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Elastodynamic infinite elements based on modified Bessel shape functions, applicable in the finite element method

K.S. Kazakov*

Department of Structural Mechanics, VSU, Sofia, Bulgaria

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Abstract. In this paper decay and mapped elastodynamic infinite elements, based on modified Bessel shape functions and appropriate for Soil-Structure Interaction problems are described and discussed. These elements can be treated as a new form of the recently proposed Elastodynamic Infinite Elements with United Shape Functions (EIEUSF) infinite elements. The formulation of 2D horizontal type infinite elements (HIE) is demonstrated, but by similar techniques 2D vertical (VIE) and 2D corner (CIE) infinite elements can also be formulated. It is demonstrated that the application of the elastodynamical infinite elements is the easier and appropriate way to achieve an adequate simulation including basic aspects of Soil-Structure Interaction. Continuity along the artificial boundary (the line between finite and infinite elements) is discussed as well and the application of the proposed elastodynamical infinite elements in the Finite Element Method is explained in brief. Finally, a numerical example shows the computational efficiency of the proposed infinite elements.

Keywords: soil-structure interaction; wave propagation; infinite elements; finite element method; Bessel functions

1. Introduction

In static SSI analysis, the simple truncation of the far field with setting of appropriate boundary conditions gives very often-good results. However, in dynamic cases, an artificial boundary made by truncation makes results to be erroneous because of reflection waves. In last decades, much works has been done on unbounded domain problems and several kinds of modeling techniques have been developed to avoid these effects. Such techniques are viscous boundary, transmitting boundary, boundary elements, infinite elements and system identification method. At the same time several numerical methods for these types of problems were suggested. The basic idea of these approaches is to divide domain Ω into two parts the bounded part Ω_c and unbounded part Ω_∞, where for the first one is valid $x_i \le c_i$. For appropriate simulations we need to set the assumption that function $u(x_i) = 0$ on Ω_∞.

Among these approaches, using infinite elements is good way to solve Soil-Structure Interaction problems since its concept and formulation are similar to those of Finite element method except for

*Corresponding author, Professor, E-mail: kskazakov@yahoo.com

the infinite extent of the element region and shape function in one direction and there is no loss of symmetry of the element matrices. The domain Ω_∞ is partitioned into a finite number of infinite elements directly incorporated with the meshes on the bounded domain Ω_c. In the numerical models these domains very often have called near (Ω_c) and far (Ω_∞) fields.

Infinite elements can be classified into five types: classical, decay, mapped, elastodynamical and envelope infinite elements (Kazakov 2009).

2. Backgrounds for infinite elements

Infinite elements are widely used in the numerical simulations when unbounded domain exists. The origin of these elements is the works (Bettess 1978, Ungless 1973). Classification of the infinite elements is proposed in Kazakov (2009). During the last three decades much element formulations have been suggested (Aubry *et al.* 2003, Genes and Kocak 2002, Kazakov 2009, Yan *et al.* 2000, Zhao and Valliappan 1993).

Soil-Structure Interaction (SSI) is a typical civil engineering problem (Bathe 1982, Basu and Chopra 2002, Fang and Brown 1995, Luco and Westmann 1972, Madabhushi 1996, Oh and Jou 2001, Pradhan *et al.* 2003, Todorovski *et al.* 2000, Tzong and Penzien 1986). The early history of SSI is summarized in work of (Kausel 2010).

The infinite elements can be integrated in the Finite element method codes (Kazakov 2010, Madabhushi 1996, Wolf and Song 1996, Wolf 1988) adequately dynamic SSI simulations to be obtained. The infinite elements as a computational technique is one of the often used since their concepts and formulations are much closed to those of the finite elements. These elements are very effective for models of structures containing a near field discretized by finite elements and a far field discretized by infinite elements. In the last two decades a lot of dynamic infinite elements were developed. Yan *et al.* (2000) proposes one of the most effective elastodynamical elements, the concept of which is used in the EIEUSF formulations. A novel numerical model for unbounded soil domain using periodic infinite elements is proposed in Bagheripour and Marandi (2005), and using infinite elements in the wavelet theory in Bagheripour (2010). Finite and infinite elements can be used in static analysis of different civil engineering structures (Patil 2010). The influence of the boundary conditions on SSI models can be seen in Wang (2005).

