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Effects of dead loads on the static analysis of plates

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Abstract. The collapse of structures due to snow loads on roofs occurs frequently for steel structures and rarely for reinforced concrete structures. Since the most significant difference between these structures is related to their ability to handle dead loads, dead loads are believed to play an important part in the collapse of structures by snow loads. As such, the effect of dead loads on displacements and stress couples produced by live loads is presented for plates with different edge conditions. The governing equation of plates that takes into account the effect of dead loads is formulated by means of Hamilton's principle. The existence and effect of dead loads are proven by numerical calculations based on the Galerkin method. In addition, a closed-form solution for simply supported plates is proposed by solving, in approximate terms, the governing equation that includes the effect of dead loads, and this solution is then examined. The effect of dead loads on static live loads can be explained explicitly by means of this closed-form solution. A method that reflects the effects of dead loads on live loads is presented as an example. The present study investigates an additional factor in lightweight roof structural elements, which should be considered due to their recent development.

Keywords: plates; dead load; initial stress; live load; snow load; Galerkin equation; linear and non linear; safety; static; roof

1. Introduction

Plates are frequently used in building structures as main structural members, which are subjected to large vertical loads. The theory of plates has progressed from the classical theory of the Kirchhoff-Love and Mindlin-Reissner plates, as shown in Timoshenko (1959) and Volterra and Zachmanoglou (1965), to more recent advanced theories. Shimpi *et al.* (2007) proposed two new displacement-based first-order shear deformation plate theories involving only two unknown functions. Wu *et al.* (2007) presented a novel Bessel function by which to obtain the exact solutions for free-vibration analysis of a rectangular plate with typical edge conditions. Boscolo and Banerjee (2011) developed a dynamic stiffness method for the accurate and efficient free vibration analysis of plates. Fang *et al.* (2007), Lee and Chen (2011) investigated problems involving circular cutouts. Tanveer and Singh (2009) investigated the linear and nonlinear forced vibration, including rotary inertia, of laminated composite rectangular and other shaped first-order shear deformable plates. Rao and Saheb (2008) developed a simple formula by which to

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investigate the large-amplitude free vibration behavior of structural members, such as beams and plates. Takabatake (1991, 1991, 1996, 1996, 1996, 1996, 1998, 1999) proposed a simplified analytical method for various plates.

However, the above-mentioned plate theories do not reflect an important characteristic of plates frequently used in building structures such as roof slabs. In the initial state, such plates are subjected to only dead loads, which are the empty weight of the actual structures and are always invariant. External loads, such as live loads, always act on the deformed state caused by the dead loads. If the dead load is extremely heavy, then the initial stress is very large in the initial dead-load-only state. The difference between a heavy concrete slab and a steel lightweight slab is obvious in building structures. Thus, the behavior of plates used in building structures should be different depending on the invariant dead load present. Compared to lightweight steel structures, heavy reinforced concrete structures collapse far less frequently due to snow loads on roofs even though, theoretically, the reinforced concrete structures and steel structures should have the same degree of safety. Why, then, do lightweight steel structures collapse more frequently? The present author believes that the dead loads of structures must play an important part in this phenomenon.

In contrast to live loads, dead loads are characterized by being stationary. Structures are always subject to dead loads and have conservative initial stresses produced by these dead loads. When structures are subjected to live loads, the strain energy produced by the initial stresses is considered, and this strain energy has the effect of decreasing external disturbances such as displacements and stress couples produced by live loads. However, due to the unknown nature of this phenomenon, current trends in structural design do not consider the effect of dead loads. If the effect of dead loads is clarified, it will be possible to estimate the effective value of live loads. As a result, it will be possible to have the same safety factor for both heavy structures and lightweight structures and to design truly safe structures. Previous studies for linear and nonlinear plate problems do not take into consideration the effect of dead loads, which is an essential problem for plates used in structures.

In a previous study (Takabatake 1990, 1991, 2010), the present author demonstrated the effect of dead loads on the static and dynamic problems of elastic beams and proposed a closed-form approximate solution of simply supported beams. This new attention became an important jumping-off point for extensions of elementary beam theory and was extended to the finite-element method using a beam element that takes the effect of dead loads into account (Zhos and Zhu 1996). The existence of an initial bend in a beam due to a dead load has been suggested to increase the natural frequencies of lateral vibrations (Kelly *et al.* 1991). The present author (Takabatake 1992) reported the effect of dead loads on the dynamic response of a uniform elastic rectangular plate and clarified the physical factors governing this effect. Mostaghel and Yu (1995) revealed that, based on the large deflection theory for thin plates and the principle of conservation of energy, preforming a thin plate into any shape has the effect of increasing its natural frequencies. Yu *et al.* (1994) reported a phenomenon whereby the natural frequencies increase when the plate is preformed into a shape with a specific mode of vibration. Zhou (2002) proposed a finite-element formulation for plates that takes into consideration the stiffening effect of dead loads. Recently, Durmaz and Dalaglon (2006) created a map to obtain the precise variation of ground snow loads in the Eastern Black Sea region, where dead loads were recognized to have a great influence on the structural behavior produced by live loads, such as snow loads.

The goal of the present paper is to clarify the effect of dead loads in static elastic plates. First, the governing equation of plates, which takes into account the effect of dead loads, from a previous study (Takabatake 1992) is presented, where the effect of dead loads is based on strain energy resulting from conservative initial stresses produced by dead loads. This strain energy supports part of the potential energy of the live loads. If the dead loads are large, then the initial stresses are large, and the energy supported by the initial stresses is large. Consequently, the displacements and stress couples produced by live loads are smaller than for those excluding the effect of dead loads. Second, from the results of numerical calculations using the Galerkin method, the effect of dead loads is shown for the case of simply supported and clamped rectangular plates subjected to static live loads. Third, in order to apply the effect of dead loads to design, the parameters used to express this effect are shown in general physically explicit expressions without numerical calculations by a closed-form solution for simply supported plates, which is proposed based on the governing equation including the effect of dead loads. Finally, a method that reflects the effect of dead loads on live loads is presented through an example.

2. Governing equations including the effect of dead loads for plates

In Fig. 1, a rectangular plate is shown along with a Cartesian coordinate system. The external forces are assumed to be transverse loads only, and axial forces are neglected. Deflections $\tilde{w}$ are produced due to the dead loads $\tilde{p}$ of the plates. This deformed state is considered as the reference state. As live loads $\bar{p}$ per unit area act on this reference state, deflections $\bar{w}$ occur, in which deflections $\bar{w}$ are measured from the reference state. Deflections $\tilde{w}$ and $\bar{w}$ and transverse loads $\tilde{p}$ and $\bar{p}$ are considered to be positive when they act toward the positive direction of the z axis.

Assuming the validity of the Kirchhoff-Love plate theory and neglecting the strains in the middle surface produced by in-plane forces, the nonlinear strain-displacement relations of plates can be obtained from Washizu (1982) as follows

$$\left.\begin{aligned} \bar{\varepsilon}_x &= -z\bar{w}_{,xx} + \underline{\frac{1}{2}(\bar{w}_{,x})^2} \\ \bar{\varepsilon}_y &= -z\bar{w}_{,yy} + \underline{\frac{1}{2}(\bar{w}_{,y})^2} \\ \bar{\gamma}_{xy} &= -2z\bar{w}_{,xy} + \underline{\bar{w}_{,x}\bar{w}_{,y}} \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \tilde{\varepsilon}_x &= -z\tilde{w}_{,xx} + \underline{\frac{1}{2}(\tilde{w}_{,x})^2} \\ \tilde{\varepsilon}_y &= -z\tilde{w}_{,yy} + \underline{\frac{1}{2}(\tilde{w}_{,y})^2} \\ \tilde{\gamma}_{xy} &= -2z\tilde{w}_{,xy} + \underline{\tilde{w}_{,x}\tilde{w}_{,y}} \end{aligned}\right\} \tag{2}$$

where $\bar{\varepsilon}_x, \bar{\varepsilon}_y, \bar{\gamma}_{xy}$ and $\tilde{\varepsilon}_x, \tilde{\varepsilon}_y, \tilde{\gamma}_{xy}$ are the strains due to live loads $\bar{p}$ and dead loads $\tilde{p}$, respectively, and z is the distance from the middle surface. The underlined terms in Eqs. (1) and (2) indicate nonlinear terms.

