DOI: http://dx.doi.org/10.12989/sem.2012.42.5.677

Secondary resonances of a microresonator under AC-DC electrostatic and DC piezoelectric actuations

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(Received April 16, 2011, Revised March 15, 2012, Accepted May 1, 2012)

Abstract. This article studies the secondary resonances of a clamped-clamped microresonator under combined electrostatic and piezoelectric actuations. The electrostatic actuation is induced by applying the AC-DC voltage between the microbeam and the electrode plate that lies at the opposite side of the microbeam. The piezoelectric actuation is induced by applying the DC voltage between upper and lower sides of piezoelectric layer. It is assumed that the neutral axis of bending is stretched when the microbeam is deflected. The drift effect of piezoelectric layer (the phenomenon where there is a slow increase of the free strain after the application of a DC field) is neglected. The equations of motion are solved by using the multiple scale perturbation method. The system possesses a subharmonic resonance of order one-half and a superharmonic resonance of order two. It is shown that using the DC piezoelectric actuation, the sensitivity of AC-DC electrostatically actuated microresonator under subharmonic and superharmonic resonances may be tuned. In addition, it is shown that the tuning domain of the microbeam under combined electrostatic and piezoelectric actuations at subharmonic and superharmonic conditions is larger than the tuning domain of microbeam under only the electrostatic actuation.

Keywords: microresonator; electrostatic actuation; piezoelectric actuation; perturbation method; superharmonic resonance; subharmonic resonance

1. Introduction

Low weight, small size, low consumption energy and high durability of microelectro-mechanical systems (MEMS) have increased the use of microresonators as a key component of pressure sensors, gyroscopes and RF systems (Liu and Paden 2002). The microresonators usually include a microbeam that is excited by an electrostatic or piezoelectric actuation. In microresonators under electrostatic actuation, an AC voltage is applied between the microbeam and the electrode plate. In microresonators under piezoelectric actuation, an AC voltage is applied between the upper and lower sides of a piezoelectric layer deposited on a part of microbeam. A DC electrostatic or piezoelectric actuation is often combined with AC voltages in order to tune the sensitivity and natural frequency of the microresonator. Many researches have been performed to study the mechanical behavior of the microresonators under electrostatic and piezoelectric actuations. Mahmoodi and Jalili (2007, 2009) studied the nonlinear vibration of a clamped-free microresonator under AC piezoelectric actuation at primary resonance. In another work, they studied the

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superharmonic and subharmonic resonances of system under piezoelectric actuation (2010). Li *et al.* (2006), Dick *et al.* (2007) studied the oscillations of a clamed-clamped microresonator under AC piezoelectric actuation. The primary resonance of clamped-clamped microresonators under combined AC-DC electrostatic actuation has been studied by Younis and Nayfeh (2003). Zamanian *et al.* (2010) considered the effect of structural viscoelastic damping on the primary resonance of clamped-clamped microbeam under AC-DC electrostatic actuation. The subharmonic and superharmonic resonances of this configuration have been studied by Abdel-Rahman and Nayfeh (2003), Nayfeh and Younis (2005). The primary resonances of a clamped-free microresonator under combined AC-DC electrostatic actuation has been studied by Zhang and Meng (2005), Lizhong and Xiaoli (2007). For the tuning of microresonator under AC-DC electrostatic actuation, the natural frequency of microbeam is changed using the DC electrostatic actuation. It has been shown by Abdel-Rahman *et al.* (2002) that in systems under AC-DC electrostatic actuation the natural frequency of microbeam about static position decreases due to the softening effect of electrostatic actuation and increases due to the midplane stretching. It means that the increase of natural frequency about static position by an increase of DC electrostatic actuation is restricted due to the system parameters. It has been shown that usually by increasing the value of DC electrostatic actuation, the softening effect overcomes to the stretching effect, and so the natural frequency decreases. It has been shown that even if in a special domain of DC electrostatic actuation the stretching effect overcomes to the softening effect, this domain is restricted. Zamanian *et al*. (2008), Rezazadeh *et al.* (2006) showed that using a combination of DC piezoelectric and DC electric actuations, the static deflection and natural frequency of system about static position may be increased or decreased. It means that the tuning domain of the microbeam under combined electrostatic and piezoelectric actuation is larger than the tuning domain of a microbeam under only the electrostatic actuation. The effect of this combination on the mechanical behavior of microresonator under AC piezoelectric actuation has been studied by Zamanian *et al*. (2009). In another work they studied the effect of this combination on the mechanical behavior of microresonator under AC electrostatic actuation (Zamanian and Khadem 2010). They showed that when the piezoelectric layer is deposited on the full length of microbeam layer, the nonlinear shift of resonance frequency may be increased or decreased using a combination of DC piezoelectric and DC electrostatic actuations. In the previous work, Zamanian and Khadem (2010) considered the effect of DC piezoelectric actuation on the primary resonance of electrostatically actuated microresonator. In some conditions, such as filtering application, the system is oscillating at subharmonic or superharmonic resonances. The effect of piezoelectric layer in this condition has not been considered, and the present paper studies this topic.

2. Modeling and formulation

The system model as shown in Fig. 1 is a clamped-clamped microbeam with a piezoelectric layer whose left and right ends are at distance l_1 and l_2 from the left end of the microbeam, respectively. A DC voltage equal to P_{dc} is applied to the upper and lower sides of the piezoelectric layer. An electrode plate lies at distance d from the microbeam, and an AC-DC voltage equal to $v_p + v_{ac}\cos(\hat{\Omega}t)$ is applied to the microbeam and the electrode plate. In this actuation, v_{ac} and $\hat{\Omega}$ are the amplitude and frequency of the AC part, respectively. Here, v_p is the magnitude of DC part, and t is time. It is assumed that one end of microbeam is stretched equal to $\bar{\varepsilon}$ in the axial direction due

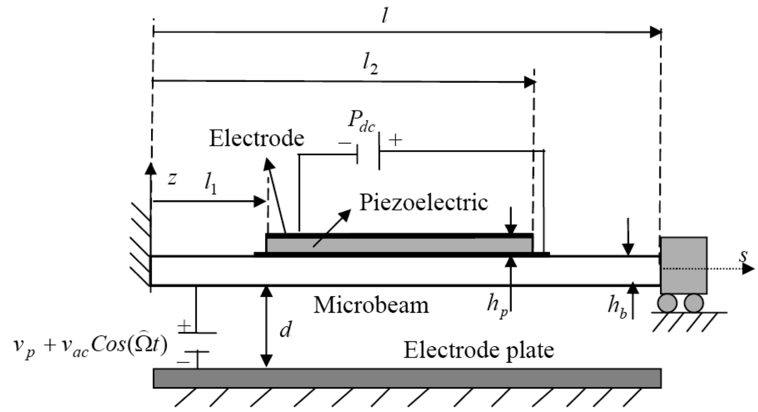

Fig. 1 The configuration of system under combined electrostatic and piezoelectric actuations

to the existence of axial load. It must be noted that the microbeam is not free to deflect in axial direction, but its right end is maintained to remain in the constant position which is obtained by applying the axial load.

It is assumed that $E_b(E_p), h_b(h_p)$ and $\rho_b(\rho_p)$ are the module of elasticity, the thickness and the mass per unit volume of microbeam (piezoelectric layer), respectively. The parameter w_b is width of microbeam and piezoelectric layer. The parameter $\bar{z}_n$ shows the distance between bending neutral axis and midline of microbeam where the piezoelectric layer is deposited. Also, the parameter $\hat{c}$ is the viscous damping per unit length of microbeam, and ε_0 is the dielectric permittivity in vacuum. As shown in Fig. 1 s and z are the axial and lateral direction, respectively and W is the lateral deflection. It must be noted that the drift effect of piezoelectric layer (the phenomenon where there is a slow increase of the free strain after the application of a DC field) will be neglected in derivation of equation. The equation of motion for the considered microbeam using the Newton's second law is written as (Zamanian and Khadem 2010).

$$\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 w}{\partial x^2}\right) + m(x)\frac{\partial^2 w}{\partial \tau^2} + c\frac{\partial w}{\partial \tau} - (\alpha_1\Gamma(w,w) + N - \alpha_3 P_{dc})\frac{\partial^2 w}{\partial x^2} =$$

$$\frac{\alpha_2(v_p + v_{ac}\cos(\Omega\tau))^2}{(1-w)^2} + \alpha_4 P_{dc}\left(\frac{d^2 H_{l_1/l}}{dx^2} - \frac{d^2 H_{l_2/l}}{dx^2}\right),$$

$$w\big|_{x=0} = 0, \quad \frac{\partial w}{\partial x}\Big|_{x=0} = 0, \quad w\big|_{x=1} = 0, \quad \frac{\partial w}{\partial x}\Big|_{x=1} = 0 \tag{1}$$

where

$$w = \frac{-W}{d}, \quad x = \frac{s}{l}, \quad \tau = \frac{t}{T}, \quad T = \sqrt{\frac{\rho_b h_b w_b l^4}{E_b I_b}}, \quad \Gamma(f_1, f_2) = \int_0^l \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x}dx$$

$$H_{l_i/l} = Heaviside\ \ function = \begin{cases} 1 & x \geq \dfrac{l_i}{l} \\ & \\ 0 & x < \dfrac{l_i}{l} \end{cases} \quad i = 1,2 \tag{2}$$