3. Elastodynamical infinite element with united shape functions (EIEUSF)

The displacement field in the elastodynamical infinite element can be described in the standard form of the shape functions based on wave propagation functions (Kazakov 2005, 2010) as

$$\mathbf{u}(x,z,\omega) = \sum_{i=1}^{n}\sum_{q=1}^{m} N_{iq}(x,z,\omega)\mathbf{p}_{iq}(\omega), \quad \text{or} \quad \mathbf{u}(x,z,\omega) = N_p(x,z,\omega)\mathbf{p}(\omega) \tag{1}$$

where $N_{iq}(x,z,\omega)$ are the standard shape displacement functions, $\mathbf{p}_{iq}(\omega)$ is the generalized coordinates associated with $N_{iq}(x,z,\omega)$, n is the number of nodes for the element and m is the number of wave functions included in the formulation of the infinite element. For horizontal wave propagation basic shape functions for the *HIE* infinite element, the local coordinate system of which

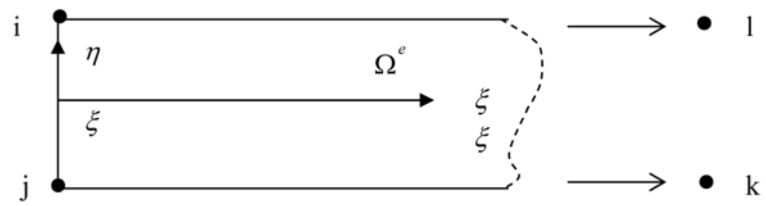

Fig. 1 Local coordinate system of horizontal infinite elements (HIE)

is given in Fig. 1, can be expressed as

$$N_{iq}(x, z, \omega) = T(x, z, \eta, \xi) N_{iq}(\eta, \xi, \omega) = T(x, z, \eta, \xi) L_i(\eta) W_q(\xi, \omega) \tag{2}$$

where $W_q(\xi, \omega)$ are horizontal wave functions and $L_i(\eta)$ are Lagrange interpolation polynomial which has unit value at *i*th node while zeros at the other nodes. For HIE infinite element the ranges of the local coordinates are: $\eta \in [-1, 1]$ and $\xi \in [0, \infty)$. Here $T(x, z, \eta, \xi)$ assures the geometrical transformations of local to global coordinates.

Taking into account only real parts of the wave functions the equations of the wave propagation can be written as

$$\mathrm{Re} W_q(\xi, \omega) = \cos\left(\frac{\omega}{c_s}\xi\right) e^{-\alpha\xi} \quad \text{or} \quad \mathrm{Re} W_q(\xi, \omega) = \cos\left(\frac{\omega}{c_p}\xi\right) e^{-\alpha\xi} \tag{3}$$

where c_s, c_p are the wave velocities for *S*-waves and *P*-waves respectively, and α is appropriate constant, called attenuation factor.

Expanding this functions in a Fourier-like series for all wave functions included in the formulation of the infinite element the shape functions for *HIE* can be written as

$$\mathrm{Re} W(\xi) = \frac{1}{m}\sum_{q=1}^{m} A_q \cos\left(\frac{\omega_q}{c_s}\xi\right) e^{-\alpha\xi} \quad \text{or} \quad \mathrm{Re} W(\xi) = \frac{1}{m}\sum_{q=1}^{m} A_q \cos\left(\frac{\omega_q}{c_p}\xi\right) e^{-\alpha\xi} \tag{4}$$