Now, assuming the stress-strain relations to be linear, the equilibrium equation and the boundary conditions of the plates, which include the effect of dead loads, are obtained from a previous study

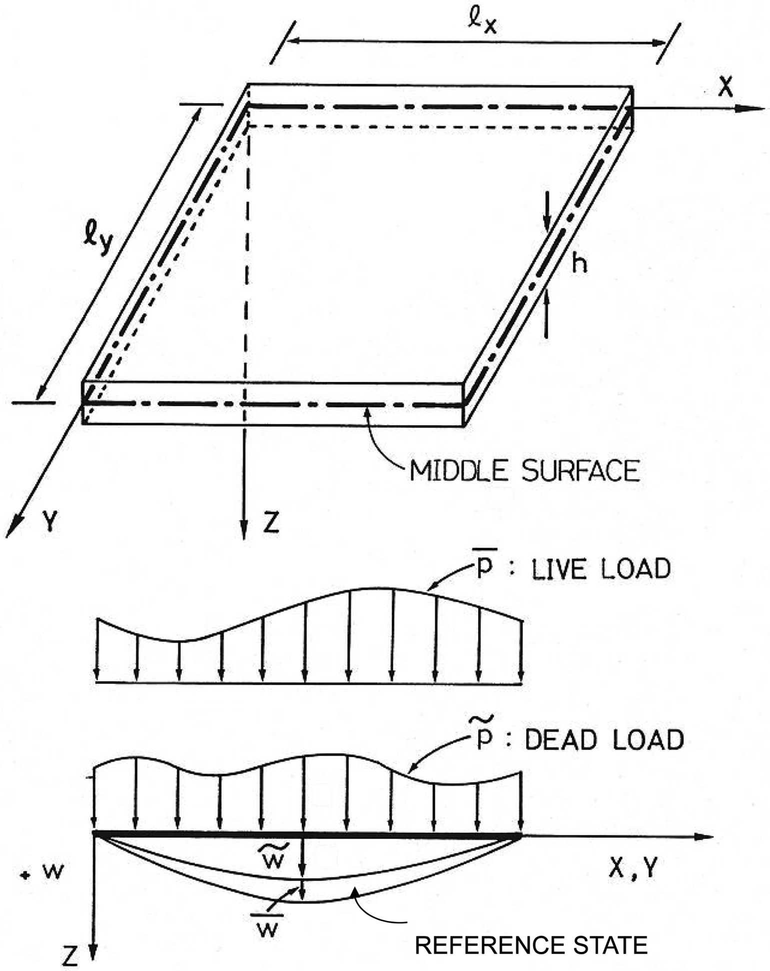

Fig. 1 Coordinates and load distribution of a plate

(Takabatake 1992) using Hamilton's principle, as follows

$$\begin{aligned} & \bar{w}_{,xxxx} + 2\bar{w}_{,xxyy} + \bar{w}_{,yyyy} - \frac{\bar{p}}{D} \\ & -\frac{6}{h^2}[2\tilde{w}_{,x}\tilde{w}_{,xx}\bar{w}_{,x} + (\tilde{w}_{,x})^2\bar{w}_{,xx} + 2\tilde{w}_{,y}\tilde{w}_{,yy}\bar{w}_{,y} + (\tilde{w}_{,y})^2\bar{w}_{,yy}] \\ & -\frac{6\nu}{h^2}[2\tilde{w}_{,y}\tilde{w}_{,xy}\bar{w}_{,x} + (\tilde{w}_{,y})^2\bar{w}_{,xx} + 2\tilde{w}_{,x}\tilde{w}_{,xy}\bar{w}_{,y} + (\tilde{w}_{,x})^2\bar{w}_{,yy}] \\ & -\frac{6(1-\nu)}{h^2}[(\tilde{w}_{,x}\tilde{w}_{,y}\bar{w}_{,y})_{,x} + (\tilde{w}_{,x}\tilde{w}_{,y}\bar{w}_{,x})_{,y}] = 0 \end{aligned} \tag{3}$$

$$
\left.
\begin{array}{l}
\left.
\begin{array}{ll}
\bar{w} = 0 \quad or & \bar{w}_{,xxx} + (2-\nu)\bar{w}_{,xyy} \\
& -\dfrac{6}{h^2}\bar{w}_{,x}[(\tilde{w}_{,x})^2 + \nu(\tilde{w}_{,y})^2] \\
& -\dfrac{6(1-\nu)}{h^2}\bar{w}_{,x}\tilde{w}_{,y}\bar{w}_{,y} = 0 \\
\bar{w}_{,x} = 0 \quad or & \bar{w}_{,xx} + \nu\bar{w}_{,yy} = 0
\end{array}
\right\} \text{at } x = 0 \text{ and } x = \ell_x \\
\left.
\begin{array}{ll}
\bar{w} = 0 \quad or & \bar{w}_{,yyy} + (2-\nu)\bar{w}_{,xxy} \\
& -\dfrac{6}{h^2}\bar{w}_{,y}[(\tilde{w}_{,y})^2 + \nu(\tilde{w}_{,x})^2] \\
& -\dfrac{6(1-\nu)}{h^2}\tilde{w}_{,x}\tilde{w}_{,y}\bar{w}_{,x} = 0 \\
\bar{w}_{,y} = 0 \quad or & \bar{w}_{,yy} + \nu\bar{w}_{,xx} = 0
\end{array}
\right\} \text{at } y = 0 \text{ and } y = \ell_y \\
\bar{w} = 0 \quad or \quad \bar{w}_{,xy} = 0 \qquad \text{at corners}
\end{array}
\right\} \tag{4}
$$

Note that Eqs. (3) and (4) are linear with respect to unknown deflections $\bar{w}$, because the deflections $\tilde{w}$ due to the dead loads are previously known. The nonlinear terms in these equations express the effects of dead loads. Neglecting these terms will provide general equations for plates subjected to only live loads $\bar{p}$.

For the reference state, in which only the dead loads $\tilde{p}$ act, the following familiar governing equation and boundary conditions can be written as

$$\tilde{w}_{,xxxx} + 2\tilde{w}_{,xxyy} + \tilde{w}_{,yyyy} - \frac{\tilde{p}}{D} = 0 \tag{5}$$

$$
\left.
\begin{array}{l}
\left.
\begin{array}{ll}
\tilde{w} = 0 \quad or & \tilde{w}_{,xxxx} + (2-\nu)\tilde{w}_{,xyy} = 0 \\
\tilde{w}_{,x} = 0 \quad or & \tilde{w}_{,xxx} + \nu\tilde{w}_{,yy} = 0
\end{array}
\right\} \text{at } x = 0 \text{ and } x = \ell_x \\
\left.
\begin{array}{ll}
\tilde{w} = 0 \quad or & \tilde{w}_{,yyy} + (2-\nu)\tilde{w}_{,xxy} = 0 \\
\tilde{w}_{,y} = 0 \quad or & \tilde{w}_{,yy} + \nu\tilde{w}_{,xx} = 0
\end{array}
\right\} \text{at } y = 0 \text{ and } y = \ell_y \\
\tilde{w} = 0 \quad or \quad \tilde{w}_{,xy} = 0 \qquad \text{at corners}
\end{array}
\right\} \tag{6}
$$

where D is the bending stiffness of the plates ($D = Eh^3/12(1-\nu^2)$), h is the thickness of the plates, and ν is Poisson's ratio of the plates.

3. Numerical results

Let us examine, by means of the Galerkin method, the effect of dead loads on a uniform rectangular plate subjected to static live loads. The Galerkin equation for the current plates may be obtained as follows

$$\int_0^{\ell_x}\int_0^{\ell_y} Q\,\delta\overline{w}\,dxdy = 0 \tag{7}$$

where the notation Q is the equilibrium equation given in Eq. (3), and ℓ_x and ℓ_y are the dimensions of the plate in the x and y directions, respectively. Now, the deflection $\overline{w}$ due to live loads is expressed by power series expansions as follows

$$\overline{w}(x,y) = \sum_{m=1}\sum_{n=1}\overline{w}_{mn} f_{mn} \quad \begin{pmatrix} m = 1,2,3,\ldots,\infty \\ n = 1,2,3,\ldots,\infty \end{pmatrix} \tag{8}$$

where $\overline{w}_{mn}$ are unknown displacement coefficients and f_{mn} are the shape functions satisfying the boundary conditions of the plates. The following functions represent f_{mn} for simply supported and clamped plates

$$f_{mn} = \sin\frac{m\pi x}{\ell_x}\sin\frac{n\pi y}{\ell_y} \quad \text{(for a simply supported plate)} \tag{9}$$

$$f_{mn} = \sin\frac{\pi x}{\ell_x}\sin\frac{m\pi x}{\ell_x}\sin\frac{\pi y}{\ell_y}\sin\frac{n\pi y}{\ell_y} \quad \text{(for a clamped plate)} \tag{10}$$

Substituting Eq. (8) into Eq. (7), the Galerkin equation can be rewritten as

$$\delta\overline{w}_{mn}\colon \int_0^{l_x}\int_0^{l_y} Q f_{mn}\,dxdy = 0 \quad \begin{pmatrix} m = 1,2,3,\ldots,\infty \\ n = 1,2,3,\ldots,\infty \end{pmatrix} \tag{11}$$

