1.5E. Details of the experimental design are listed in Table 1. The test layout is shown in Fig. 9.

The axial compression load, midspan lateral deflections and lateral bracing force were the main

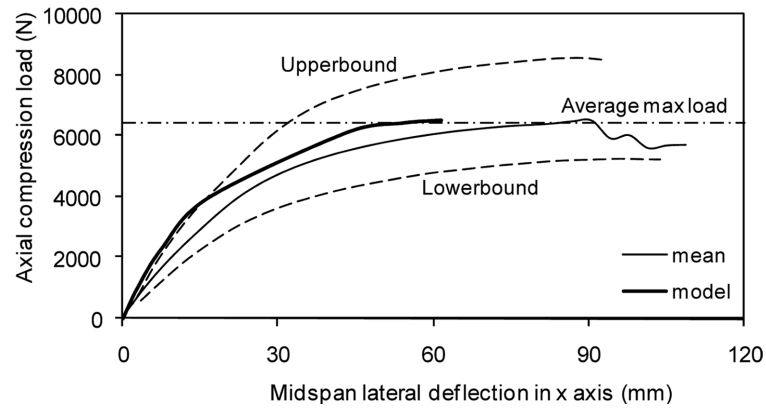


Fig. 10 Axial compression load and midspan deflection in x axis of specimen group 2

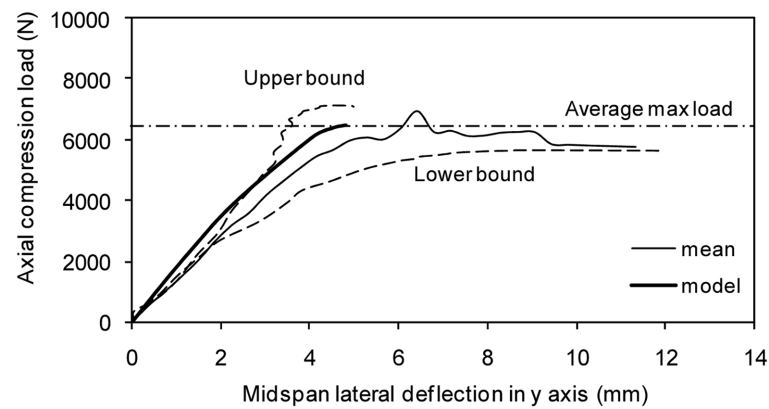


Fig. 11 Axial compression load and midspan deflection in y axis of specimen group 2