The coefficients A_q can be written as

$$A_q = \int_0^{T_x} \mathrm{Re} W(\xi, t) \cos\left(\frac{\omega_q}{c_s}\xi\right) dt \quad \text{or in the form} \quad A_q = \frac{1}{\Omega_e}\int_0^{\Omega_e} \mathrm{Re} W(\xi, t) \cos\left(\frac{\omega_q}{c_s}\xi\right) dt \tag{5}$$

Because, m is a finite number and A_q can be treated as weight coefficients, so that $\sum_{q=1}^{m} A_q = 1$ for shape functions than Eq. (4) can be expressed as

$$\mathrm{Re} W(\xi) = \frac{1}{m}\sum_{q=1}^{m}\sum_{n=1}^{\infty} 1 \cos\left(\frac{\omega_q}{c_s}\xi\right) e^{-\alpha\xi}, \quad \mathrm{Re} W(\xi) = \frac{1}{m}\sum_{q=1}^{m}\sum_{n=1}^{\infty} 1 \cos\left(\frac{\omega_q}{c_p}\xi\right) e^{-\alpha\xi} \tag{6}$$

Using this approach can be written

$$N_i(x, z) = \sum_{q=1}^{m} N_{iq}(x, z, \omega) = T(x, z, \eta, \xi) L_i(\eta) \mathrm{Re} W(\xi) \tag{7}$$

and

$$N_i(x,z)\mathbf{p}_i = \sum_{q=1}^{m} N_{iq}(x,z,\omega)\mathbf{p}_{iq}(\omega) = T(x,z,\eta,\xi)L_i(\eta)\mathrm{Re}W(\xi)\mathbf{p}_i \tag{8}$$

Then Eq. (1) can be expressed as

$$\mathbf{u}(x,z) = N_p(x,z)\mathbf{p} \tag{9}$$

The procedure described by the above equations can be treated as a superposing procedure based on a finite number of terms, where real components of the wave functions $\mathrm{Re}W_q(\xi,\omega)$ are preliminary shape functions or basis functions from mathematical point of view, and coefficients A_q are generalized coordinates with only one component, corresponding to the node i or weight coefficients from mathematical point of view.

4. Element shape functions, based on Bessel functions

The idea and concept of the EIEUSF class infinite elements are presented in (Kazakov 2005, 2009, 2010). Several EIEUSF formulations are discussed and have been demonstrated that the shape functions, related to nodes k and l (the nodes, situated in infinity, Fig. 1) are not necessary to be constructed, because corresponding to these shape functions generalized coordinates or weights, see Eq. (1), are zeros. The displacements in infinity are vanished, and these shape functions must be omitted.

For horizontal wave propagation the basic shape functions for the *HIE* infinite element can be expressed using Bessel functions as

$$N_{iq}(\eta,\xi,\omega) = L_i(\eta)\tilde{J}_0^q(\psi\xi) \tag{10}$$

where $\tilde{J}_0^q(\psi\xi)$ are modified Bessel functions of first kind. These functions can be written as

$$\tilde{J}_0^q(\psi\xi) = J_0^q(\psi\xi)\exp(-\beta\xi) \tag{11}$$

where $J_0^q(\psi\xi)$ are standard Bessel functions of first kind. In Eq. (11) ψ and β are constants, chosen in such a way that the length of the wave and the attenuation of the wave respectively, are identical with those, if Eq. (2) is used. This means that the following two relations are valid

$$\psi = \frac{\omega}{\varpi} \tag{12}$$

or

$$\psi = \frac{L_w}{\bar{L}_w} \tag{13}$$

where ϖ is the wave frequency corresponding to ω if Bessel functions are used to approximate the displacements in the infinite element domain, and

$$\exp(-\beta\xi) = \frac{1}{\sqrt{\xi}}\exp(-\alpha\xi) \tag{14}$$

because the Bessel functions of first kind decay proportionally to $1/\sqrt{\xi}$. Although the roots of Bessel functions are not generally periodic, except asymptotically for large ξ, such functions give acceptable results. And what is more, using Bessel functions one can approximate change of the wave length in the far field region.