In Eq. (1) the terms including $H(x), m(x)$ and c are the term of bending stiffness, mass inertia, and viscous damping, respectively. These terms are (Zamanian and Khadem 2010)

$$m(x) = 1 + \frac{\rho_p h_p}{\rho_b h_b}(H_{l_1/l} - H_{l_2/l}), \qquad c = \frac{\hat{c} l^4}{E_b I_b T}$$

$$H(x) = (1 - H_{l_1/l}) + \frac{\bar{I}_b}{I_b}(H_{l_1/l} - H_{l_2/l}) + \frac{E_p I_p}{E_b I_b}(H_{l_1/l} - H_{l_2/l}) + H_{l_2/l}$$

$$I_b = \frac{1}{12} w_b h_b^3, \qquad \bar{I}_b = \frac{1}{12} w_b h_b^3 + h_b w_b \bar{z}_n^2, \qquad \bar{z}_n = \frac{E_p h_p (h_p + h_b)}{2(E_b h_b + E_p h_p)}$$

$$\bar{A}_p = \frac{w_b}{2}(h_p^2 + h_p h_h - 2 h_p \bar{z}_n)$$

$$I_p = w_b \left[\frac{1}{3}\left(h_p^3 + \frac{3}{2} h_b h_p^2 + \frac{3}{4} h_p h_b^2 \right) + h_p \bar{z}_n^2 - (h_p^2 + h_p h_b)\bar{z}_n \right] \tag{3}$$

In Eq. (1) the term including N is due to the axial load, and the term including α_1 is due to stretching effect. These terms are (Zamanian and Khadem 2010)

$$N = N_b \left(\frac{1 + \dfrac{E_p h_p}{E_b t_b}}{\left(1 + \dfrac{E_p h_p}{E_b h_b}\right)\left(1 - \dfrac{l_2 - l_1}{l}\right) + \dfrac{l_2 - l_1}{l}} \right), \qquad N_b = \frac{\bar{\varepsilon} A_b l}{I_b}, \quad A_b = h_b w_b$$

$$\alpha_1 = \alpha_b \left(\frac{1 + \dfrac{E_p h_p}{E_b h_b}}{\left(1 + \dfrac{E_p h_p}{E_b h_b}\right)\left(1 - \dfrac{l_2 - l_1}{l}\right) + \dfrac{l_2 - l_1}{l}} \right), \qquad \alpha_b = 6\left(\frac{d}{h_b}\right)^2 \tag{4}$$

In addition, in Eq. (1) the term including $\alpha_3 P_{dc}$ is due to the axial effect of piezoelectric layer and the terms including α_2 and α_4 are due to the lateral effect of electrostatic and bending effect of piezoelectric actuation, respectively. These terms are (Zamanian and Khadem 2010)

$$\alpha_3 = \alpha_5 \left(\frac{6\left(\dfrac{l_2 - l_1}{l}\right)\dfrac{E_p}{E_b}}{\left(1 + \dfrac{E_p h_p}{E_b h_b}\right)\left(1 - \dfrac{l_2 - l_1}{l}\right) + \dfrac{l_2 - l_1}{l}} \right), \qquad \alpha_5 = \frac{2 l^2 d_{31}}{h_b^3}$$

$$\alpha_4 = 3\alpha_5\sqrt{\frac{6}{\alpha_b}}\left(\frac{E_p h_p}{E_b h_b} + \frac{E_p}{E_b} - \frac{\left(\frac{E_p}{E_b}\right)^2 \frac{h_p}{h_b}\left(1+\frac{h_p}{h_b}\right)}{1+\frac{E_p}{E_b}\frac{h_p}{h_b}}\right), \quad \alpha_2 = \frac{6\varepsilon_0 l^4}{E_b h_b^3 d^3}, \quad \Omega = \hat{\Omega} T \tag{5}$$

By equating the terms including v_{ac} or the differentiation of w with respect to time to zero in Eq. (1), the differential equation of static deflection w_s is obtained as bellow

$$\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 w_s}{\partial x^2}\right) - (\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{\partial^2 w_s}{\partial x^2} = \frac{\alpha_2 {v_p}^2}{(1-w_s)^2} +$$

$$\alpha_4 P_{dc}\left(\frac{d^2 H_{l_1/l}}{dx^2} - \frac{d^2 H_{l_2/l}}{dx^2}\right), \quad w_s\big|_{x=0} = 0, \quad \frac{dw_s}{dx}\big|_{x=0} = 0, \quad w_s\big|_{x=1} = 0, \quad \frac{dw_s}{dx}\big|_{x=1} = 0 \tag{6}$$

The displacement of system is the sum of static deflection w_s, and the dynamic deflection $u(x,\tau)$, so

$$w(x,\tau) = u(x,\tau) + w_s \tag{7}$$

By substituting Eq. (7) into Eq. (1), and expanding the electrostatic actuation about the static position, and by using Eq. (6), it is found

$$\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 u}{\partial x^2}\right) + m(x)\frac{\partial^2 u}{\partial \tau^2} + c\frac{\partial u}{\partial \tau} = (\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{\partial^2 u}{\partial x^2} +$$

$$2\alpha_1\Gamma(w_s,u)\frac{d^2 w_s}{dx^2} + \alpha_1\Gamma(u,u)\frac{d^2 w_s}{dx^2} + 2\alpha_1\Gamma(w_s,u)\frac{\partial^2 u}{\partial x^2} +$$

$$\alpha_1\Gamma(u,u)\frac{\partial^2 u}{\partial x^2} + \frac{2\alpha_2 v_p^2}{(1-w_s)^3}u + \frac{3\alpha_2 v_p^2}{(1-w_s)^4}u^2 + \frac{4\alpha_2 v_p^2}{(1-w_s)^5}u^3$$

$$+\frac{2\alpha_2 v_p v_{ac}\cos(\Omega\tau)}{(1-w_s)^2} + \frac{4\alpha_2 v_p v_{ac}\cos(\Omega\tau)}{(1-w_s)^3}u + \frac{\alpha_2(v_{ac}\cos(\Omega\tau))^2}{(1-w_s)^2}$$

$$u\big|_{x=0} = u\big|_{x=1} = 0, \qquad \frac{\partial u}{\partial x}\big|_{x=0} = \frac{\partial u}{\partial x}\big|_{x=1} = 0 \tag{8}$$

If one set v_{ac} and c in Eq. (8) equal to zero, and keep its linear terms, the linear equation of free vibration of undamped system about the static position will be

$$\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 u}{\partial x^2}\right) + m(x)\frac{\partial^2 u}{\partial \tau^2} - (\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{\partial^2 u}{\partial x^2} - 2\alpha_1\Gamma(w_s,u)\frac{d^2 w_s}{dx^2} -$$

$$\frac{2\alpha_2 v_p^2}{(1-w_s)^3}u = 0, \qquad u\big|_{x=0} = u\big|_{x=1} = 0, \qquad \frac{\partial u}{\partial x}\big|_{x=0} = \frac{\partial u}{\partial x}\big|_{x=1} = 0 \tag{9}$$

Here, if the system oscillates about the static position by nth natural frequency, then the vibration of the system may be assumed as

$$u = \varphi_n(x) e^{i\omega_n \tau} \tag{10}$$

where $\varphi_n(x)$ is the nth linear mode shape and ω_n is nth natural frequency of vibration of the deflected microbeam about the static position. By substituting Eq. (10) into Eq. (9), and multiplying it by $e^{-i\omega_n \tau}$, the differential equation governed by linear mode shapes of system will be

$$\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 \phi_n}{\partial x^2}\right) - (\alpha_1 \Gamma(w_s, w_s) + N - \alpha_3 P_{dc})\frac{d^2\phi_n}{dx^2} - 2\alpha_1 \Gamma(w_s, \phi_n)\frac{d^2 w_s}{dx^2} -$$

$$\left(\frac{2\alpha_2 v_p^2}{(1-w_s)^3} + m(x)\omega_n^2\right)\phi_n = 0, \quad \phi_n|_{x=0} = 0, \quad \frac{d\phi_n}{dx}|_{x=0} = 0, \quad \phi_n|_{x=1} = 0, \quad \frac{d\phi_n}{dx}|_{x=1} = 0 \tag{11}$$

Zamanian *et al.* (2008) studied the static deflection, mode shape of vibration about static deflection and natural frequency of system, i.e. the solutions of Eqs. (6) and (11). Also, the solution of Eq. (8) at primary resonance has been studied by Zamanian and Khadem (2010). Here solutions of Eq. (8) at subharmonic and superharmonic resonances are considered.

3. Subharmonic resonance

By considering $T_0 = \tau$, $T_1 = \varepsilon\tau$ and $T_2 = \varepsilon^2\tau$, where ε is small non-dimensional bookkeeping parameter, the solution of Eq. (8) is assumed as follows (Abdel-Rahman and Nayfeh 2003)

$$u(x,\tau,\varepsilon) = \varepsilon u_1(x,T_0,T_1,T_2) + \varepsilon^2 u_2(x,T_0,T_1,T_2) + \varepsilon^3 u_3(x,T_0,T_1,T_2) + \ldots \tag{12}$$

In order to balance the nonlinear terms with the terms of air damping c and excitation v_{ac}, these terms are considered as order ε^2 and ε^3, respectively. The following equation is obtained by substituting Eq. (12) into Eq. (8), and equating the terms with the same power of ε^1.

order (ε^1)

$$L(u_1) = m(x)\frac{\partial^2 u_1}{\partial T_0^2} + \frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2 u_1}{\partial x^2}\right) - (\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{\partial^2 u_1}{\partial x^2} -$$

$$2\alpha_1\Gamma(w_s,u_1)\frac{d^2 w_s}{dx^2} - \frac{2\alpha_2 v_p^2}{(1-w_s)^3}u_1 = 0,$$

$$u_1|_{x=0} = 0, \quad \frac{\partial u_1}{\partial x}|_{x=0} = 0, \quad u_1|_{x=1} = 0, \quad \frac{\partial u_1}{\partial x}|_{x=1} = 0 \tag{13}$$