Since the governing equation Q contains deflections $\tilde{w}$ due to dead loads $\tilde{p}$, it is necessary to determine beforehand deflections $\tilde{w}$ from Eqs. (5) and (6), which can be expressed generally as follows

$$\tilde{w}(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{w}_{mn} f_{mn}(x,y) \tag{12}$$

where $\tilde{w}_{mn}$ are unknown displacement coefficients. For simply supported plates, $\tilde{w}_{mn}$ and f_{mn} are easily determined from Navier's double Fourier series. For clamped plates, for simplicity, the shape functions f_{mn} in Eq. (9) are used. Substituting Eqs. (8) and (12) into Eq. (11), the Galerkin equation reduces to linear, nonhomogeneous, algebraic equations with respect to unknown displacement coefficients $\overline{w}_{mn}$, as follows

$$\begin{aligned}
\delta\overline{w}_{mn}:\quad & \sum_{\overline{m}=1}\sum_{\overline{n}=1}\overline{w}_{\overline{m}\overline{n}}\Big[\int_0^{\ell_x}\int_0^{\ell_y}\big[f_{\overline{m}\overline{n},xxxx} + 2f_{\overline{m}\overline{n},xxyy} + f_{\overline{m}\overline{n},yyyy} \\
& - \sum_{\tilde{m}=1}\sum_{\tilde{n}=1}\sum_{\hat{m}=1}\sum_{\hat{n}=1}\tilde{w}_{\tilde{m}\tilde{n}}\,\tilde{w}_{\hat{m}\hat{n}}\,\frac{6}{h^2}\big[2f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},xx}f_{\overline{m}\overline{n},x} \\
& + f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},x}f_{\overline{m}\overline{n},xx} + 2f_{\tilde{m}\tilde{n},y}f_{\hat{m}\hat{n},yy}f_{\overline{m}\overline{n},y} + f_{\tilde{m}\tilde{n},y}f_{\hat{m}\hat{n},y}f_{\overline{m}\overline{n},yy} \\
& + \nu[2f_{\tilde{m}\tilde{n},y}f_{\hat{m}\hat{n},xy}f_{\overline{m}\overline{n},x} + f_{\tilde{m}\tilde{n},y}f_{\hat{m}\hat{n},y}f_{\overline{m}\overline{n},xx} + 2f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},xy}f_{\overline{m}\overline{n},y} + f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},x}f_{\overline{m}\overline{n},yy}] \\
& + (1-\nu)[(f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},y}f_{\overline{m}\overline{n},y})_{,x} + (f_{\tilde{m}\tilde{n},x}f_{\hat{m}\hat{n},y}f_{\overline{m}\overline{n},x})_{,y}]\}]f_{mn}\,dxdy\Big] = \int_0^{\ell_x}\int_0^{\ell_y}\frac{\overline{p}}{D}f_{mn}\,dxdy
\end{aligned} \tag{13}$$

Solving Eq. (13) for $\tilde{w}_{\bar{m}\bar{n}}$ and applying the result to Eq. (8), the displacements $\bar{w}$ due to live loads are obtained. In addition, stress couples $\bar{M}_x, \bar{M}_y$, and $\bar{M}_{xy}$ due to live loads are given by the well-known relations $\bar{M}_x = -D(\bar{w}_{,xx} + \nu\bar{w}_{,yy})$, $\bar{M}_y = -D(\bar{w}_{,yy} + \nu\bar{w}_{,xx})$, and $\bar{M}_{xy} = -(1-\nu)D\bar{w}_{,xy}$.

Next, the effect of dead loads on simply supported and clamped plates is examined. Two types of rectangular plates are considered: a reinforced concrete plate and a steel plate. The steel plate is considered equivalent to a plate obtained from a steel structure. It is assumed that dead loads $\tilde{p}$ and live loads $\bar{p}$ are uniformly distributed loads, with the constant live load of $\bar{p}$ = 5.88 kN/m^2 being a snow depth of 2 m with a snow density of 2.94 kN/m^3. The modulus of elasticity E and Poisson's ratio ν are $E = 21 \times 10^{10}$ N/m^2 and $\nu = 0.3$ for the steel plate and $E = 2.1 \times 10^9$ N/m^2 and $\nu = 0.17$ for the reinforced concrete plate. The standard thicknesses h_0 of the plates are 0.07 m and 0.14 m for simply supported steel and reinforced concrete plates, respectively, and 0.05 m and 0.10 m for the clamped steel and reinforced concrete plates, respectively. In addition, the standard length of the plate, ℓ_x, in the x direction is 5 m. The length of the plate, ℓ_y, in the y direction and the thickness h are given by

$$\left.\begin{aligned} \ell_y &= \alpha_\ell \ell_x \qquad (\alpha_\ell = 1 \text{ and } 2) \\ h &= \alpha_h h_0 \qquad (\alpha_h = 1 \text{ and } 2) \end{aligned}\right\} \tag{14}$$

where the span ratio α_ℓ and the thickness ratio α_h are parameters having values of 1 and 2. Numerical calculations are obtained by varying the dead load $\tilde{p}$ and parameters α_ℓ and α_h under a constant live load $\bar{p}$.

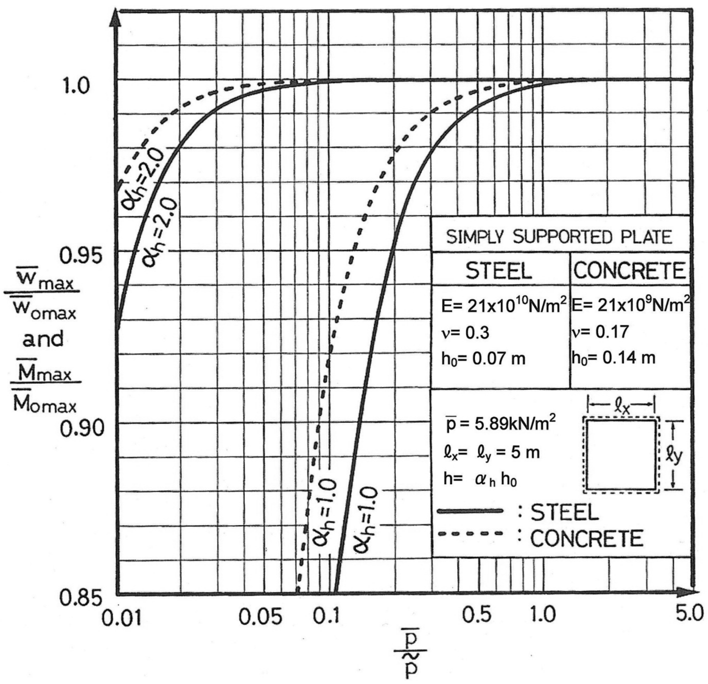

Fig. 2 Relationship among $\bar{w}_{max}/\bar{w}_{0max}$, $\bar{M}_{max}/\bar{M}_{0max}$ and $\bar{p}/\tilde{p}$ for a simply supported plate

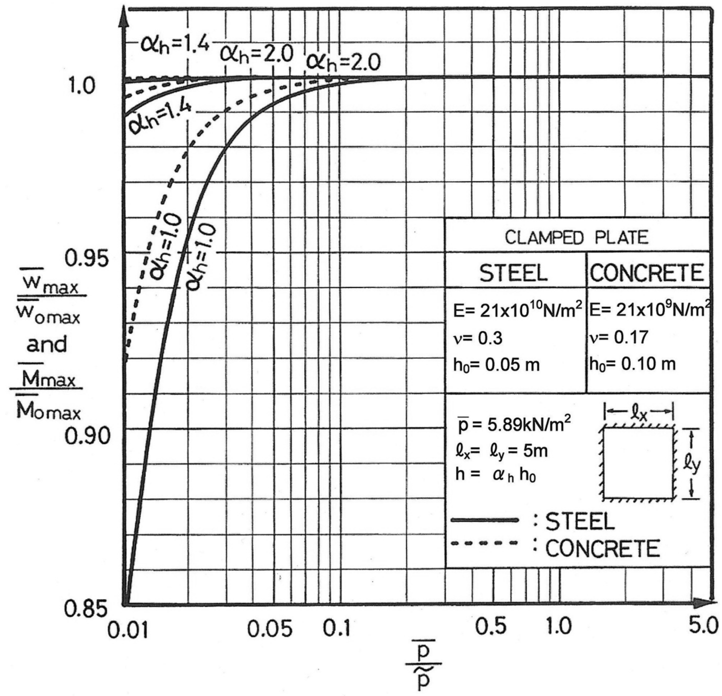

Fig. 3 Relationship among $\overline{w}_{max}/\overline{w}_{0max}$, $\overline{M}_{max}/\overline{M}_{0max}$ and $\overline{p}/\tilde{p}$ for a clamped plate