If the element has four nodes and eight DOF only four shape functions can be used to approximate the displacements, related to one frequency. These functions can be written as

$$N_{1q}(\eta, \xi, \omega) = N^{u}_{iq}(\eta, \xi, \omega) = L_i(\eta)J^{q}_{0}(\psi\xi)\exp(-\beta\xi) \tag{15.a}$$

$$N_{2q}(\eta, \xi, \omega) = N^{v}_{iq}(\eta, \xi, \omega) = L_i(\eta)J^{q}_{0}(\psi\xi)\exp(-\beta\xi) \tag{15.b}$$

and

$$N_{3q}(\eta, \xi, \omega) = N^{u}_{jq}(\eta, \xi, \omega) = L_j(\eta)J^{q}_{0}(\psi\xi)\exp(-\beta\xi) \tag{16.a}$$

$$N_{4q}(\eta, \xi, \omega) = N^{v}_{jq}(\eta, \xi, \omega) = L_j(\eta)J^{q}_{0}(\psi\xi)\exp(-\beta\xi) \tag{16.b}$$

In the above equations, Eq. (15.a) is identical to Eq. (15.b) and Eq. (16.a) is identical to Eq. (16.b). If rotational DOF are used then the element has four nodes and ten DOF. Two additional shape functions must be used, written as

$$N_{5q}(\eta, \xi, \omega) = N^{\varphi}_{iq}(\eta, \xi, \omega) = L_j(\eta)[J^{q}_{1}(\psi\xi)\exp(-\beta\xi) - \beta J^{q}_{0}(\psi\xi)\exp(-\beta\xi)] \tag{17.a}$$

and

$$N_{6q}(\eta, \xi, \omega) = N^{\varphi}_{jq}(\eta, \xi, \omega) = L_j(\eta)[J^{q}_{1}(\psi\xi)\exp(-\beta\xi) - \beta J^{q}_{0}(\psi\xi)\exp(-\beta\xi)] \tag{17.b}$$

Here $J^{q}_{0}(\psi\xi)$ and $J^{q}_{1}(\psi\xi)$ are Bessel functions of first kind. The Taylor series indicates that $J^{q}_{1}(\psi\xi)$ is the derivative of $J^{q}_{0}(\psi\xi)$.

The function $L_j(\eta)$ is linear if no mid-nodes. Finally, if mid-node on the side *i-j* is used, then the Lagrange interpolation polynomials must be quadratic. Modified Bessel functions of first kind, in

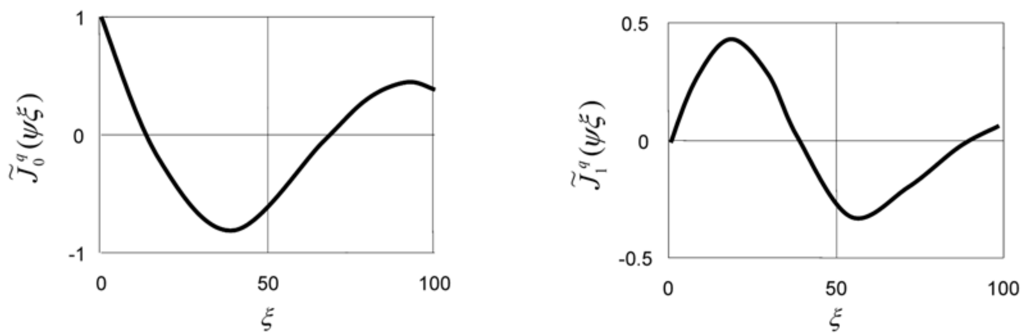

Fig. 2 $\tilde{J}^{q}_{0}(\psi\xi)$ and $\tilde{J}^{q}_{1}(\psi\xi)$ modified Bessel functions accordance with Eq. (11) ($\tilde{J}_0^q(\psi\xi)$ and $\tilde{J}_0^q(\psi\xi)$), are illustrated in Fig. 2.

The continuity along the artificial boundary (the line between finite and infinite elements, see Fig. 3 line $-x_b$ and line x_b) is assured in the same way as between two plane finite elements (Kazakov 2008). The application of the proposed infinite elements in the Finite element method is discussed below.