The solution of Eq. (13) is

$$u_1 = A(T_1,T_2)e^{i\omega T_0}\varphi(x) + \overline{A}(T_1,T_2)e^{-i\omega T_0}\varphi(x) \tag{14}$$

where $\varphi(x)$ and ω are the first normalized mode shape and first natural frequency of the system, respectively. Also, $A(T_1, T_2)$ is a complex function obtained by imposing the solvability condition. If one assumes the frequency of AC voltage as $\Omega = 2\omega + \varepsilon^2\sigma$, then a secular terms will be appeared in the equation obtained by equating the terms with same power of ε^3. It means that system has a subharmonic resonance frequency at $\Omega \cong 2\omega$. In this condition the term $v_{ac}\cos(\Omega\tau)$ will be presented as a excitation term in order (ε^2), so

order (ε^2)

$$L(u_2) = -m(x)\frac{2\partial^2 u_1}{\partial T_0 \partial T_1} + \alpha_1 \Gamma(u_1,u_1)\frac{d^2 w_s}{dx^2} + 2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_1}{\partial x^2} + \frac{3\alpha_2 {v_p}^2}{(1-w_s)^4}{u_1}^2 +$$
$$\frac{2\alpha_2}{(1-w_s)^2}v_p v_{ac}\cos(\Omega T_0), \quad u_2\big|_{x=0} = 0, \quad \frac{\partial u_2}{\partial x}\big|_{x=0} = 0, \quad u_2\big|_{x=1} = 0, \quad \frac{\partial u_2}{\partial x}\big|_{x=1} = 0 \quad (15)$$

order (ε^3)

$$L(u_3) = -2m(x)\frac{\partial^2 u_2}{\partial T_0 \partial T_1} - m(x)\left(\frac{\partial^2 u_1}{\partial T_1^2} + 2\frac{\partial^2 u_1}{\partial T_0 \partial T_2}\right) - c\frac{\partial u_1}{\partial T_0} + 2\alpha_1\Gamma(u_1,u_2)\frac{d^2 w_s}{dx^2} +$$
$$2\alpha_1\Gamma(w_s,u_2)\frac{\partial^2 u_1}{\partial x^2} + 2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_2}{\partial x^2} + \alpha_1\Gamma(u_1,u_1)\frac{\partial^2 u_1}{\partial x^2} + \frac{6\alpha_2 v_p^2}{(1-w_s)^4}u_1 u_2 +$$
$$\frac{4\alpha_2 {v_p}^2}{(1-w_s)^5}{u_1}^3 + \frac{4\alpha_2 v_p}{(1-w_s)^3}v_{ac}\cos(\Omega T_0)u_1,$$
$$u_3\big|_{x=0} = 0, \quad \frac{\partial u_3}{\partial x}\big|_{x=0} = 0, \quad u_3\big|_{x=1} = 0, \quad \frac{\partial u_3}{\partial x}\big|_{x=1} = 0 \quad (16)$$

By substituting Eq. (14) into Eq. (15), it is found

$$L(u_2) = (A^2 e^{2i\omega T_0} + 2A\bar{A} + \bar{A}^2 e^{-2i\omega T_0})h(x) + 2f(x)\cos(\Omega T_0) -$$
$$2m(x)\omega i\left(\frac{\partial A(T_1,T_2)}{\partial T_1}e^{i\omega T_0} - \frac{\partial \bar{A}(T_1,T_2)}{\partial T_1}e^{-i\omega T_0}\right) \quad (17)$$

where

$$h(x) = \alpha_1\Gamma(\varphi,\varphi)\frac{d^2 w_s}{dx^2} + 2\alpha_1\Gamma(w_s,\varphi)\frac{d^2\varphi}{dx^2} + \frac{3\alpha_2 {v_p}^2}{(1-w_s)^4}\varphi^2, \quad f(x) = \frac{\alpha_2 v_p v_{ac}}{(1-w_s)^2} \quad (18)$$

If A depends only on T_2, then the secular term does not arise in Eq. (17). By using this assumption, the particular solution of Eq. (17) is as follows

$$u_2 = \psi_1(x)A^2 e^{2i\omega T_0} + 2\psi_2(x)A\bar{A} + \psi_1(x)\bar{A}^2 e^{-2i\omega T_0} + \psi_3(x)(e^{i\Omega T_0} + e^{-i\Omega T_0}) \quad (19)$$

where, ψ_1, ψ_2 and ψ_3 are obtained by solving the following equations

$$\frac{d^2}{dx^2}\left(H(x)\frac{d^2\psi_j}{dx^2}\right) - 4m(x)\omega^2\delta_{1j}\psi_j - \delta_{3j}\left(m(x)\Omega^2 + \frac{2\alpha_2 v_p^2}{(1-w_s)^3}\right)\psi_j -$$
$$(\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{d^2\psi_j}{dx^2} - 2\alpha_1\Gamma(w_s,\psi_j)\frac{d^2 w_s}{dx^2} - \frac{2\alpha_2 v_p^2}{(1-w_s)^3}\psi_j = h(x)\delta_{1j} +$$
$$h(x)\delta_{2j} + f(x)\delta_{3j}, \quad j = 1,2,3$$
$$\psi_j\big|_{x=0} = 0, \quad \frac{d\psi_j}{dx}\big|_{x=0} = 0, \quad \psi_j\big|_{x=1} = 0, \quad \frac{d\psi_j}{dx}\big|_{x=1} = 0 \tag{20}$$

In the above equations δ_{ij}, $i,j = 1,2,3$ is Kronecker delta function. The solution of Eq. (20) may be obtained by using the linear symmetric mode shapes of vibration of the microbeam about the static position as comparison functions in Galerkin method. So, it is assumed that

$$\psi_j = \sum_{k=1}^{M} c_k \varphi_k \tag{21}$$

where φ_k shows kth linear symmetric mode shape of deflected microbeam about static position, and c_k are coefficients that must be obtained by using the Galerkin method. Substituting Eq. (21) into Eq. (20), multiplying the results by φ_n, $n = 1,2,\ldots,M$, and integrating the results from $x = 0$ to $x = 1$, one obtains

$$\int_0^1 \sum_{k=1}^{M} c_k \frac{d^2}{dx^2}(H(x)\frac{d^2\varphi_k}{dx^2})\varphi_n\, dx - 4\omega^2\delta_{1j}\int_0^1 \sum_{k=1}^{M} c_k\, m(x)\varphi_k\,\varphi_n\, dx -$$
$$\delta_{3j}\int_0^1 \sum_{k=1}^{M} c_k\,\varphi_k\,\varphi_n\,(m(x)\Omega^2 + \frac{2\alpha_2 v_p^2}{(1-w_s)^3})dx - \int_0^1 \sum_{k=1}^{M} c_k(\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{d^2\varphi_k}{dx^2}\varphi_n$$
$$-2\alpha_1\int_0^1 \sum_{k=1}^{M} c_k\,\Gamma(\varphi_k,w_s)\varphi_n\frac{d^2 w_s}{dx^2}dx - 2\alpha_2 {v_p}^2\int_0^1 \sum_{k=1}^{M} c_k\frac{\varphi_k\,\varphi_n}{(1-w_s)^3}\,dx -$$
$$\int_0^1 (h(x)\delta_{1j} - \int_0^1 (h(x)\delta_{1j} - f(x)\delta_{3j})\varphi_n dx = 0, \quad j = 1,2 \tag{22}$$

By substituting Eq. (19) into Eq. (16), and considering that $\Omega = 2\omega + \varepsilon^2\sigma$, one may find

$$L(u_3) = \left[-2\omega i\,\frac{dA}{dT_2}m(x)\varphi(x) - i\omega A(c\varphi(x)) + \chi(x)A^2\bar{A} + \bar{A}\zeta_1(x)e^{i\sigma T_2}\right]e^{i\omega T_0} + cc + NST \tag{23}$$

where

$$\zeta_1(x) = 2\alpha_1\Gamma(\varphi,\psi_3)\frac{d^2 w_s}{dx^2} + 2\alpha_1\Gamma(w_s,\psi_3)\frac{d^2\varphi}{dx^2} + 2\alpha_1\Gamma(w_s,\varphi)\frac{d^2\psi_3}{dx^2} + \frac{2\alpha_2 v_p v_{ac}\varphi}{(1-w_s)^3} + \frac{6\alpha_2 v_p^2}{(1-w_s)^4}\varphi\psi_3$$
$$\chi(x) = \chi_q^G + \chi_c^G + \chi_q^E + \chi_c^G \tag{24}$$

and

$$\chi_q^G = (2\alpha_1\Gamma(\psi_1,\varphi) + 4\alpha_1\Gamma(\psi_2,\varphi))\frac{d^2w_s}{dx^2} + \left(2\alpha_1\frac{d^2\psi_1}{dx^2} + 4\alpha_1\frac{d^2\psi_2}{dx^2}\right)\Gamma(w_s,\varphi) +$$

$$(2\alpha_1\Gamma(w_s,\psi_1) + 4\alpha_1\Gamma(w_s,\psi_2))\frac{d^2\varphi}{dx^2}$$

$$\chi_c^G(x) = 3\alpha_1\Gamma(\varphi,\varphi)\frac{d^2\varphi}{dx^2}$$

$$\chi_q^E = \frac{6\alpha_2 v_p^2}{(1-w_s)^4}(2\varphi\psi_2 + \varphi\psi_1)$$

$$\chi_c^E = \frac{12\alpha_2 v_p^2}{(1-w_s)^5}\varphi^3 \qquad (25)$$

In Eq. (23), *NST* stands for non-secular terms and *cc* denotes the complex conjugate terms. The left side of Eq. (23) is self-adjoint, so the solvability condition is obtained by multiplying the right side of Eq. (23) by $e^{-i\omega T_0}\varphi(x)$ and integrating the result from $x = 0$ to $x = 1$. So

$$2i\omega\left(\overline{m}\frac{dA}{dT_2} + \frac{\mu A}{2}\right) + 8SA^2\overline{A} - \Lambda_1\overline{A}e^{i\sigma T_2} = 0 \qquad (26)$$

where

$$\mu = \int_0^1 c\varphi^2 dx$$

$$S_q^G = -\frac{1}{8}\int_0^1 \chi_q^G\varphi\, dx$$

$$S_c^G = -\frac{1}{8}\int_0^1 \chi_c^G\varphi\, dx$$

$$S_q^E = -\frac{1}{8}\int_0^1 \chi_q^E\varphi\, dx$$

$$S_c^E = -\frac{1}{8}\int_0^1 \chi_c^E\varphi\, dx$$

$$\overline{m} = \int_0^1 m(x)\varphi^2 dx$$

$$\Lambda_1 = \int_0^1 \zeta_1(x) dx$$

$$S = S_q^G + S_c^G + S_q^E + S_c^E \qquad (27)$$

By substituting $A = (1/2)ae^{i(\gamma+\sigma T_2)/2}$ into Eq. (26) and separating the real and imaginary parts, it results that

$$\bar{m}\frac{da}{dT_2} = -\frac{\mu}{2}a - \frac{\Lambda_1 a}{2\omega}\sin\gamma = f_1$$

$$\bar{m}a\frac{d\gamma}{dT_2} = -a\bar{m}\sigma + \frac{2Sa^3}{\omega} - \frac{\Lambda_1 a}{\omega}\cos\gamma = g_1 \tag{28}$$