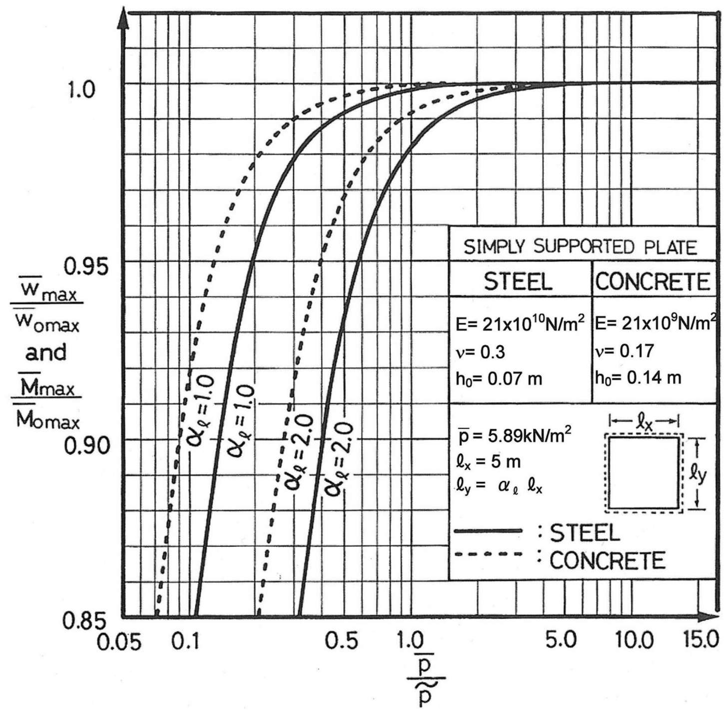

Fig. 4 Relationship among $\overline{w}_{max}/\overline{w}_{0max}$, $\overline{M}_{max}/\overline{M}_{0max}$ and $\overline{p}/\tilde{p}$ for a simply supported plate

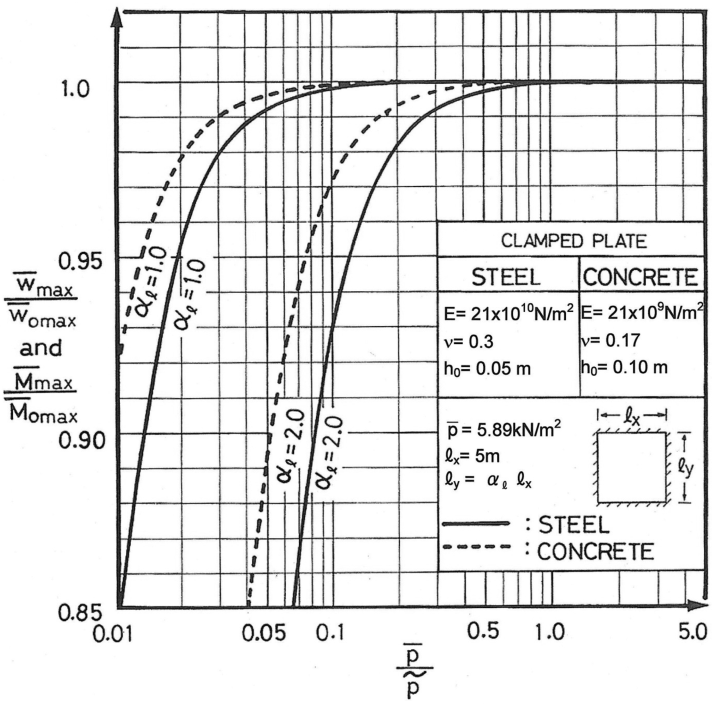

Fig. 5 Relationship among $\overline{w}_{max}/\overline{w}_{0max}$, $\overline{M}_{max}/\overline{M}_{0max}$ and $\overline{p}/\tilde{p}$ for a clamped plate

Figs. 2 and 3 show the results for rectangular plates with the variation of the thickness h and dead loads $\tilde{p}$ under a constant live load $\overline{p}$. In addition, Figs. 4 and 5 show the results for the plates with the variation of the span length ℓ_y and dead loads $\tilde{p}$ under a constant live load $\overline{p}$. These figures reveal that each maximum value of deflection and stress couple for a plate with live-load-to-dead-load ratio, $\overline{p}/\tilde{p}$, is reduced to the value indicated by the ratios $\overline{w}_{max}/\overline{w}_{0max}$ and $\overline{M}_{max}/\overline{M}_{0max}$ by considering the effect of dead loads. If the effect of dead loads does not exist, the values of these ratios are always equal to 1. Since dead loads $\tilde{p}$ are large in comparison with the constant live loads $\overline{p}$, i.e., $\overline{p}/\tilde{p} < 1$, the ratios $\overline{w}_{max}/\overline{w}_{0max}$ and $\overline{M}_{max}/\overline{M}_{0max}$ are smaller than 1. Here, $\overline{w}_{max}$ and $\overline{M}_{max}$ indicate the respective maximum values of deflections $\overline{w}$ and stress couples $\overline{M}$ due to the live load $\overline{p}$ in each plate, respectively, and are obtained from the governing equation that includes the effect of dead loads. On the other hand, $\overline{w}_{0max}$ and $\overline{M}_{0max}$ are the maximum values corresponding to $\overline{w}_{max}$ and $\overline{M}_{max}$, respectively, and are obtained from the governing equation that excludes the effect of dead loads. These values agree with values obtained from the general plate theory. There is little difference between the ratios $\overline{w}_{max}/\overline{w}_{0max}$ and $\overline{M}_{max}/\overline{M}_{0max}$ in the numerical results, but this difference can be neglected in practice. This fact is also explained in the subsequent approximate solution. The scale of transverse axis in Fig. 4 differs from Figs. 2, 3, and 5, because the simply supported plate is greatly affected by the dimensions of the plate. Now, calculating the reductions from the ratios $\overline{w}_{max}/\overline{w}_{0max}$ and $\overline{M}_{max}/\overline{M}_{0max}$ given in Figs. 2 through 4, the effect of dead loads on simply supported plates is obtained as shown in Fig. 6.

These results indicate that an increase in dead load under a constant live load decreases the deflections and stress couples due to the live load and that this effect is large for a thin plate or a

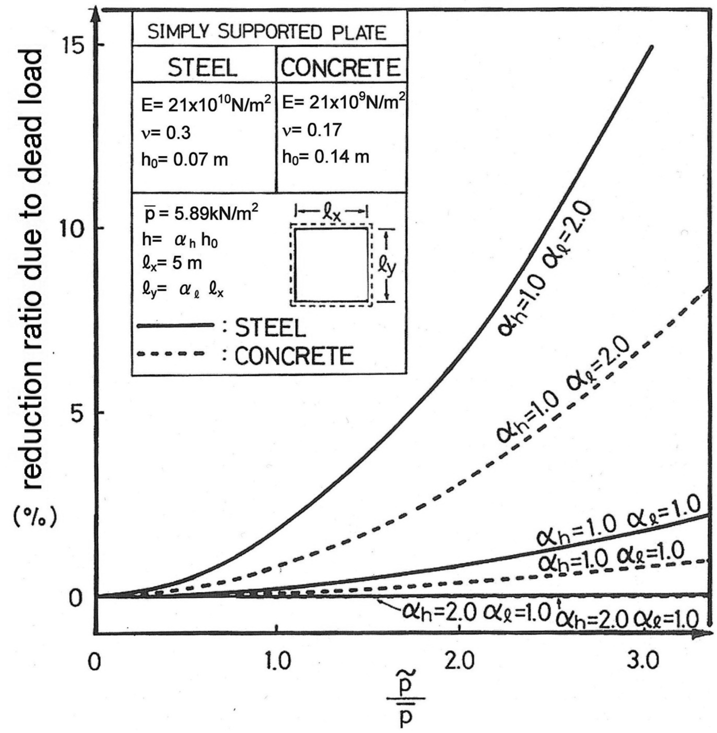

Fig. 6 Relationship between reduction ratio due to a dead load and $\tilde{p}/\bar{p}$ for a simply supported plate

plate with a large span length. Thus, the effect of dead loads is evident in these plates.