Using the procedure, given in details in Kazakov (2005) and briefly described here, mapped EIEUSF finite elements, based on modified Bessel functions, can be formulated, based on Eq. (18)

$$N_i(x,z) = \sum_{q=1}^{m} N_{iq}(x,z,\omega) = \sum_{q=1}^{m} T(x,z,\eta,\xi)N_{iq}(\eta,\xi,\omega) = \sum_{q=1}^{m} T(x,z,\eta,\xi)L_i(\eta)\tilde{J}_0^q(\psi\xi) \tag{18}$$

where $\tilde{J}_0^q(\psi\xi) = J_0^q(\psi\xi)\exp(-\beta\xi)$.

5. Stiffness and mass matrices of the element

By analogy with EIEUSF (Kazakov 2010) and since each shape function $N_{iq}(\eta,\xi,\omega)$ is associated with only one frequency $\mathbf{p}_{iq}(\omega)$ is a generalized coordinate involving a single wave component only. Then the component matrices k_{iq} and m_{iq} can be written as

$$k_{iq} = \int_{\Omega_e} \bar{B}_i^T D \bar{B}_q d\Omega_e$$

and

$$m_{iq} = \left(\int_{\Omega_e} \bar{N}_i^T N_q d\Omega_e\right) I \tag{19}$$

where $\bar{B}_i = [\partial](\bar{N}_i) = [\partial](L_i W)$; $[\partial]$ is a linear differential operator matrix. If Bessel functions are used, the first derivative of $J_0^q(\psi\xi)$ is $J_1^q(\psi\xi)$ (The Taylor series indicate that $J_1^q(\psi\xi)$ is derivative of $J_0^q(\psi\xi)$) and can be expressed as $\frac{d}{d\xi}J_0^q(\psi\xi) = (J_{-1}^q(\psi\xi) + J_1^q(\psi\xi))/2$.

6. Equation of motion of the entire system

The equation of motion for the whole Soil-Structure Interaction system including far field soil region can be written as

$$\begin{bmatrix} S_{ss}(\omega) & S_{sb}(\omega) \\ S_{bs}(\omega) & S_{bb}(\omega) + S_{bb}^g(\omega) \end{bmatrix} \begin{Bmatrix} U_s(\omega) \\ U_b(\omega) \end{Bmatrix} = \begin{Bmatrix} F_s(\omega) \\ F_b(\omega) \end{Bmatrix} \tag{20}$$

where $U(\omega), F(\omega)$ and $S(\omega)$ are respectively displacement vector, force vector and dynamic stiffness matrix in frequency domain. Subscripts s and b stand for the nodes along the artificial boundary between the near and the far field soil region and for those of the structure and near field soil region respectively. This equation can be transformed into time domain by inverse Fourier transformation as

$$\begin{bmatrix} M_{ss} & M_{sb} \\ M_{bs} & M_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{u}_s(t) \\ \ddot{u}_b(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S_1^g \end{bmatrix} \begin{Bmatrix} \dot{u}_s(t) \\ \dot{u}_b(t) \end{Bmatrix} + \begin{bmatrix} K_{ss} & K_{sb} \\ K_{bs} & K_{bb} + S_0^g \end{bmatrix} \begin{Bmatrix} u_s(t) \\ u_b(t) \end{Bmatrix}$$
$$= \begin{Bmatrix} f_s(t) \\ f_b(t) - \int_0^t \{ S_2^g + (t-\tau) S_3^g (-a(t-\tau)) u_b(\tau) d\tau \} \end{Bmatrix} \tag{21}$$

where $u(t)$ and $f(t)$ are respectively displacement and force vectors, and S_j^g are mechanical characteristics of the far field soil region.