By substituting Eqs. (14) and (19) into Eq. (12), and considering that $\Omega = 2\omega + \varepsilon^2\sigma$, and substituting $\varepsilon = 1$, the solution of Eq. (8) at subharmonic resonance will be

$$u(x,\tau) = a\cos\frac{1}{2}(\Omega\tau+\gamma)\varphi(x) + \frac{1}{2}a^2[\psi_2(x) + \cos(\Omega\tau+\gamma)\psi_1(x)] + 2\Lambda\cos(\Omega\tau)\psi_3(x) \tag{29}$$

By letting da/dT_2 and $d\gamma/dT_2$ be equal to zero in Eq. (28), one can obtain the equilibrium point (a_0, γ_0) as

$$\frac{\Lambda_1}{2\omega}\sin\gamma = -\frac{\mu}{2}, \qquad \frac{\Lambda_1}{\omega}\cos\gamma_0 = -\bar{m}\sigma + \frac{2Sa_0^2}{\omega}$$

$$a_0^2[\mu^2 + (\sigma\bar{m} - \frac{2Sa_0^2}{\omega})^2] = \frac{\Lambda_1^2 a_0^2}{\omega^2} \tag{30}$$

This equation has a trivial solution i.e., $a_0 = 0$ and the two nontrivial solutions as follow

$$a_0^2 = \frac{\omega}{2S}\left(\sigma\bar{m} \pm \sqrt{\frac{\Lambda_1^2}{\omega^2} - \mu^2}\right) \tag{31}$$

The stability of the trivial solution may be studied by substituting $A = (1/2)(p+iq)e^{\sigma T_2/2}$ into Eq. (28), and computing the Jacobian matrix. So

$$\bar{m}p' = -\frac{\mu}{2}p - \frac{q\bar{m}\sigma}{2} + \frac{\Lambda_1 q}{2\omega} + \frac{S}{\omega}(p^2+q^2)q = f_2$$

$$\bar{m}q' = \frac{p\bar{m}\sigma}{2} - \frac{\mu}{2}q + \frac{\Lambda_1 p}{2\omega} - \frac{S}{\omega}(p^2+q^2)p = g_2 \tag{32}$$

Considering Eq. (31), the characteristic equation and its roots at the trivial solution $p_0 = q_0 = 0$ are

$$\lambda^2 + \frac{\mu}{2}\lambda + \frac{\mu^2}{4} + \left(\frac{\sigma^2\bar{m}^2}{4} - \frac{\Lambda^2}{4\omega^2}\right) = 0$$

$$\lambda = -\frac{\mu}{2} \pm \frac{1}{2\omega}\sqrt{\Lambda^2 - \sigma^2\omega^2\bar{m}^2}$$

Instability condition occurs when one of the eigenvalues of the system gets a positive value. So, Eq. (33) demonstrates that the threshold for unstable trivial solution is

$$\Lambda_1^2 > \mu^2\omega^2 + \sigma^2\omega^2\overline{m}^2 \tag{34}$$

The bifurcation point occurs when one of the eigenvalues is equal to zero. So

$$(\frac{\mu}{2})^2 + (\frac{\sigma^2\overline{m}^2}{4} - \frac{\Lambda_A^2}{4\omega^2}) = 0$$
$$\Lambda_A = \omega\sqrt{\mu^2 + \sigma^2\overline{m}^2} \tag{35}$$

where $\Lambda = \Lambda_A$ is the bifurcation point. Since at the trivial solution $(p, q) = (0, 0)$ Eq. (32) results in

$$\frac{\partial}{\partial\Lambda}\begin{bmatrix} f_2 \\ g_2 \end{bmatrix}_{\substack{p=p_0 \\ q=q_0}} = \begin{bmatrix} \dfrac{q_0}{2\omega} \\ \dfrac{p_0}{2\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{36}$$

Since the rank of matrix, which is resulted from multiplying Jacobian matrix of Eq. (32) by matrix of Eq. (36), is not equal to 2, the bifurcation point is not a saddle-node bifurcation point (Nayfeh and Balachandran 1995). It is resulted from center manifold theorem that this bifurcation point is a pitchfork bifurcation point.

Now, the stability of the non-trivial solution is studied by computing the Jacobian matrix by using Eq. (28). The characteristic equation and its roots on the nontrivial solution are

$$\lambda^2 + \mu\lambda - \frac{2S\sigma\overline{m}}{\omega}a_0^2 + \frac{4S^2a_0^4}{\omega^2} = 0$$
$$\lambda = \frac{-\mu \pm \sqrt{\mu^2 - 4\left(\dfrac{4S^2a_0^4}{\omega^2} - \dfrac{2S\sigma\overline{m}}{\omega}a_0^2\right)}}{2} \tag{37}$$

This equation demonstrates that for existence of an unstable solution, it must

$$a_0^2 < \frac{\omega\sigma\overline{m}}{2S} \tag{38}$$

In the bifurcation point, one of the eigenvalues of system is equal to zero, so by considering Eq. (37), one may obtain

$$-\frac{2S\sigma\overline{m}}{\omega} + \frac{4S^2a_0^2}{\omega^2} = 0 \tag{39}$$

By taking derivative from Eq. (30) with respect to the variable a_0, it is resulted that

$$-4S\omega(\sigma\overline{m} - \frac{2Sa_0^2}{\omega}) = 2\Lambda_1\frac{d\Lambda_1}{da_0} \tag{40}$$

Comparing Eqs. (40) and (39) shows that by altering Λ_1 as a control parameter, a bifurcation occurs when $d\Lambda_1/da_0$ is equal to zero as shown in Fig. 8 (below). Also, it is resulted by taking derivative of Eq. (28) with respect to σ

$$\frac{\partial}{\partial \Lambda_1}\begin{bmatrix}\dfrac{f_1}{a}\\ \dfrac{g_1}{a}\end{bmatrix}_{\substack{\gamma=\gamma_0\\ a=a_0}} = \begin{bmatrix}-\dfrac{a_0}{2\omega}\sin\gamma_0\\ -\dfrac{\cos\gamma_0}{\omega}\end{bmatrix} = \begin{bmatrix}\neq 0\\ \neq 0\end{bmatrix} \tag{41}$$

Since the rank of the matrix which is resulted from multiplying Jacobian matrix of Eq. (28) by matrix of Eq. (41), is equal to 2, the bifurcation point is a saddle-node bifurcation point (Nayfeh and Balachandran 1995). Same as the above process must be performed for σ as a control parameter, and so for sake of brevity, it is not given here.

4. Superharmonic resonance

The system has a resonance frequency at $\Omega = \omega/2 + \varepsilon^2\sigma$. In order to balance the nonlinear terms with the terms of air damping c and excitation v_{ac}, these terms are considered as order $O(\varepsilon^2)$ and $O(\varepsilon^{3/2})$, respectively. In this condition the term $2\alpha_2 v_p v_{ac}\cos(\Omega\tau)/(1-w_s)^2$ would be realized as a excitation force by order $O(\varepsilon^{3/2})$. It is not possible to balance this term with the terms including order $O(\varepsilon^1)$ or $O(\varepsilon^2)$. Therefore, for solving this problem the solution of Eq. (8) is assumed as

$$u(x,\tau) = v_{ac}\cos(\Omega\tau)\psi_4(x) + \hat{u}(x,\tau) \tag{42}$$

By substituting Eq. (42) into Eq. (8), and equating the terms including power one of $v_{ac}\cos(\Omega\tau)$, one obtains

$$\frac{d^2}{dx^2}\left(H_1(x)\frac{d^2\psi_4}{dx^2}\right) - (m(x)\Omega^2 + \frac{2\alpha_2 v_p^2}{(1-w_s)^3})\psi_4 - (\alpha_1\Gamma(w_s,w_s) + N - \alpha_3 P_{dc})\frac{d^2\psi_4}{dx^2} -$$
$$2\alpha_1\Gamma(w_s,\psi_4)\frac{d^2 w_s}{dx^2} = g(x), \qquad g(x) = \frac{2\alpha_2 v_p}{(1-w_s)^2}$$
$$\psi_4\big|_{x=0} = 0, \quad \frac{d\psi_4}{dx}\Big|_{x=0} = 0, \quad \psi_4\big|_{x=1} = 0, \quad \frac{d\psi_4}{dx}\Big|_{x=1} = 0 \tag{43}$$