The above results indicate that the effect of dead loads is large for a plate designed to have the lowest sectional efficiency. A plate designed to have the lowest thickness essentially has a lower load-carrying capacity than a thick plate, and in this case, the dead load is reduced as much as possible. Since this reduction in dead loads results in a reduction in the effect of dead loads, the plate becomes weaker for live loads. Present trends in structural design are based on a system in which structures are designed in terms of the sum of dead loads and live loads. Since live loads acting on structures are generally stipulated in the structural design code and are unchangeable in structural design, structural engineers are concerned about reducing the dead load. As a result, almost all new structures are being designed as light structures. Although the same degree of safety is desired for all structures, lightweight structures are deficient in terms of safety because the effect of dead loads in lightweight structures is smaller than that in heavy structures. In order to reflect this effect in structural design, it is necessary to reflect the reduction effect due to the existence of the dead loads on the live load, because the magnitude of live load is predetermined based on the intended used of the structure and independents of the existing dead loads. The present paper does not suggest increasing the dead load of structures. Rather, in order to avoid the collapse of the plate due to live loads, the safety factor for lightweight structures should be increase to coincide with the safety factor for heavy structures.

4. Approximate solution

The effect of dead loads on static live loads has been demonstrated numerically using the Galerkin method. The use of this effect in design requires a general parametric expression in terms of parameters that do not require numerical calculation. Therefore, it is necessary to solve analytically the governing equation that includes the effect of dead loads for static problems. However, because of the difficulty in solving this equation analytically, a closed-form solution is proposed by making some assumptions. Simply supported rectangular uniform plates are considered, because based on the preceding numerical results, the effect of dead loads is remarkable in simply supported plates.

From Eq. (3), the current governing equation, which includes the effect of dead loads, can be rewritten as follows

$$\bar{w}_{,xxxx} + 2\bar{w}_{,xxyy} + \bar{w}_{,yyyy} = \frac{1}{D}(\bar{p} - DR) \tag{15}$$

where R denotes the effect of dead loads and is a function of $\tilde{w}(x,y)$ and $\bar{w}(x,y)$, as defined by

$$\begin{aligned} R = &-\frac{6}{h^2}[2\tilde{w}_{,x}\tilde{w}_{,xx}\bar{w}_{,x} + (\tilde{w}_{,x})^2\bar{w}_{,xx} + 2\tilde{w}_{,y}\tilde{w}_{,yy}\bar{w}_{,yy} + (\tilde{w}_{,y})^2\bar{w}_{,yy}] \\ &-\frac{6\nu}{h^2}[2\tilde{w}_{,y}\tilde{w}_{,xy}\bar{w}_{,x} + (\tilde{w}_{,y})^2\bar{w}_{,xx} + 2\tilde{w}_{,x}\tilde{w}_{,xy}\bar{w}_{,y} + (\tilde{w}_{,x})^2\bar{w}_{,yy}] \\ &-\frac{6(1-\nu)}{h^2}[(\tilde{w}_{,x}\tilde{w}_{,y}\bar{w}_{,y})_{,x} + (\tilde{w}_{,x}\tilde{w}_{,y}\bar{w}_{,x})_{,y}] \end{aligned} \tag{16}$$

On the other hand, the general governing equation excluding the effect of dead loads can be written as follows

$$\bar{w}_{0,xxxx} + 2\bar{w}_{0,xxyy} + \bar{w}_{0,yyyy} = \frac{\bar{p}}{D} \tag{17}$$

where the displacements $\bar{w}_0$ due to live loads exclude the effect of dead loads. Here, the displacements $\bar{w}_0$ for a simply supported plate are obtained using Navier's solution as follows

$$\bar{w}_0 = \sum_{m=1}\sum_{n=1}\bar{w}_{0mn}f_{mn} \tag{18}$$

where f_{mn} is given in Eq. (9) and $\bar{w}_{0mn}$ is obtained as follows

$$\bar{w}_{0mn} = \frac{\bar{p}_{mn}}{D\pi^4\left[\left(\frac{m}{\ell_x}\right)^2 + \left(\frac{n}{\ell_y}\right)^2\right]^2} \tag{19}$$

where the coefficients $\bar{p}_{mn}$ of the double Fourier expansion of live loads $\bar{p}$ are determined from Table 1.5.1 in Szilard (1974).

The above numerical results indicate that, for practical load ratios $\bar{p}/\tilde{p}$, dead loads have a reduction ratio $\bar{w}_{max}/\bar{w}_{0max}$ of between 1.0 and 0.85. Hence, it is assumed that the unknown displacements $\bar{w}$ in R given by Eq. (16) are related to the displacements $\bar{w}_0$ excluding the effect of dead loads as follows

$$\bar{w}(x,y) = \beta(x,y)\bar{w}_0(x,y) \tag{20}$$

where $\beta(x,y)$ is a coefficient indicating the effect of dead loads and is a function of x and y. Here, $\beta(x,y)$ always takes a value of less than 1. The accuracy of the approximate solution proposed here depends on $\beta(x,y)$. Then, based on the preceding numerical results and the trial-and-error results for the proposed solution, the coefficient $\beta(x,y)$ for uniform live loads is assumed to be

$$\beta(x,y) = \beta_0 \sin\left(\frac{3\pi x}{\ell_x}\right)\sin\left(\frac{3\pi y}{\ell_y}\right) \tag{21}$$

where β_0 is the effect of dead loads on the displacements $\bar{w}$ at the midpoint ($x = \ell_x/2$ and $y = \ell_y/2$) of the plate and is assumed to be approximately 1.

Using the above approximation, $R(\bar{w}, \tilde{w})$, which is a function of both unknown displacements $\bar{w}$ and known displacement $\tilde{w}$, is changed to $R(\bar{w}_0, \tilde{w})$, which is a function of all known displacements $\bar{w}_0$ and $\tilde{w}$. Consequently, R reduces to a function composed of all known displacements. Therefore it means that Eq. (15) changes from the nonlinear equation to the linear one. Eq. (15) is reduced to an analytically solvable form. Then, let us give the right-hand side of Eq. (15) as

$$\bar{P} = \bar{p} - DR \tag{22}$$

The quantities $\bar{P}, \bar{p}$, and R are expanded into a double Fourier series as follows

$$\bar{P} = \sum_{m=1}\sum_{n=1} \bar{P}_{mn} f_{mn} \tag{23}$$

$$\bar{p} = \sum_{m=1}\sum_{n=1} \bar{p}_{mn} f_{mn} \tag{24}$$

$$R = \sum_{m=1}\sum_{n=1} R_{mn} f_{mn} \tag{25}$$

Substituting Eqs. (23) to (25) into Eq. (22), the Fourier coefficients $\bar{P}_{mn}$ can be written as

$$\bar{P}_{mn} = \bar{p}_{mn}\left[1 - \frac{DR_{mn}}{\bar{p}_{mn}}\right] \tag{26}$$

The second term in the right-hand side of the above equation indicates a reduction in the Fourier coefficients $\bar{p}_{mn}$ of live loads $\bar{p}$ due to the existence of dead loads. If the coefficients R_{mn} are known, then the coefficients $\bar{P}_{mn}$ become known and the displacements $\bar{w}$ can be easily obtained by replacing $\bar{p}_{mn}$ in Eq. (19) with $\bar{P}_{mn}$. Thus, let us define the coefficients R_{mn}.

The Fourier coefficients R_{mn} can be given by

$$R_{mn} = \frac{4}{\ell_x \ell_y}\int_0^{\ell_x}\int_0^{\ell_y} R f_{mn}\,dx\,dy \tag{27}$$

where R is a function of the known displacements $\bar{w}_0$ and $\tilde{w}_0$. Then, expressing the ratio of dead loads $\tilde{p}$ to live loads $\bar{p}$ as

$$\alpha = \frac{\bar{p}}{\tilde{p}} \tag{28}$$

the displacements $\tilde{w}$ due to uniform dead loads $\tilde{p}$ become $1/\alpha$ times the displacements $\bar{w}_0$ due to uniform live loads $\bar{p}$, because only uniformly distributed loads are considered in the current problem. Substituting the span ratio $\alpha_\ell(=\ell_y/\ell_x)$ and the expression

$$\bar{p}_{mn} = \bar{p}\bar{p}^*_{mn} \tag{29}$$

in Eq. (19), the coefficients $\bar{w}_{0mn}$ can be expressed in a non-dimensional form as follows

$$\bar{w}_{0mn} = \frac{\bar{p}(\ell_x)^4}{D}\bar{w}^*_{0mn} \tag{30}$$

where the superscript * denotes a non-dimensional value, and $\bar{w}^*_{0mn}$ and $\bar{p}^*_{mn}$ are defined as follows

$$\bar{w}^*_{0mn} = \frac{\bar{p}^*_{mn}}{\pi^4\left[m^2+\left(\frac{n}{\alpha_\ell}\right)^2\right]^2} \tag{31}$$

$$\bar{p}^*_{mn} = \frac{16}{\pi^2 mn} \quad (\text{for } m,\ n = 1,\ 3,\ 5,...) \tag{32}$$

Applying these relations into R given in Eq. (27), the Fourier coefficients R_{mn} can be written as

$$R_{mn} = \frac{\beta_0(\ell_x)^8\bar{p}^3}{h^2D^3\alpha^2}g_{mn} \tag{33}$$

where g_{mn} is a non-dimensional quantity, as given in Eq. (63) in the Appendix, and depends only on the span ratio $\alpha_\ell = \ell_y/\ell_x$.