7. Numerical example

Structure with rigid strip foundation on a homogeneous half-space is modeled as shown in Fig.3. Four models of the far field are used, briefly described as:

- *model 1* - elastic springs with stiffness k_b^2, calculated using the Gorbunov-Possadov relation (Kazakov 2009) modified in accordance with the mesh as $k_b^2 = E_b \cdot l_\eta (1-\upsilon)/d \cdot (1+\upsilon)(1-2\upsilon)$;
- *model 2* - elastic springs with stiffness k_b^1, calculated using the Tsitovich relation (Kazakov 2009), modified in accordance with the mesh as $k_b^1 = E_b \cdot l_\eta / 0.87 \cdot d \cdot (1-\upsilon^2)$, where l_η is the element size and b = 1- thickness of the element;
- *model 3* - the far field is desctitezed by massless EIEUSF infinite elements with only one frequency (single wave component);
- *model 4* - the far field is desctitezed by massless infinite elements with Bessel shape functions.

The stiffness matrices of the infinite elements, used in models *model 3* and *model 4* are calculated by *EIEUSF matrix module*, leading to Eq. (22) and Eq. (23). Horizontal harmonic displacements

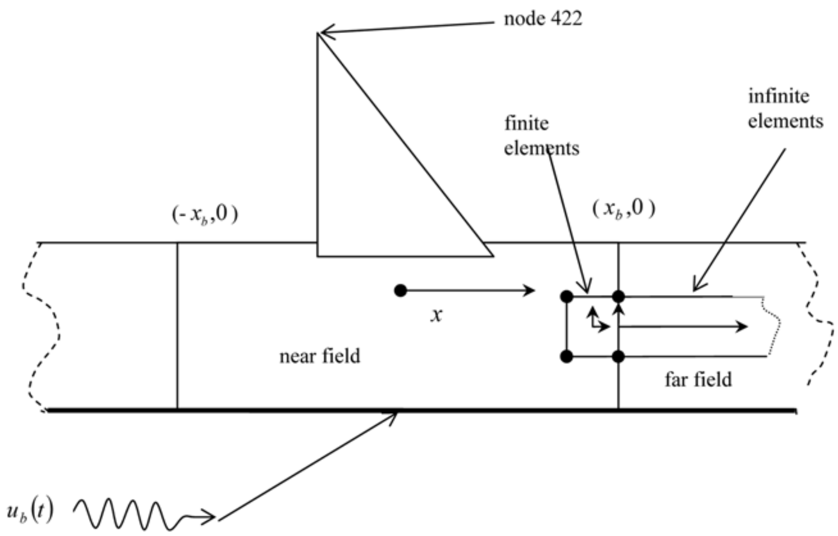

Fig. 3 Computational model

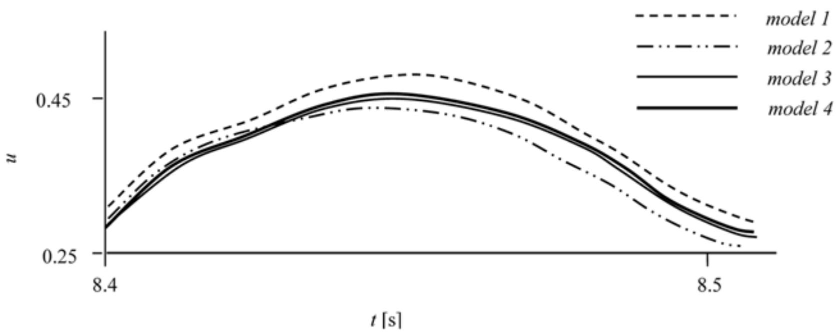

Fig. 4 Time history of the displacements of node 422

Table 1 Natural periods of vibration and max displacement of node 422

Models	model 1	model 2	model 3	model 4
natural periods of vibration	1.2608	1.2598	1.2606	1.2606
	0.7215	0.7215	0.7215	0.7215
	0.6105	0.5946	0.6052	0.6068
	0.5680	0.4359	0.5514	0.5523
	0.5448	0.4185	0.5152	0.5171
	0.3774	0.3515	0.3716	0.3722
	0.3635	0.3514	0.3575	0.3586
	0.3521	0.3402	0.3511	0.3514
	0.3521	0.3394	0.3506	0.3509
	0.3348	0.3123	0.3338	0.3329
max displacement [m]	**0.4679**	**0.4359**	**0.4518**	**0.4522**

with period $T_\theta = 0.8$ s and amplitude $u_b^{\max} = 0.1$ m are applied on the nodes as shown in Fig. 3.