The solution of Eq. (43) may be obtained using the Galerkin method, similar to the process which has been performed for obtaining the solution of Eq. (20). The only difference is that ψ_j must be replaced by ψ_4, the right side of Eq. (20) must be replaced by $g(x)$, and the terms including δ_{1j} and δ_{3j} in left hand side of Eq. (20) must be replaced by $-(m(x)\Omega^2 + 2\alpha_2 v_p^2/(1-w_s)^3)\psi_4$. Now, by substituting Eq. (42) into Eq. (8), and eliminating the terms of Eq. (43) from the resulted equation, one obtains

$$\frac{\partial^2}{\partial x^2}\left(H_1(x)\frac{\partial^2\hat{u}}{\partial x^2}\right)+m(x)\frac{\partial^2\hat{u}}{\partial\tau^2}+c\frac{\partial\hat{u}}{\partial\tau}=[\alpha_1\Gamma(w_s,w_s)+N-\alpha_3P_{dc}]\frac{\partial^2\hat{u}}{\partial x^2}+$$
$$2\alpha_1\Gamma(w_s,\hat{u})\frac{d^2w_s}{dx^2}+\alpha_1\Gamma(\hat{u},\hat{u})\times\frac{d^2w_s}{dx^2}+\alpha_1\Gamma(\hat{u},\hat{u})\frac{\partial^2u}{\partial x^2}+$$
$$(v_{ac}\cos(\Omega\tau))^2\left[2\alpha_1\Gamma(w_s,\psi_4)\frac{d^2\psi_4}{dx^2}+\alpha_1\Gamma(\psi_4,\psi_4)\frac{d^2w_s}{dx^2}+\right.$$
$$\left.\frac{\alpha_2}{(1-w_s)^2}+\frac{4\alpha_2v_p\psi_4}{(1-w_s)^3}+\frac{3\alpha_2v_p^2\psi_4^2}{(1-w_s)^2}\right]+\frac{2\alpha_2v_p^2}{(1-w_s)^3}\hat{u}+\frac{3\alpha_2v_p^2}{(1-w_s)^4}\hat{u}^2+\frac{4\alpha_2v_p^2}{(1-w_s)^5}\hat{u}^3$$
$$u|_{x=0}=u|_{x=1}=0,\qquad \frac{\partial u}{\partial x}|_{x=0}=\frac{\partial u}{\partial x}|_{x=1}=0 \tag{44}$$

By considering $T_0=\tau, T_1=\varepsilon\tau$ and $T_2=\varepsilon^2\tau$ the solution of Eq. (44) may be assumed as (Abdel-Rahman and Nayfeh 2003)

$$\hat{u}(x,\tau,\varepsilon)=\varepsilon u_1(x,T_0,T_1,T_2)+\varepsilon^2u_2(x,T_0,T_1,T_2)+\varepsilon^3u_3(x,T_0,T_1,T_2)+.... \tag{45}$$

The following equations will be obtained by substituting Eq. (45) into Eq. (44), and equating the terms with same power of ε.

order (ε)

$$L(u_1)=m(x)\frac{\partial^2u_1}{\partial T_0^2}+\frac{\partial^2}{\partial x^2}\left(H(x)\frac{\partial^2u_1}{\partial x^2}\right)-(\alpha_1\Gamma(w_s,w_s)+N-\alpha_3P_{dc})\frac{\partial^2u_1}{\partial x^2}-$$
$$2\alpha_1\Gamma(w_s,u_1)\frac{d^2w_s}{dx^2}-\frac{2\alpha_2{v_p}^2}{(1-w_s)^3}u_1=0$$
$$u_1|_{x=0}=0,\quad \frac{\partial u_1}{\partial x}|_{x=0}=0,\quad u_1|_{x=1}=0,\quad \frac{\partial u_1}{\partial x}|_{x=1}=0 \tag{46}$$

order (ε^2)

$$L(u_2)=\alpha_1\Gamma(u_1,u_1)\frac{d^2w_s}{dx^2}+2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2u_1}{\partial x^2}+\frac{3\alpha_2v_p^2}{(1-w_s)^4}{u_1}^2-2m(x)\frac{\partial^2u_1}{\partial T_0\partial T_1}$$
$$u_2|_{x=0}=0,\quad \frac{\partial u_2}{\partial x}|_{x=0}=0,\quad u_2|_{x=1}=0,\quad \frac{\partial u_2}{\partial x}|_{x=1}=0 \tag{47}$$

order (ε^3)

$$L(u_3) = -2m(x)\frac{\partial^2 u_2}{\partial T_0 \partial T_1} - m(x)\left(\frac{\partial^2 u_2}{\partial T_1^2} + 2\frac{\partial^2 u_1}{\partial T_0 \partial T_2}\right) - c\frac{\partial u_1}{\partial T_0} + 2\alpha_1\Gamma(u_1,u_2)\frac{d^2 w_s}{dx^2} +$$
$$2\alpha_1\Gamma(w_s,u_2)\frac{\partial^2 u_1}{\partial x^2} + 2\alpha_1\Gamma(w_s,u_1)\frac{\partial^2 u_2}{\partial x^2} + \alpha_1\Gamma(u_1,u_1)\frac{\partial^2 u_1}{\partial x^2} +$$
$$+\frac{6\alpha_2 v_p^2}{(1-w_s)^4}u_1 u_2 + \frac{4\alpha_2 v_p{}^2}{(1-w_s)^5}u_1{}^3 + (v_{ac}\cos(\Omega T_0))^2 \times$$
$$\left[2\alpha_1\Gamma(w_s,\psi_4)\frac{d^2\psi_4}{dx^2} + \alpha_1\Gamma(\psi_4,\psi_4)\frac{d^2 w_s}{dx^2} + \frac{\alpha_2}{(1-w_s)^2} + \frac{4\alpha_2 v_p \psi_4}{(1-w_s)^3} + \frac{3\alpha_2 v_p^2 \psi_4^2}{(1-w_s)^4}\right],$$
$$u_3\big|_{x=0} = 0, \quad \frac{\partial u_3}{\partial x}\big|_{x=0} = 0, \quad u_3\big|_{x=1} = 0, \quad \frac{\partial u_3}{\partial x}\big|_{x=1} = 0 \tag{48}$$

Eq. (46) and Eq. (13) are identical. Also, Eq. (47) without considering the external force is similar to Eq. (15). So, the solution of Eqs. (46) and (47) is similar to Eqs. (14) and (19) without considering the external force. By substituting the solution of Eqs. (46) and (47) into Eq. (48), and introducing the excitation frequency by detuning parameter σ as $\Omega = \omega/2 + \varepsilon^2\sigma$, and keeping only the terms that produce secular terms, it results that

$$L(u_3) = \left[-2\omega i\frac{dA}{dT_2}m(x)\phi(x) - i\omega A(c\phi(x)) + \chi(x)A^2\bar{A} + v_{ac}^2\bar{A}\zeta_2(x)e^{2i\sigma T_2}\right]\times$$
$$e^{i\omega T_0} + cc + NST \tag{49}$$

where, $\chi(x)$ is obtained from Eq. (25), and

$$\zeta_2(x) = \frac{1}{4}\alpha_1\Gamma(\psi_4,\psi_4)\frac{d^2 w_s}{dx^2} + \frac{1}{2}\alpha_1\Gamma(w_s,\psi_4)\psi_4'' + \frac{\alpha_2}{4(1-w_s)^2} + \frac{\alpha_2 v_p}{(1-w_s)^3}\psi_4 + \frac{3\alpha_2 v_p^2}{4(1-w_s)^4}\psi_4^2 \tag{50}$$

The left hand side of Eq. (49) is self-adjoint, so the solvability condition is obtained by multiplying its right hand side with $e^{-i\omega T_0}\varphi(x)$ and integrating the result from $x = 0$ to $x = 1$. So

$$2i\omega\left(\bar{m}\frac{dA}{dT_2} + \frac{\mu A}{2}\right) + 8SA^2\bar{A} - \Lambda_2 v_{ac}^2\bar{A}e^{2i\sigma T_2} = 0 \tag{51}$$

where, $\mu, \bar{m}$ and S are same as the obtained expression in Eq. (27), and

$$\Lambda_2 = \int_0^1 \zeta(x)dx \tag{52}$$

By substituting $A = 1/2ae^{i(2\sigma T_2 - \gamma)}$ into Eq. (52) and separating the real and imaginary parts, it results that

$$\bar{m}\frac{da}{dT_2} = -\frac{\mu}{2}a + \frac{\Lambda_2 v_{ac}^2}{\omega}\sin\gamma = f_3$$
$$\bar{m}\frac{d\gamma}{dT_2} = 2\sigma\bar{m} - \frac{Sa^2}{\omega} + \frac{\Lambda_2 v_{ac}^2}{a\omega}\cos\gamma = g_3 \tag{53}$$

By substituting Eqs. (14) and (19) into (45), and then substituting the result into Eq. (42) and substituting $\varepsilon = 1$, the solution of Eq. (8) at superharmonic resonance is

$$\hat{u}(x,t) = v_{ac}\cos(\Omega\tau)\psi_4(x) + a\cos(2\Omega\tau + \gamma)\phi(x) + \frac{1}{2}a^2[\psi_2(x) + \cos 2(4\Omega\tau + 2\gamma)\psi_1(x)] \tag{54}$$

By letting da/dT_2 and $d\gamma/dT_2$ being equal to zero in Eq. (53), one can obtain the equilibrium point (a_0, γ_0) as

$$a_0^2\left[\frac{\mu^2}{4} + \left(2\sigma\bar{m} - \frac{Sa_0^2}{\omega}\right)^2\right] = \frac{v_{ac}{}^4\Lambda_2^2}{\omega^2} \tag{55}$$

This equation shows that the amplitude a_0 is maximum, when the expression in the parenthesis vanishes. So, it results that

$$a_0 = \frac{2v_{ac}{}^2|\Lambda_2|}{\omega\mu},$$
$$\sigma = \frac{Sa_0^2}{2\omega\bar{m}} \tag{56}$$

Also, by considering that $\sigma = \Omega - \varepsilon^2\omega/2$ and combining it with the obtained results, the nonlinear resonance frequency is obtained as

$$\Omega = \frac{\omega}{2} + \frac{2S\,v_{ac}{}^4\Lambda^2}{\omega^3\mu^2\bar{m}} \tag{57}$$

The characteristic equation of Jacobian matrix of Eq. (53) is

$$\lambda^2 + \mu_1\lambda + \left[\frac{\mu_1^2}{4} + \left(\sigma\bar{m} - \frac{3Sa_0^2}{\omega}\right)\left(\sigma\bar{m} - \frac{Sa_0^2}{\omega}\right)\right] = 0 \tag{58}$$

Considering Eq. (58), the eigenvalue of Jacobian matrix will be as follows

$$\lambda_{1,2} = -\mu_1 \pm \sqrt{-4\left(\sigma\bar{m} - \frac{3Sa_0^2}{\omega}\right)\left(\sigma\bar{m} - \frac{Sa_0^2}{\omega}\right)} \tag{59}$$