Since the coefficients R_{mn} have been given, the coefficients $\bar{P}_{mn}$ are known. Hence, the displacements $\bar{w}$ due to live loads, taking into consideration the effect of dead loads, yield

$$\bar{w} = \sum_{m=1}\sum_{n=1}\bar{w}_{mn}f_{mn} \tag{34}$$

where $\bar{w}_{mn}$ are

$$\bar{w}_{mn} = \frac{(\ell_x)^4\bar{P}_{mn}}{D\pi^4\left[m^2+\left(\frac{n}{\alpha_\ell}\right)^2\right]^2} \tag{35}$$

Substituting Eqs. (26) and (33) into Eq. (35), the displacements $\bar{w}$ can be written as

$$\bar{w} = [1-k\alpha_w]\bar{w}^*_0\frac{\bar{p}(\ell_x)^4}{D} \tag{36}$$

where k and α_w are defined as

$$k = \beta_0\left[\frac{(\ell_x)^4\tilde{p}}{hD}\right]^2 \tag{37}$$

$$\alpha_w = \frac{\bar{w}^*_1}{\bar{w}^*_0} \tag{38}$$

and

$$\overline{w}_0^* = \sum_{m=1}\sum_{n=1}\overline{w}_{0mn}^* f_{mn} \tag{39}$$

$$\overline{w}_1^* = \sum_{m=1}\sum_{n=1}\overline{w}_{1mn}^* f_{mn} \tag{40}$$

Here, $\overline{w}_{1mn}^*$ is defined as

$$\overline{w}_{1mn}^* = \frac{g_{mn}}{\pi^4\left[m^2+\left(\frac{n}{\alpha_\ell}\right)^2\right]^2} \tag{41}$$

The non-dimensional parameter α_w depends on only the span ratio α_ℓ and can be evaluated beforehand.

Similarly, the stress couples $\overline{M}_x, \overline{M}_y$, and $\overline{M}_{xy}$ due to live loads can be given as

$$\overline{M}_x = [1-k\alpha_{Mx}]\overline{M}_{x0}^*\overline{p}\ell_x^2 \tag{42}$$

$$\overline{M}_y = [1-k\alpha_{My}]\overline{M}_{y0}^*\overline{p}\ell_x^2 \tag{43}$$

$$\overline{M}_{xy} = [1-k\alpha_{Mxy}]\overline{M}_{xy0}^*\overline{p}\ell_x^2 \tag{44}$$

where

$$\alpha_{Mx} = \frac{\overline{M}_{x1}^*}{\overline{M}_{x0}^*}, \quad \alpha_{My} = \frac{\overline{M}_{y1}^*}{\overline{M}_{y0}^*}, \quad \alpha_{Mxy} = \frac{\overline{M}_{xy1}^*}{\overline{M}_{xy0}^*} \tag{45}$$

and

$$\left.\begin{aligned}
\overline{M}_{x0}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^2+\nu\left(\frac{n\pi}{\alpha_\ell}\right)^2\right]\overline{w}_{0mn}^* f_{mn}\\
\overline{M}_{x1}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^2+\nu\left(\frac{n\pi}{\alpha_\ell}\right)^2\right]\overline{w}_{1mn}^* f_{mn}\\
\overline{M}_{y0}^* &= \sum_{m=1}\sum_{n=1}\left[\left(\frac{n\pi}{\alpha_\ell}\right)^2+\nu(m\pi)^2\right]\overline{w}_{0mn}^* f_{mn}\\
\overline{M}_{y1}^* &= \sum_{m=1}\sum_{n=1}\left[\left(\frac{n\pi}{\alpha_\ell}\right)^2+\nu(m\pi)^2\right]\overline{w}_{1mn}^* f_{mn}\\
\overline{M}_{xy0}^* &= -(1-\nu)\sum_{m=1}\sum_{n=1}\frac{mn\pi^2}{\alpha_\ell}\overline{w}_{0mn}^*\cos\left(\frac{m\pi x}{\ell_x}\right)\cos\left(\frac{n\pi y}{\ell_y}\right)\\
\overline{M}_{xy1}^* &= -(1-\nu)\sum_{m=1}\sum_{n=1}\frac{mn\pi^2}{\alpha_\ell}\overline{w}_{1mn}^*\cos\left(\frac{m\pi x}{\ell_x}\right)\cos\left(\frac{n\pi y}{\ell_y}\right)
\end{aligned}\right\} \tag{46}$$

Similarly, the transverse shear forces $\bar{q}_x$ and $\bar{q}_y$ due to live loads can be obtained as

$$\bar{q}_x = [1-k\alpha_{qx}]\bar{q}_{x0}^*\bar{p}\ell_x \tag{47}$$

$$\bar{q}_y = [1-k\alpha_{qy}]\bar{q}_{y0}^*\bar{p}\ell_y \tag{48}$$

where

$$\alpha_{qx} = \frac{\bar{q}_{x1}^*}{\bar{q}_{x0}^*}, \quad \alpha_{qy} = \frac{\bar{q}_{y1}^*}{\bar{q}_{y0}^*} \tag{49}$$

and

$$\left.\begin{aligned}
\bar{q}_{x0}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^3 + m\pi\left(\frac{n\pi}{\alpha_\ell}\right)^2\right]\bar{w}_{0mn}^*\cos\left(\frac{m\pi x}{\ell_x}\right)\sin\left(\frac{n\pi y}{\ell_y}\right)\\
\bar{q}_{x1}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^3 + m\pi\left(\frac{n\pi}{\alpha_\ell}\right)^2\right]\bar{w}_{1mn}^*\cos\left(\frac{m\pi x}{\ell_x}\right)\sin\left(\frac{n\pi y}{\ell_y}\right)\\
\bar{q}_{y0}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^2\left(\frac{n\pi}{\alpha_\ell}\right) + \left(\frac{n\pi}{\alpha_\ell}\right)^3\right]\bar{w}_{0mn}^*\sin\left(\frac{m\pi x}{\ell_x}\right)\cos\left(\frac{n\pi y}{\ell_y}\right)\\
\bar{q}_{y1}^* &= \sum_{m=1}\sum_{n=1}\left[(m\pi)^2\left(\frac{n\pi}{\alpha_\ell}\right) + \left(\frac{n\pi}{\alpha_\ell}\right)^3\right]\bar{w}_{1mn}^*\sin\left(\frac{m\pi x}{\ell_x}\right)\cos\left(\frac{n\pi y}{\ell_y}\right)
\end{aligned}\right\} \tag{50}$$

In the plate theory, the deflection $\bar{w}_0$, stress couples $\bar{M}_{x0}, \bar{M}_{y0}$, and $\bar{M}_{xy0}$, and transverse shear forces $\bar{q}_{x0}$ and $\bar{q}_{y0}$, which exclude the reduction effect of dead loads, are related as follows

$$\begin{gathered}
\bar{w}_0 = \bar{w}_0^*\frac{\bar{p}\,\ell_x^{\,4}}{D}\\
\bar{M}_{x0} = \bar{M}_{x0}^*\bar{p}\,\ell_x^{\,2}, \quad \bar{M}_{y0} = \bar{M}_{y0}^*\bar{p}\,\ell_x^{\,2}, \quad \bar{M}_{xy0} = \bar{M}_{xy0}^*\bar{p}\,\ell_x^{\,2}\\
\bar{q}_{x0} = \bar{q}_{x0}^*\bar{p}\,\ell_x, \quad \bar{q}_{y0} = \bar{q}_{y0}^*\bar{p}\,\ell_x, \quad \bar{q}_{xy0} = \bar{q}_{xy0}^*\bar{p}\,\ell_x
\end{gathered} \tag{51}$$

Thus, applying Eq. (51) to Eqs. (36), (42) through (44), (47), and (48), the reduction ratios due to the dead loads $\tilde{p}$ on the action of live loads $\bar{p}$ are expressed as follows

$$\frac{\bar{w}}{\bar{w}_0} = 1-k\alpha_w \tag{52}$$

$$\frac{\bar{M}_x}{\bar{M}_{x0}} = 1-k\alpha_{Mx} \tag{53}$$

$$\frac{\bar{M}_y}{\bar{M}_{y0}} = 1-k\alpha_{My} \tag{54}$$

$$\frac{\overline{M}_{xy}}{\overline{M}_{xy0}} = 1 - k\alpha_{Mxy} \tag{55}$$

$$\frac{\overline{q}_x}{\overline{q}_{x0}} = 1 - k\alpha_{qx} \tag{56}$$

$$\frac{\overline{q}_y}{\overline{q}_{y0}} = 1 - k\alpha_{qy} \tag{57}$$

Thus, considering of the effect of dead loads, it is clarified that the reductions in the displacements, stress couples, and transverse shear forces produced by live loads always depend on k in the second terms of the right-hand sides of the above equations. Then, since the non-dimensional coefficients α_w, α_{Mx}, α_{My}, α_{Mxy}, α_{qx}, and α_{qy} depend only on the span ratio α_ℓ (= ℓ_y/ℓ_x), they can be given as shown in Tables 1 and 2 for $\nu = 0.3$ and 0.17, respectively. The effect of dead loads is obtained from the product of k and the values given in Tables 1 and 2. The product is small, as is the effect itself. Consequently, the slight variation in the values shown in Tables 1 and 2 may be disregarded in practical use. This explains the phenomenon whereby there is little difference between the displacements and stress couples due to the effect of dead loads, as indicated by the preceding numerical results.