The results for the first 10 natural periods, corresponding to the models and max displacement of node 422, are given in Table 1. The time history of the displacements of node 422, see Fig. 3, between 8.4 s and 8.5 s are illustrated in Fig. 4.

$$[k]_{IE\_m3} = \begin{bmatrix} 3.1278 & 1.5129 & -3.1278 & -1.5129 \\ 1.5129 & 3.1278 & -1.5129 & -3.1278 \\ -3.1278 & 1.5129 & 3.1278 & -1.5129 \\ -1.5129 & -3.1278 & -1.5129 & 3.1278 \end{bmatrix} \cdot 10^5 \tag{22}$$

$$[k]_{IE\_m4} = \begin{bmatrix} 3.0258 & 1.1159 & -3.0258 & -1.1159 \\ 1.1159 & 3.0258 & -1.1159 & -3.0258 \\ -3.0258 & 1.1159 & 3.0258 & -1.1159 \\ -1.1159 & -3.0258 & -1.1159 & 3.0258 \end{bmatrix} \cdot 10^5 \tag{23}$$

8. Conclusions

In this paper a formulation of elastodynamical infinite element, based on Bessel shape functions appropriate for Soil-Structure Interaction problems is presented. This element is a new form of the infinite element, given in (Kazakov 2005). The base of the development is new shape functions, obtained by modification of the standard Bessel functions of first kind. The stiffness matrices of the examined infinite elements are calculated by *EIEUSF matrix module*, developed by the same author.

The numerical example shows the computational efficiency of the proposed infinite elements. Such elements can be directly used in the FEM code. The results are in a good agreement with the results, obtained by EIEUSF infinite elements.

The formulation of 2D horizontal type infinite elements (*HIE)* is demonstrated, but by similar techniques 2D vertical (*VIE*) and 2D corner (*CIE*) infinite elements can also be formulated. It was demonstrated that the application of the elastodynamical infinite elements is the easier and appropriate way to achieve an adequate simulation (2D elastic media) including basic aspects of Soil-Structure Interaction. Continuity along the artificial boundary (the line between finite and infinite elements) is discussed as well and the application of the proposed elastodynamical infinite elements in the Finite element method is explained in brief.

References

Aubry, D., Clouteau, D. and D'Azemar, P. (2003), "A dynamic substructure approach to soil-structure interaction", *Computat. Mech. Publ.*, **3**, Springer Verlag.

Bagheripour, M.H., Rahgozar, R. and Malekinejad, M. (2010), "Efficient analysis of SSI problems using infinite elements and wavelet theory", *Geothech. Eng.,* **2**(4), 229-252.

Bagheripour, M.H. and Marandi, S.M. (2005), "A numerical model for unbounded soil domain in earthquake SSI analysis using periodic infinite elements", *Int. J. Civil Eng*., **3**(2), 96-111.

Basu, U. and Chopra A.K. (2002), "Numerical evaluation of the damping-solvent extraction method in the frequency domain", *Earthq. Eng. Struct. Dyn.*, **31**(6), 1231-1250.

Bathe, K.J. (1982), *Finite Element Procedures in Engineering Analysis*, New Jersey, Prentice-Hill.

Bettess, P. (1978), "Infinite elements", *Int. J. Numer. Meth. Eng.*, **11**, 54-64.

Fang, K. and Brown, R. (1995), "Numerical simulation of wave propagation in anisotropic media", *CREWES Research Report*, **7**, 101-122.

Genes, M.C. and Kocak, S. (2002), "A combined finite element based soil-structure interaction model for large-scale system and applications on parallel platforms", *Eng. Struct.*, **10**, 154-172.