Instability condition occurs when one of the eigenvalues of the system gets a positive value, and the bifurcation point occurs when one of the eigenvalues is equal to zero. It is obtained from Eq. (58) that

$$\left[\frac{\mu^2}{4} + \left(\sigma\bar{m} - \frac{3Sa_0^2}{\omega}\right)\left(\sigma\bar{m} - \frac{Sa_0^2}{\omega}\right)\right] = 0 \tag{60}$$

By taking derivative from Eq. (60) with respect to the variable a_0, it is resulted that

$$\frac{d\sigma}{da_0} = \frac{-\left[\frac{\mu_1^2}{4} + \left(\sigma\bar{m} - \frac{Sa_0^2}{\omega}\right)\left(\sigma\bar{m} - \frac{3Sa_0^2}{\omega}\right)\right]}{a_0\left(\sigma - \frac{Sa_0^2}{\omega}\right)} \tag{61}$$

Comparing Eqs. (61) and (60) shows that by altering σ as a controller parameter, the bifurcation point occurs when $d\sigma/da_0$ is equal to zero as shown in Figs. 12-13, below. It is also resulted by taking derivative of Eq. (53) with respect to σ

$$D_\sigma\begin{bmatrix} f_3(a,\gamma) \\ g_3(a,\gamma) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\bar{m} \end{bmatrix} \tag{62}$$

The rank of matrix resulted from multiplying jacobian matrix of Eq. (53) by matrix of Eq. (62) is equal to 2; therefore the bifurcation point is a saddle-node bifurcation point (Nayfeh and Balachandran 1995).

5. Results and discussions

Eqs. (30) and (55) show that the equilibrium solution of system at subharmonic (superharmonic) resonance depends on the values of $S, \omega, \bar{m}, \mu$ and $\Lambda_1(\Lambda_2)$. The value of S depends on the nonlinear geometric and electrostatic terms, $\bar{m}$ depends on the mass distribution, μ depends on the damping factor, and ω is natural frequency. Also, Λ_1 and Λ_2 are resulting from the terms including the AC voltage. The solution of this paper may be validated by comparing the obtained nonlinear coefficients S and Λ_1 for the microbeam using $h_p/h_b = 0$ with previous works. The validation of coefficient S shown in Fig. 2 has been performed in previous work. The comparison between the value of Λ_1 and Λ_2 obtained in this work by Galerkin method and the value obtained by Abdel-Rahman and Nayfeh (2003) using numerical shooting method is shown in Table 1. It shows an excellent agreement between the values of this paper and previous work. It must be noted that that the coefficient S is same as the values obtained by Zamanian and Khadem (2010). Here the variations of new coefficients Λ_1 and Λ_2 against to the variation of system parameters are studied. It is assumed that in all figures $\alpha_b = 3.7$, $E_p/E_b = 1$. This study is necessary for better understanding of the variations of response which is studied after this part.

Figs. 3 and 4 show the variations of Λ_1 and Λ_2 with respect to the variations of system parameters. These figures show that the variation range of Λ_1 and Λ_2 is at order 10^0. Fig. 3 shows that if the value of v_{ac} is assumed constant, then by increasing the absolute value of $\alpha_5 P_{dc}$ from $\alpha_5 P_{dc} = 0$ to $\alpha_5 P_{dc} = -2$, the value of Λ_1 increases. Also, it shows that by increasing the value of $\alpha_2 v_p^2$, from 35 into 55, the values of Λ_1 increases more, and by increasing the value of V_b from 8.7 to 20, its value decreases. Fig. 4 shows that if the value of $\alpha_2 v_p^2$ is assumed constant, then by

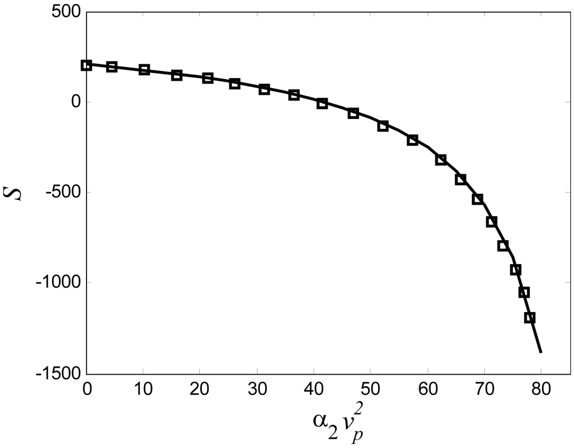

Fig. 2 Variations of S with respect to $\alpha_2 v_p^2$ for a microbeam without piezoelectric layer, N = 8.7, and a_b = 3.7; the square points belong to Younis and Nayfeh (2003), the solid lines belong to Zamanian and Khadem (2010)

Table 1 Comparison between the values of Λ_1 obtained in this paper and numerical values obtained by Abdel-Rahman and Nayfeh (2003)

Numerical method	Galerkin method	System parameters
α_b = 3.7, α_2 = 3.9, v_p = 3.08, N = 8.7, v_{ac} = 0.042, $l_2 - l_1$ = 0	Λ_1 = 2.42	Λ_1 = 2.42
α_b = 3.7, α_2 = 3.9, v_p = 3.4, N = 8.7, v_{ac} = 0.038, $l_2 - l_1$ = 0	Λ_1 = 2.52	Λ_1 = 2.52

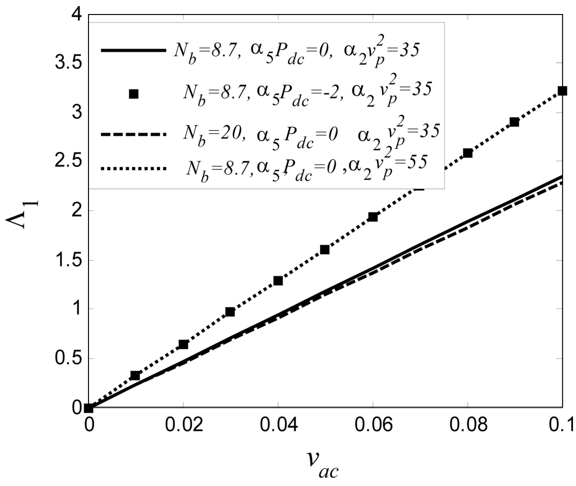

Fig. 3 Variations of Λ_1 with respect to the variations of v_{ac} for different values of system parameters, where, α_2 = 2.95, h_p/h_b = 0.1, v_{ac} = 0.02, μ = 0.04, $l_2 - l_1$ = 0.4l

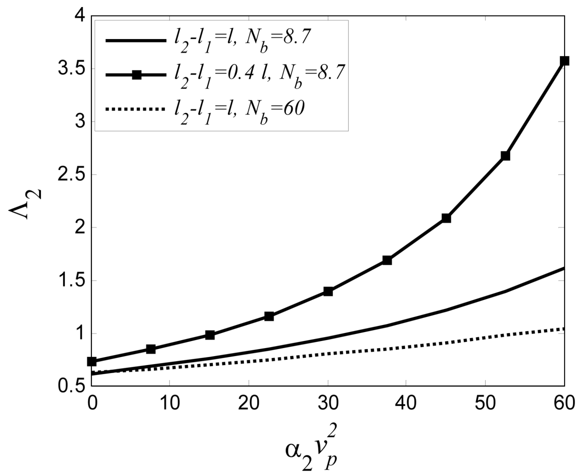

Fig. 4 Variations of Λ_2 with respect to the variations of $\alpha_2 v_p^2$ for different values of system parameters, where v_{ac} = 0.5, h_p/h_b = 0.1, μ = 0.04, α_2 = 2.95, $\alpha_5 P_{dc}$ = –2

decreasing the length of piezoelectric layer from $l_2 - l_1 = l$ to $l_2 - l_1 = 0.4l$, the value of Λ_2 increases, and by increasing the value of N_b from 8.7 to 60 its value decreases. Also, Fig. 4 shows that the variations Λ_2 against to the variations of $\alpha_2 v_p^2$ when $l_2 - l_1 = 0.4l$ are larger than its variations when $l_2 - l_1 = l$. The main reason of all above variations lies in the calculation of $\zeta_1(x)$ and $\zeta_2(x)$. Eqs. (24) and (50) show that the variations of $\zeta_1(x)$ and $\zeta_2(x)$ are mainly depend on the variations of terms including $1/(1-w_s)^n$, which have been multiplied by α_2 or $\alpha_2 v_p$. It is clear that these terms increase by increasing the value of w_s. So, the variations of system parameters that induce a large increase in the value of w_s cause a large increase in the value of Λ_1 and Λ_2. The previous work showed that if the DC electrostatic and piezoelectric actuations is not zero to zero then by increasing the value of $\alpha_2 v_p^2$ or decreasing the length of piezoelectric layer from $l_2 - l_1 = l$ to $l_2 - l_1 \cong 0.25l$, the value of w_s increases (Zamanian *et al.* 2008). They showed that when $0.25l < l_2 - l_1 < 0.8l$, then increasing the value of $\alpha_5 P_{dc}$ increases the value of w_s. Also in previous work, it has been shown that increasing the value of axial load N_b decreases the value of w_s.