Subsequently, in order to examine the exactness of the approximate solution proposed here, the results of the proposed solution were compared with the above-described numerical results obtained by the Galerkin method, in which the coefficient β_0 is assumed to be 1. All of the results showed excellent agreement, except in the case of α_ℓ = 2.0 in Fig. 4, for simply supported plates. However, this difference is negligible for practical use and shows error toward the side of safety. The exactness of the proposed solution has been proven. Thus, the effect of dead loads is governed by the value of k given in Eq. (37). Namely, the effect is proportional to the eighth power of the span

Table 1 Values of α_w, α_{Mx}, α_{My}, α_{Mxy}, α_{qx}, and α_{qy} ($\nu = 0.3$)

$\alpha_\ell = \ell_y/\ell_x$	α_w	α_{Mx}	α_{My}	α_{Mxy}	α_{qx}	α_{qy}
1.0	0.280×10^{-4}	0.295×10^{-4}	0.295×10^{-4}	0.263×10^{-4}	0.538×10^{-4}	0.538×10^{-4}
1.1	0.403×10^{-4}	0.445×10^{-4}	0.422×10^{-4}	0.376×10^{-4}	0.846×10^{-4}	0.703×10^{-4}
1.2	0.547×10^{-4}	0.624×10^{-4}	0.618×10^{-4}	0.499×10^{-4}	0.123×10^{-3}	0.864×10^{-4}
1.3	0.708×10^{-4}	0.829×10^{-4}	0.861×10^{-4}	0.624×10^{-4}	0.170×10^{-3}	0.101×10^{-3}
1.4	0.881×10^{-4}	0.105×10^{-3}	0.115×10^{-3}	0.742×10^{-4}	0.222×10^{-3}	0.114×10^{-3}
1.5	0.106×10^{-3}	0.130×10^{-3}	0.148×10^{-3}	0.867×10^{-4}	0.280×10^{-3}	0.125×10^{-3}
1.6	0.125×10^{-3}	0.155×10^{-3}	0.185×10^{-3}	0.105×10^{-3}	0.341×10^{-3}	0.155×10^{-3}
1.7	0.144×10^{-3}	0.180×10^{-3}	0.226×10^{-3}	0.124×10^{-3}	0.405×10^{-3}	0.188×10^{-3}
1.8	0.163×10^{-3}	0.206×10^{-3}	0.270×10^{-3}	0.144×10^{-3}	0.471×10^{-3}	0.221×10^{-3}
1.9	0.183×10^{-3}	0.232×10^{-3}	0.315×10^{-3}	0.165×10^{-3}	0.537×10^{-3}	0.255×10^{-3}
2.0	0.202×10^{-3}	0.258×10^{-3}	0.363×10^{-3}	0.186×10^{-3}	0.602×10^{-3}	0.289×10^{-3}
3.0	0.379×10^{-3}	0.475×10^{-3}	0.763×10^{-3}	0.436×10^{-3}	0.114×10^{-2}	0.570×10^{-3}
4.0	0.541×10^{-3}	0.626×10^{-3}	0.988×10^{-3}	0.669×10^{-3}	0.145×10^{-2}	0.724×10^{-3}
5.0	0.690×10^{-3}	0.736×10^{-3}	0.110×10^{-2}	0.834×10^{-3}	0.163×10^{-2}	0.799×10^{-3}

Table 2 Values of α_w, α_{Mx}, α_{My}, α_{Mxy}, α_{qx}, and α_{qy} ($\nu = 0.17$)

$\alpha_\ell = \ell_y/\ell_x$	α_w	α_{Mx}	α_{My}	α_{Mxy}	α_{qx}	α_{qy}
1.0	0.280×10^{-4}	0.299×10^{-4}	0.299×10^{-4}	0.263×10^{-4}	0.538×10^{-4}	0.538×10^{-4}
1.1	0.403×10^{-4}	0.448×10^{-4}	0.418×10^{-4}	0.376×10^{-4}	0.846×10^{-4}	0.703×10^{-4}
1.2	0.547×10^{-4}	0.625×10^{-4}	0.617×10^{-4}	0.499×10^{-4}	0.123×10^{-3}	0.864×10^{-4}
1.3	0.708×10^{-4}	0.826×10^{-4}	0.868×10^{-4}	0.624×10^{-4}	0.170×10^{-3}	0.101×10^{-3}
1.4	0.881×10^{-4}	0.104×10^{-3}	0.117×10^{-3}	0.742×10^{-4}	0.222×10^{-3}	0.114×10^{-3}
1.5	0.106×10^{-3}	0.128×10^{-3}	0.153×10^{-3}	0.867×10^{-4}	0.280×10^{-3}	0.125×10^{-3}
1.6	0.125×10^{-3}	0.152×10^{-3}	0.195×10^{-3}	0.105×10^{-3}	0.341×10^{-3}	0.155×10^{-3}
1.7	0.144×10^{-3}	0.177×10^{-3}	0.243×10^{-3}	0.124×10^{-3}	0.405×10^{-3}	0.188×10^{-3}
1.8	0.163×10^{-3}	0.201×10^{-3}	0.298×10^{-3}	0.144×10^{-3}	0.471×10^{-3}	0.221×10^{-3}
1.9	0.183×10^{-3}	0.226×10^{-3}	0.349×10^{-3}	0.165×10^{-3}	0.537×10^{-3}	0.255×10^{-3}
2.0	0.202×10^{-3}	0.251×10^{-3}	0.397×10^{-3}	0.186×10^{-3}	0.602×10^{-3}	0.289×10^{-3}
3.0	0.379×10^{-3}	0.458×10^{-3}	0.845×10^{-3}	0.436×10^{-3}	0.114×10^{-2}	0.570×10^{-3}
4.0	0.541×10^{-3}	0.606×10^{-3}	0.114×10^{-2}	0.669×10^{-3}	0.145×10^{-2}	0.724×10^{-3}
5.0	0.690×10^{-3}	0.717×10^{-3}	0.130×10^{-2}	0.834×10^{-3}	0.163×10^{-2}	0.799×10^{-3}

length ℓ_x and the second power of the dead load $\tilde{p}$ and inversely proportional to the second power of the thickness h and of the bending rigidity of the plate, D.

Then, the factor R obtained by substituting Eq. (33) into Eq. (25) decreases live loads $\bar{p}$, as estimated from the right-hand side of Eq. (15). Consequently, this reduction can be explained by the effects of dead loads for the displacements, stress couples, and transverse shear forces caused by live loads. Since heavy plates have a larger value of R than lightweight plates, the effect of dead loads is greater for heavy plates than for lightweight plates. As such, in order to achieve parity in safety, the safety factor for lightweight plates must be increased by considering the effect of dead loads.

Next, the reduction effect due to the dead loads on the total maximum deflections of simply supported plates is accounted for in the relation $(\bar{w}+\tilde{w})_{\max}/(\bar{w}_0+\tilde{w})_{\max}$. Considering the relation $\bar{w}_0/\tilde{w} = \bar{p}/\tilde{p}$ $(= \alpha)$ and of Eq. (36), we have

$$\frac{(\bar{w}+\tilde{w})_{\max}}{(\bar{w}_0+\tilde{w})_{\max}} = 1 - \frac{\alpha k \alpha_w}{1+\alpha} \tag{58}$$

Hence, the reduction effect of dead loads on the total maximum deflections of the plate subjected to the dead load $\tilde{p}$ and live load $\bar{p}$ is governed by the factors $\alpha (= \bar{p}/\tilde{p})$ and k. Similar relationships for the stress couples $\overline{M}_x, \overline{M}_y$, and $\overline{M}_{xy}$, and the transverse shear forces $\bar{q}_x$ and $\bar{q}_y$ can be obtained by replacing α_w with α_{Mx}, α_{My}, α_{Mxy}, α_{qx}, and α_{qy} , respectively.