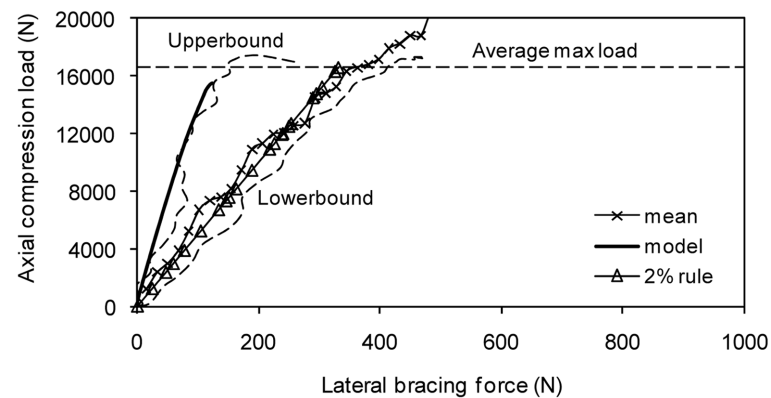


Fig. 12 Axial compression load and lateral bracing force of group 8

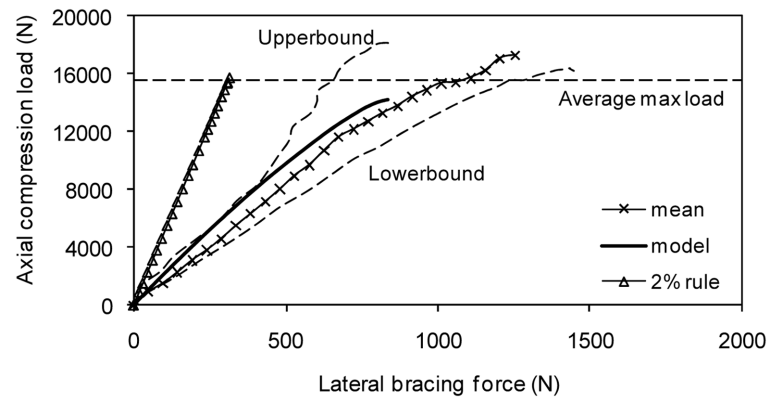


Fig. 13 Axial compression load and lateral bracing force of group 9

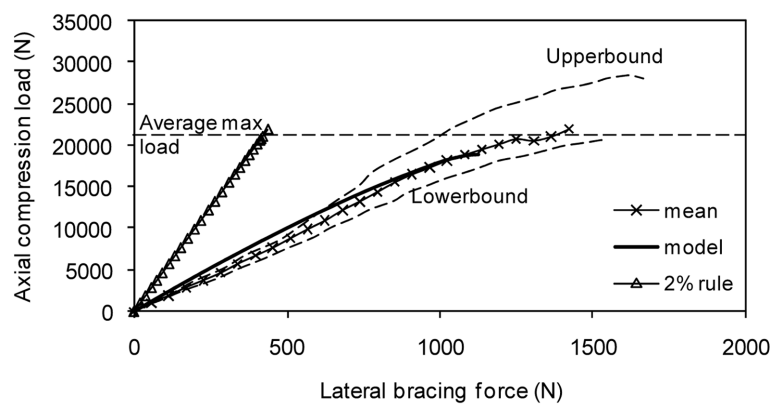


Fig. 14 Axial compression load and lateral bracing force of group 10

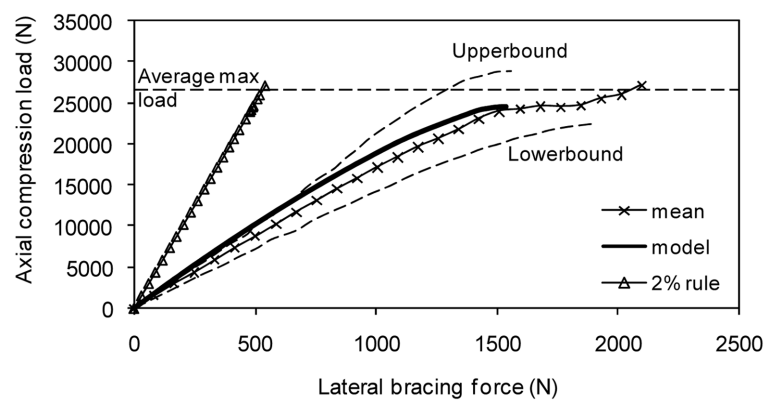


Fig. 15 Axial compression load and lateral bracing force of group 11

Table 2 Maximum load carrying capacity of unbraced wood beam-columns

Group	Maximum load (kN)			Midspan bending moments ($\times 10^6$ N.mm)					
				M_x			M_y		
	Test	Model	Error (%)	Test	Model	Error (%)	Test	Model	Error (%)
1	9	9.01	0.11	0.24	0.238	-0.83	0.354	0.338	-4.52
2	6.4	6.504	1.62	0.163	0.161	-1.23	0.687	0.530	-22.85
3	6.2	6.259	0.95	0.39	0.374	-4.10	0.532	0.446	-16.17
4	3.2	3.477	8.66	-*	0.088	-	-	0.487	-
5	9.5	9.713	2.24	0.203	0.211	3.94	1.01	0.82	-18.81
6	4.8	4.592	-4.33	0.122	0.117	-4.10	0.793	0.621	-21.69
7	6.3	6.014	-4.54	0.133	0.132	-0.75	1.081	0.876	-18.96

Note: * test results of the bending moments of specimen group 9 are not available as the lateral deflection exceeded the effective range of the string pots.

concerns for the verification study. The proposed method was implemented based on the beam-column dimensions and load eccentricities. The input parameters of the model were based on the mean values of the material property test results. The test results and model predictions are shown in Figs. 10 to 15 and Table 2.

It can be seen that the model agreed well with the test results in terms of the stiffness of the load-and-deflection curves, lateral bracing force and the maximum compression load. The model predictions of the midspan deflection in the x axis turned out to be stiffer than the test results, due to the assumption of a constant (linear) lateral bracing stiffness. On the other hand, the model predictions agreed very well with the test results on the ratio between the lateral bracing forces and the axial compression load.

3. Conclusions

This paper presents a curvature method for analysis of beam-columns of different materials with arbitrary cross-section shapes. The methodology focuses on the development of column deflection curve based on the curvatures at discrete cross-sections, whereas the moment-curvature relationship of complex cross-sections can be constructed following methods in literature. Verification with analytical solution and experimental results implies that the proposed method has satisfactory accuracy, high computation efficiency and is reliability in modeling the structural behaviour of braced wood beam-columns subjected to eccentric compression load. The generated method and knowledge in this study can be used as an alternative to FEM models in analysis of beam-columns of complex configurations and subjected to combined loading.

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