The variations of equilibrium solution as a function of system parameters at subharmonic and superharmonic conditions are studied in Figs. 5 to 8. In these figures α_2 = 2.95, h_p/h_b = 0.1, μ = 0.04, and $l_2 - l_1 = 0.4l$. Figs. 5 and 6 show the variations of equilibrium points a_0 as a function of detuning parameter σ at the subharmonic condition. In the first system $\alpha_2 v_p^2$ = 35, $S > 0$ and in the second system $\alpha_2 v_p^2$ = 55, $S < 0$. These figures show that the system has two nontrivial branches which are approximately parallel. The branches are inclined to the right side for $S > 0$, and inclined to the left side for $S < 0$. It shows that the lower branch of nontrivial solution is unstable. Also, these figures show that the trivial solution is unstable for all values of σ in interval $[\sigma_1 \;\; \sigma_2]$, where $[\sigma_1 \;\; \sigma_2]$ is the distance between the cross points of trivial solution by nontrivial solution. These figures show that when $S > 0$ and $\sigma > 0$ ($S < 0$ and $\sigma < 0$), and the response of system is on the stable trivial branch $a_0 = 0$, then by decreasing (increasing) the value of σ slowly, the amount of a_0 moves to stable trivial branch until it arrives to $\sigma_2(\sigma_1)$, then it jumps to the upper nontrivial branch. It demonstrates that when $S > 0$ ($S < 0$) and the stationary solution is on the upper branch,

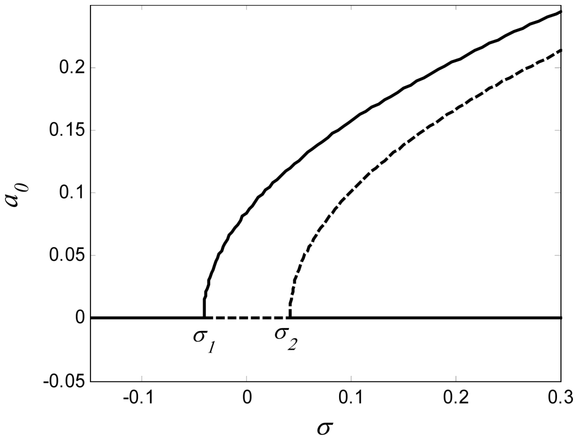

Fig. 5 Variations of a_0 with respect to the variations of σ for system with $\alpha_2 v_p^2 = 35$, v_{ac} = 0.06, $S > 0$. Solid lines belong to stable solution, and dashed lines belong to unstable solution

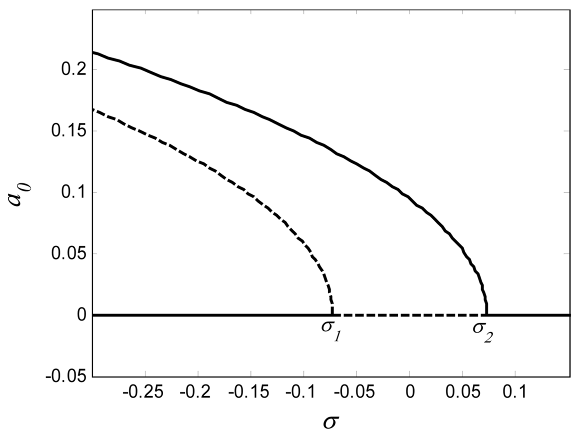

Fig. 6 Variations of a_0 with respect to the variations of σ for the system with $\alpha_2 v_p^2 = 55$, v_{ac} = 0.06, $S < 0$ Solid lines belong to stable solution, and dashed lines belong to unstable solution

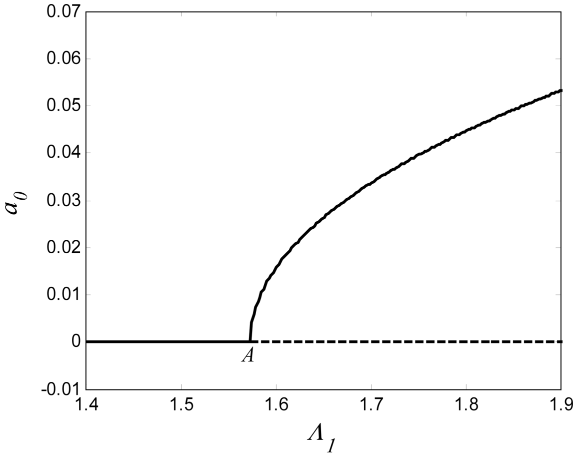

Fig. 7 Variations of a_0 with respect to the variations of Λ_1 for the system with $\alpha_2 v_p^2 = 35$ and $\sigma = -0.05$. Solid lines belong to stable solution, and dashed lines belong to unstable solution

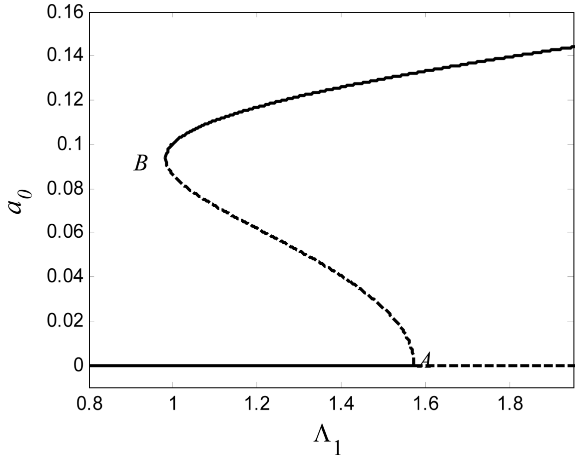

Fig. 8 Variations of a_0 with respect to the variations of Λ_1 for system with $\alpha_2 v_p^2 = 35$, $\sigma = 0.05$. Solid lines belong to stable solution, and dashed lines belong to unstable solution

then, by increasing (decreasing) σ, the value of a_0 increases until it arrives $\sigma_2(\sigma_1)$, then it may jump to the lower trivial branch. By more decreasing (increasing) of σ, the value of a_0 decreases on the upper branch.

Fig. 7 shows the variations of a_0 with respect to the variations of Λ_1 for $\sigma < 0$, $S > 0$. It shows that for values of Λ_1 smaller than Λ_A, system has only stable trivial solution, and for the values higher than Λ_A, the system has unstable trivial solution and stable nontrivial solution. Also, by considering Eq. (30), it may be resulted that a similar behavior would occur for the system with $\sigma > 0$, $S < 0$.

Fig. 8 shows that the system has only stable trivial solution at $\Lambda > \Lambda_B$, unstable nontrivial solution, stable nontrivial solution and stable trivial solution at $\Lambda_B < \Lambda < \Lambda_A$, unstable trivial solution and stable nontrivial solution at $\Lambda > \Lambda_A$. It shows that if $\Lambda < \Lambda_A$, then a_0 will be equal to zero. By increasing the value of Λ_1 until it reaches to $\Lambda = \Lambda_A$, the value of a_0 will be equal to zero. By more increase of Λ_1 from Λ_A, the value of a_0 jumps on the stable nontrivial branch, and by more increase of Λ_1 the value of a_0 increases according to stable nontrivial branch. If $\Lambda > \Lambda_A$, by decreasing the value of Λ_1 until it reaches to bifurcation point Λ_B the value of a_0 decreases on stable nontrivial branch. By more decrease of Λ from Λ_B, the value of a_0 jumps to stable trivial branch. Considering Eq. (30), it may be resulted that a similar behavior would occur for the system with $\sigma < 0$, $S < 0$.

Concisely, Figs. 5-8 show that depending on the value of parameters σ_1 and Λ_1, the response of system may be zero or a non-zero value. This behavior is useful for the design of pressure sensor, filtering system and many of MEMS devices (Younis 2011). For example in a pressure sensor when the pressure exerted on the system is changed or in the mass sensor when a mass is added to the system, then the parameters of system such as static deflection, natural frequency and finally the value of Λ_1 is changed. The subject that system shows a response or does not to this variation is depend on the value of Λ_1. It has been shown in Fig. 3 that the piezoelectric actuation and DC electrostatic actuation have a main effect on the value of Λ_1. It means that using the combination of DC electrostatic actuation and DC piezoelectric actuation, the sensitivity of system under subharmonic resonance may be tuned.

Figs. 9-13 show the effect of system parameters on the value of vibration amplitude or nonlinear shift of resonance frequency. The parameters of these figures are: $l_2 - l_1 = l$, $h_p/h_b = 0.1$, $\mu = 0.04$, $E_p/E_b = 1$, $\alpha_b = 3.7$, $\alpha_2 = 2.95$, $\alpha_5 P_{dc} = -2$ and $N_b = 8.7$.

Fig. 9 shows the effect of $\alpha_2 v_p^2$ and v_{ac} on the value of nonlinear shift of resonance frequency in superharmonic condition. It shows that by increasing the value of AC electrostatic actuation, the absolute value of nonlinear shift of resonance frequency i.e., the distance between Ω/ω and 0.5, increases. This variation may be verified by using Eq. (57) which shows that the nonlinear shift of resonance frequency directly depends on the value of v_{ac}^3. Also, Fig. 9 shows that when $\alpha_2 v_p^2 = 0$, system has hardening behavior i.e., $\Omega/\omega > 0.5$. By assuming that v_{ac} is constant, by increasing $\alpha_2 v_p^2$ to a special value (here $\alpha_2 v_p^2 \cong 30$), the value of Ω/ω increases more. It shows that by increasing the value of $\alpha_2 v_p^2$ from this special value, the hardening shift of resonance frequency decreases. When $\alpha_2 v_p^2 \cong 50$ the value of Ω/ω would be equal to 0.5, which means a linear behavior. By more increasing of $\alpha_2 v_p^2$, the value of Ω/ω alters from $\Omega/\omega > 0.5$ to $\Omega/\omega < 0.5$, which means a softening behavior. It is due to the fact that by increasing the value of $\alpha_2 v_p^2$ from zero to 30, the value of natural frequency decreases because the electrostatic actuation has a softening effect (Zamanian and Khadem 2010). Also, considering Fig. 4, the value of Λ_2 increases. Considering Eq. (57), an increase of Λ_2 and a decrease of ω, increases the value of Ω/ω, and since the value of S is positive in this region, the value of Ω/ω increases. It must be noted that the value of S decreases by increasing the value of $\alpha_2 v_p^2$, but its effect is smaller than the effect of Λ_2 and ω. By more increasing the value of $\alpha_2 v_p^2$, the effect of S would be more affected, and so, Ω/ω decreases. For the value of $\alpha_2 v_p^2$ that S is equal to zero, the value of Ω/ω would be 0.5. If the parameter of $\alpha_2 v_p^2$ increases more, then the value of S changes to a negative value. So the value of Ω/ω decreases to smaller values than 0.5. Figs. 10 and 11 show the effect of N_b and $\alpha_5 P_{dc}$ on the nonlinear shift of resonance frequency of the system. It shows that by increasing N_b or $\alpha_5 P_{dc}$, the nonlinear shift of resonance frequency decreases, i.e., the linear behavior of system increases. It means that if the