Kausel, E. (2010), "Early history of soil-structure interaction", *J. Soil Dyn. Earthq. Eng*., **30**, 822-832.

Kazakov, K.S. (2008), "Mapping functions for 2D elastodynamic infinite element with united shape function", *Slovak Journal of Civil Engineering, Bratislava*, **XVI**(4), 17-25.

Kazakov, K.S. (2009), *The Finite Element Method for Structural Modeling*, Bulgarian Academy of Science (BAS) Publishing House "Prof. Marin Drinov", Second Edition.

Kazakov, K.S. (2010), *Infinite Elements in the Finite Element Method*, VSU Publishing House, Third Edition.

Kazakov, K. (2005), "On an elastodynamic infinite element, appropriate for an soil-structure interaction models", *Proceedings of 10th Jubilee National Congress on Theoretical and Applied Mechanics*, BAS, Varna, September.

Kazakov, K. (2009), "Stiffness and mass matrices of FEM-applicable dynamic infinite element with unified shape basis", *American Institute of Physics, Current Themes in Engineering Science 2008*, Selected Papers at the World Congress on Engineering, England, London.

Kazakov, K. (2010), "Elastodynamic infinite elements with united shape function for soil-structure interaction", *Finite Elem. Analy. Des.*, **46**, 936-942.

Luco, J.E. and Westmann, R.A. (1972), "Dynamic response of a rigid footing bonded to an elastic half space", *J. Appl. Mech.-ASME*, **18**, 92-108.
Madabhushi, S.P.G. (1996), Modeling of Deformations in Dynamic Soil-Structure Interaction Problems, VELACS, Technical Report TR277, Cambridge University, England.
Oh, H.S. and Jou, Y. Ch. (2001), The Weighted Riesz-Galerkin Method for Elliptic Boundary Value Problems on Unbounded Domain, NC 28223-0001.
Patil, V.A., Sawant, V.A. and Kousik, D. (2010), "Use of finite and infinite elements in static analysis of pavement", *Interact. Multis. Mech.*, **3**(1), 95-110.
Pradhan, P.K., Baidya, D.K. and Ghosh, D.P. (2003), "Impedance functions of circular foundation resting on soil using cone model", *EJGE*, **6**, 67-80.
Todorovski, L.I., Andersen, G.R. and Likos, W.J. (2000), A New Approach to the Physical Modeling of Dynamic Soil-Structure Interaction, www.ce.jhu.edu.
Tzong, T.J. and Penzien, J. (1986), "Hybrid-modeling of a single layer half-space system in soil-structure interaction", *Earthq. Eng. Struct. D.*, **14**, 92-108.
Ungless, R.F. (1973), "Infinite elements", M.A. Sc. Dissertation, University of British Columbia.
Wang, J. (2005), "Influence of different boundary conditions on analysis of SSI", *Proceedings of 18th International Conference on Structural Mechanics in Reactor Technology (SMIRT 18),* Beijing, August.
Wolf, J.P. and Song, C. (1996), *Finite-element Modeling of Unbounded Media*, England, Wiley.
Wolf, J.P. (1988), *Soil-Structure Interaction Analysis in a Time Domain*, Englewood Cliffs, Prentice-Hill, N.J.
Yan, Ch.B., Kim, D.K. and Kim, J.N. (2000), "Analytical frequency-dependent infinite elements for soil-structure interaction analysis in a two-dimensional medium", *Eng. Struct.,* **22**, 258-271.
Zhao, Ch. and Valliappan, S. (1993), "A dynamic infinite element for three-dimensional infinite domain wave problems", *Int. J. Numer. Meth. Eng.*, **36**, 2567-2580.
Zienkiewicz, O.C., Bando, K., Bettess, P., Emson, C. and Chiam, T.C. (1985), "Mapped infinite elements for exterior wave problems", *Int. J. Numer. Meth. Eng.*, **21**, 1229-1251.