5. Example

Here, let us describe how to apply the effect of dead loads to the modification of live loads. We consider an equivalent steel plate with simply supported ends to have span lengths ℓ_x and ℓ_y of 6 m, a thickness h of 0.05 m, a Young's modulus E of 21×10^{10} N/m^2, and a Poisson's ratio ν of 0.3. The bending rigidity D is 236×10^4 N·m.

STEP 1. Evaluate the value of k from Eq. (37), in which β_0 is assumed to be 1, to obtain

$$k = 0.0001163\tilde{p}^2 \tag{59}$$

where $\tilde{p}$ (N/m^2) is a uniform dead load in the current plate. Also, from Table 1, since $\alpha_\ell = \ell_y/\ell_x = 1$, the value of α_w is 0.280×10^{-4}.

STEP 2. From Eq. (52), the reduction ratio of the maximum deflections produced by live loads, which are determined by considering the effect of dead loads, takes the following value

$$\frac{\overline{w}}{\overline{w}_0} = 1 - k\alpha_W = 1 - 3.256 \times 10^{-9}\tilde{p}^2 \tag{60}$$

STEP 3. Since this reduction effect influences the magnitude of dead loads $\tilde{p}$, the action of live loads is reduced on heavy plates to a greater degree than on lightweight plates. For the current plate, it is assumed that preventive measures have been taken previously with respect to the live load problem. These measures will be obtained from the investigation of structures that have collapsed as a result of live loads or from experiments. The dead loads of safe plates are denoted by $\tilde{p}_0$. The reduction ratios of the maximum deflections produced by live loads for safe plates are given by

$$\frac{\overline{w}}{\overline{w}_0} = 1 - 3.256 \times 10^{-9}\tilde{p}_0^2 \tag{61}$$

STEP 4. The magnitude of live loads acting on a plate with dead loads $\tilde{p}$ must increase corresponding to the reduction ratio. The reduction effect due to dead loads also contributes to the safety of the structure. The incremental ratio for safety is given by

$$\text{incremental ratio for safety} = \frac{1 - 3.256 \times 10^{-9}\tilde{p}^2}{1 - 3.256 \times 10^{-9}\tilde{p}_0^2} \tag{62}$$

For example, it is assumed that $\tilde{p}_0$ = 3,922 N/m^2 and $\tilde{p}$ = 196 N/m^2. Substituting these values into Eq. (62) reveals that the live loads $\overline{p}$ acting on the current plate with $\tilde{p}$ = 196 N/m^2 must be increased by 1.053 times.

STEP 5. The effect of dead loads on the total deflection of the current plate subjected to the dead load and live load can be obtained from Eq. (58) for arbitrary live loads.

Thus, the increase of live loads in order to prevent the collapse of structures due to live loads (snow loads) depends on the span length, the bending rigidity, the thickness of the plates used, and the standard value $\tilde{p}_0$ for dead loads. This standard value of dead loads is determined experimentally or through the investigation of collapsed structures, as mentioned above. However, for practical use, it is recommended that dead loads of reinforced concrete plates be considered as the standard value of dead loads, rather than the dead loads used commonly for plates in steel structures. This method will have the same effect when used on live loads for steel plates and reinforced concrete plates, except for different treatment of the redundancy of structural materials used to ensure safety.

6. Conclusions

The phenomenon whereby the deflection, stress couples, and shear forces produced by static live loads decrease due to the existence of the dead loads on elastic and static plates has been confirmed by means of the governing equation for plates that includes the effect of dead loads. Key factors dominating the effect of dead loads in simply supported plates have been clearly shown in a closed-form solution. From Eq. (37), the effect due to dead loads is demonstrated to be proportional to the eighth power of the span length ℓ_x and the second power of the dead load $\tilde{P}$, and inversely proportional to the second power of the thickness h and bending stiffness D. Finally, a method reflecting the effect of dead loads on live loads has been proposed for practical use. The effect of dead loads is demonstrated to be an important consideration with respect to the safety of simply supported and lightweight plates.

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Appendix 1. – Values of g_{mn}

The notation g_{mn} used in Eq. (33) is defined as follows

$$g_{mn}=24\pi^4\sum_{\bar{m}=1}\sum_{\bar{n}=1}\sum_{\tilde{m}=1}\sum_{\tilde{n}=1}\sum_{\hat{m}=1}\sum_{\hat{n}=1}\frac{\bar{p}^*_{\bar{m}\bar{n}}}{\pi^4\left[\bar{m}^2+(\frac{\bar{n}}{\alpha_\ell})^2\right]^2}$$

$$\frac{\bar{p}^*_{\tilde{m}\tilde{n}}}{\pi^4[\tilde{m}^2+(\frac{\tilde{n}}{\alpha_\ell})^2]^2}\frac{\bar{p}^*_{\hat{m}\hat{n}}}{\pi^4[\hat{m}^2+(\frac{\hat{n}}{\alpha_\ell})^2]^2}k_0(\bar{m},\bar{n};\tilde{m},\tilde{n};\hat{m},\hat{n};m,n) \tag{63}$$

in which $k_0(\bar{m},\bar{n};\tilde{m},\tilde{n};\hat{m},\hat{n};m,n)$ takes the value

$$k_0(\bar{m},\bar{n};\tilde{m},\tilde{n};\hat{m},\hat{n};m,n)=$$

$$2\bar{m}\tilde{m}\hat{m}^2F_{sc}(\hat{m},m,\tilde{m},\bar{m})F_{ss}(\bar{n},\tilde{n},\hat{n},n)+\bar{m}^2\tilde{n}\hat{m}F_{sc}(\bar{m},m,\tilde{m},\hat{m})F_{ss}(\bar{n},\tilde{n},\hat{n},n)$$

$$+\frac{2\bar{n}\tilde{n}\hat{n}^2}{\alpha_\ell^4}F_{ss}(\bar{m},\tilde{m},\hat{m},m)F_{sc}(\hat{n},n,\bar{n},\tilde{n})+\frac{\bar{n}^2\tilde{n}\hat{n}}{\alpha_\ell^4}F_{ss}(\bar{m},\tilde{m},\hat{m},m)F_{sc}(\bar{n},n,\tilde{n},\hat{n})$$

$$+\frac{\nu}{\alpha_\ell^2}[-2\bar{m}\bar{n}\tilde{m}\hat{n}F_{sc}(\tilde{m},m,\bar{m},\hat{m})F_{sc}(\bar{n},n,\tilde{n},\hat{n})+\bar{m}^2\tilde{n}\hat{n}F_{ss}(\bar{m},\tilde{m},\hat{m},m)F_{sc}(\bar{n},n,\tilde{n},\hat{n})$$

$$-2\bar{n}\tilde{m}\hat{m}\hat{n}F_{sc}(\bar{m},m,\tilde{m},\hat{m})F_{sc}(\tilde{n},n,\bar{n},\hat{n})+\bar{n}^2\tilde{m}\hat{m}F_{sc}(\bar{m},m,\tilde{m},\hat{m})F_{ss}(\tilde{n},\hat{n},\bar{n},n)]$$

$$+\frac{(1-\nu)}{\alpha_\ell^2}[\bar{n}\bar{m}^2\hat{n}F_{ss}(\bar{m},\tilde{m},\hat{m},m)F_{sc}(\tilde{n},n,\bar{n},\hat{n})-\bar{n}\tilde{m}\hat{m}\hat{n}F_{sc}(\bar{m},m,\tilde{m},\hat{m})F_{sc}(\tilde{n},n,\bar{n},\hat{n})$$

$$-2\bar{m}\bar{n}\hat{m}\hat{n}F_{sc}(\hat{m},m,\bar{m},\tilde{m})F_{sc}(\tilde{n},n,\bar{n},\hat{n})-\bar{m}\tilde{m}\tilde{n}\hat{n}F_{sc}(\hat{m},m,\bar{m},\tilde{m})F_{sc}(\bar{n},n,\tilde{n},\hat{n})$$

$$+\bar{m}\tilde{m}\hat{n}^2F_{sc}(\hat{m},m,\bar{m},\tilde{m})F_{ss}(\tilde{n},\hat{n},\bar{n},n)] \tag{64}$$

Here F_{ss} and F_{sc} indicate integrals given by

$$F_{ss}(m_1,m_2,m_3,m_4)=\int_0^1\sin 3\pi\xi\sin m_1\pi\xi\sin m_2\pi\xi\sin m_3\pi\xi\sin m_4\pi\xi d\xi \tag{65}$$

$$F_{sc}(m_1,m_2,m_3,m_4)=\int_0^1\sin 3\pi\xi\sin m_1\pi\xi\sin m_2\pi\xi\cos m_3\pi\xi\cos m_4\pi\xi d\xi \tag{66}$$