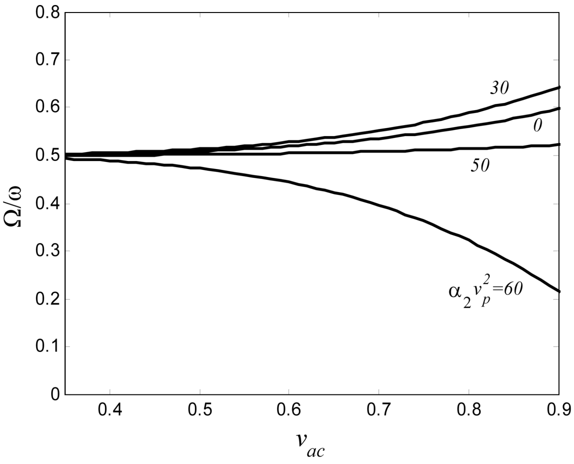

Fig. 9 The effect of v_{ac} and $\alpha_2 v_p^2$ on the nonlinear shift of resonance frequency

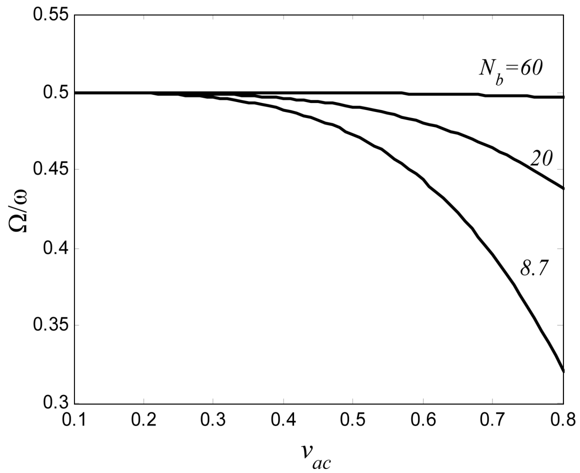

Fig. 10 Variations of Ω/ω with respect to the variations of v_{ac} for different values of N_b, where $\alpha_2 v_p^2 = 60$

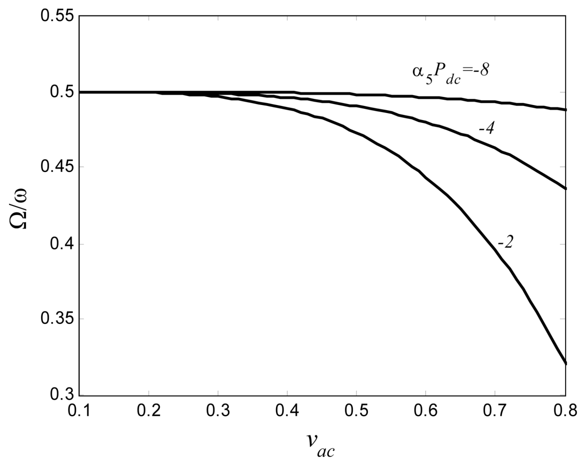

Fig. 11 Variations of Ω/ω with respect to v_{ac} for different values of $\alpha_5 P_{dc}$, where $\alpha_2 v_p^2 = 60$

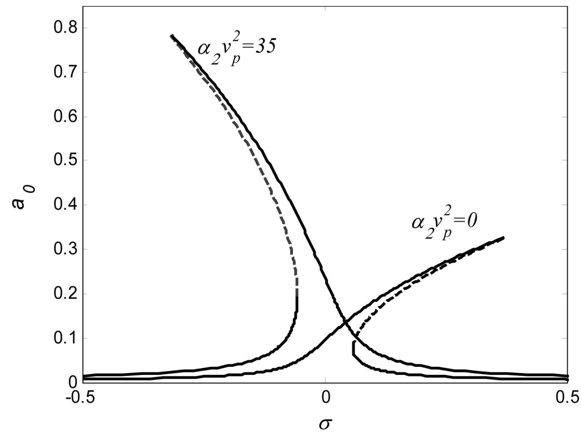

Fig. 12 Variations of a_0 with respect to the variations of σ for different values of $\alpha_2 v_p^2$ where v_{ac} = 0.2. Solid lines belong to stable solution, and dashed lines belong to unstable solution

length of piezoelectric layer is equal to the length of microbeam, then, by applying the DC voltage to the piezoelectric layer, the linear behavior of system increases. It is due to the fact that if the length of piezoelectric layer is equal to the length of microbeam, then, by increasing $\alpha_5 P_{dc}$, the axial effect of piezoelectric layer i.e., $\alpha_3 P_{dc}$ increases where the bending effect of piezoelectric layer is equal to zero (Zamanian *et al.* 2008). It has been shown by Zamanian *et al.* (2008) that by increasing N_b or $\alpha_3 P_{dc}$, the natural frequency of system increases, and so, the nonlinear shift of resonance frequency decreases.

Fig. 12 shows the variations of a_0 with respect to the variations of σ for different values of $\alpha_2 v_p^2$. It shows that for $S < 0$, the unstable solution occurs on the domain of $\sigma > 0$, and for $S < 0$, it occurs on the domain of $\sigma < 0$. It demonstrates that if $S > 0$ and $\sigma < 0$ ($S < 0$ and $\sigma > 0$), by increasing (decreasing) the value of σ slowly, the value of a_0 moves to the upper branch until it reaches to the saddle-node bifurcation point, then it jumps to the lower branch. It demonstrates that when $S > 0$ ($S < 0$) and the stationary solution is on the lower branch, then, by decreasing (increasing) σ, the value of a_0 increases until it reaches to saddle-node bifurcation point then it jumps to the upper branch. By more decreasing (increasing) of σ, the value of a_0 decreases on the upper branch. Also, Fig. 12 shows that when $\alpha_2 v_p^2 = 0$, then system under superharmonic resonance has a nonzero response. It must be noted that at this condition the response of system under primary resonance is equal to zero (Zamanian *et al.* 2010). The reason is the difference between the amplitude of secular terms at superharmonic and primary resonances. In the former, the secular term has an amplitude of v_{ac}^2 which is not depended on $\alpha_2 v_p^2$, where in the later, it depends on $\alpha_2 v_p v_{ac}$ and so it has a zero amplitude for $\alpha_2 v_p^2 = 0$.

It must be noted that in many conditions, it is needed for a system to oscillate about a static deflection (Li *et al.* 2006, Younis and Nayfeh 2003). This study demonstrates that using the proposed configuration when DC electrostatic actuation is equal to zero, the system can oscillate about static deflection due to the piezoelectric actuation. In this condition, the natural frequency of system about static deflection is larger than the natural frequency of vibration of straight microbeam

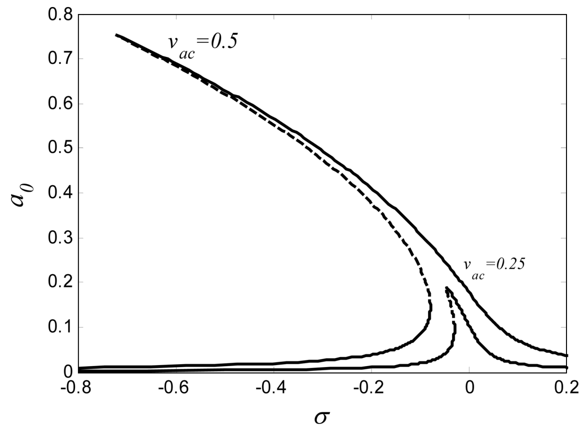

Fig. 13 Variations of a_0 with respect to to the variations of σ for different values of v_{ac} where $\alpha_2 v_p^2$ = 60. Solid lines belong to stable solution, and dashed lines belong to unstable solution

(Zamanian *et al.* 2008). It means that one can make a system that oscillates about static deflection by a resonance frequency higher than natural frequency of straight microbeam. When DC piezoelectric actuation is equal to zero then system can oscillate about static deflection due to the electrostatic actuation. In this condition, the natural frequency of system about static deflection is lower than the natural frequency of vibration of straight microbeam (Abdel-Rahman *et al.* 2002). It means, the system oscillates about static deflection by a resonance frequency lower than natural frequency of straight microbeam. Using the combination of DC electric and piezoelectric actuation, one has a system with two above advantages.

Fig. 13 shows the variation of equilibrium solution a_0 against to the variation of detuning parameters σ for different values of v_{ac}. It shows that by increasing the value of v_{ac}, the shift of resonance frequency and amplitude increases. It is due to the fact that by increasing the value of v_{ac}, the value of $S, \omega, \bar{m}$ remains constant, where considering Fig. 3 the value of Λ_2 increases.

6. Conclusions

In this article the secondary resonances of a clamped-clamped microresonator under AC-DC electrostatic and DC piezoelectric actuations has been studied. It was shown that system has subharmonic and superharmonic resonances. In addition, there are four competing terms that control the amplitude of equilibrium solution and the value of nonlinear shift of the resonance frequency. Theses terms are the nonlinear coefficient S, the natural frequency ω, the value of Λ_1, Λ_2, and the value of $\bar{m}$. It was observed that the microresonator under subharmonic and superharmonic resonances may oscillate about the static deflection due to the piezoelectric actuation. Also, it was shown that using the DC piezoelectric actuation, the sensitivity of AC-DC electrically actuated microresonator at subharmonic and superharmonic resonances may be tuned. This property may be used in the design of filtering system and pressure sensors. This study demonstrates that using the proposed configuration when DC electrostatic actuation is equal to zero, the system can oscillate about static deflection by a resonance frequency higher than natural frequency of straight microbeam, and when DC piezoelectric actuation is equal to zero, then the system can oscillate by a resonance frequency higher than natural frequency of straight microbeam. Using the combination of DC electric and piezoelectric actuation one has a system with two above advantages.

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