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An Extended Force Density Method for the form finding of cable systems with new forms

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Abstract. The Force Density Method (FDM) is a well known and extremely versatile tool in form finding of cable nets. In its linear formulation such method makes it possible to find all the possible equilibrium configurations of a net of cables having a certain given connectivity and given boundary conditions on the nodes. Each singular configuration corresponds to an assumed force density distribution. Its improvement as Non-Linear Force Density Method (NLFDM) introduces the possibility of imposing assigned relative distances among the nodes, the tensile level in the elements and/or their initial undeformed length. In this paper an Extended Force Density Method (EFDM) is proposed, which makes it possible to set conditions in terms of given fixed nodal reactions or, in other words, to fix the positions of a certain number of nodes and, at the same time, to impose the intensity of the reaction force. Through such extension, the (EFDM) enables us to deal with form findings problems of cable nets subjected to given constraints and, in particular, with mixed structures, made of cables and struts. The efficiency and the robustness of method are assessed through comparisons with other form finding techniques in dealing with characteristic applications to the prestress design of cable systems. As a further extension, the EFDM is applied to structures having some parts not yet geometrically defined, as can happen in designing new creative forms.

Keywords: cable structures; form finding; force finding; force density method; new structural forms

1. Introduction

Cable structures differ from the conventional ones for their lightness and for the versatility of their shapes. As they work only by axial tensile forces, the structural geometry and the pretensioning intensity applied to the cables are closely related. Being the geometry depending on the relationship between form and forces, it is impossible a direct design of such structures, as it happens in the case of the conventional ones. In sixties, when the first lightweight structures of this type were built, the only way for the design cable nets was the experimental one. Through physical models the cable net form, the cutting pattern and the behavior under external load were measured by means of photogrammetric methods and then assumed as basis for the design. In the same years the first

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rational solutions of the form finding problem were introduced. Barnes proposed a dynamical relaxation method (Barnes 1975), Argyris developed a F.E.M. approach suitable to deal with prestressed cable nets (Argyris 1974).

At the beginning of seventies, Schek proposed the so called Force Density Method (Schek 1974). Through the FDM the geometry of a pin-jointed network structures is found when the internal forces balance the external ones. In its linear formulation such method makes it possible to find all the possible equilibrium configurations of a net of cables having a certain given connectivity and given boundary conditions on the nodes. Each singular configuration corresponds to an assumed force density distribution. Such method still remains one of the most used tools for finding the initial basic geometry and the initial prestressing set and suggested several improvements and refinements. The so called Non-Linear Force Density Method (NLFDM) introduces the possibility of imposing assigned relative distances among the nodes, the tensile level in the elements and/or their initial undeformed length.

In parallel, Pellegrino (1993) presented a method based on the Singular Value Decomposition (SVD) technique, focused to identify both the independent self stress modes and the independent displacement modes.

Vassart and Motro (1999) presented a multiparametric form finding method, specialized to tensegrity structures. The form finding of tensegrity structures are studied by Zhang and Ohsaky (2006) who, through iterative eigenvalue analysis and spectral decomposition, find the feasible set of force densities that satisfies the requirements on rank deficiency of the equilibrium matrix with respect to the nodal coordinates. Recently, Yuan *et al.* (2007) proposed the concept of feasible integral pretensioned modes, defined through a general method based on a Double Singular Value Decomposition (DSVD), particularly efficient in determining the initial prestressing distribution of cable domes having different shapes and connectivities.

Spectacular applications of the tensegrity concepts, like the tensegrity cable domes, had been proposed by Geiger (1986), Levy (1994) and Kiewitt (1960).

In this paper an Extended Force Density Method (EFDM) is proposed, which makes it possible to set conditions in terms of given fixed nodal reactions or, in other words, to fix the positions of a certain number of nodes and, at the same time, to impose the intensity of the reaction force. Through such extension, the (EFDM) enables us to deal with form findings problems of cable nets subjected to given constraints and, in particular, with mixed structures, made of cables and struts. We mean mixed systems those made of tensioned cables and compressed struts and subjected to the different types of the constraints usually imposed by the actual design conditions. Such an approach, in addition to provide the same results of those based on the SVD, is actually more general, as it does not require the definition of the entire structural geometry. The efficiency and the robustness of method are assessed through comparisons with other form finding techniques in dealing with characteristic applications to the prestress design of cable systems. As a further extension, the EFDM is applied to structures having some parts not yet geometrically defined, as can happen in designing new creative structural forms.

2. An outline of the force density method (FDM)

With reference to a generic net, having n free nodes and n_f fixed nodes (the total number of nodes is $n_s = n + n_f$), connected by m cable elements, it is assumed that: a) the net is made of straight

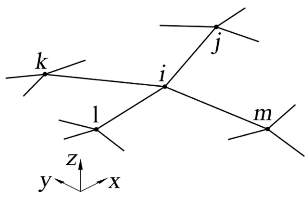


Fig. 1 Generical free node

cable elements, connected at the nodes; b) the net connectivity is known and its geometry is defined by the nodal coordinates; c) the cable elements are weightless; d) the net is subjected to concentrated forces, applied at the nodes. With reference to the *i*th node of Fig. 1, the equilibrium equations in the *x*, *y*, *z*, directions are respectively

$$T_{ij}\frac{x_j - x_i}{L_{ij}} + T_{ik}\frac{x_k - x_i}{L_{ik}} + T_{il}\frac{x_l - x_i}{L_{il}} + T_{im}\frac{x_m - x_i}{L_{im}} + F_{xi} = 0$$

$$T_{ij}\frac{y_j - y_i}{L_{ij}} + T_{ik}\frac{y_k - y_i}{L_{ik}} + T_{il}\frac{y_l - y_i}{L_{il}} + T_{im}\frac{y_m - y_i}{L_{im}} + F_{yi} = 0$$

$$T_{ij}\frac{z_j - z_i}{L_{ij}} + T_{ik}\frac{z_k - z_i}{L_{ik}} + T_{il}\frac{z_l - z_i}{L_{il}} + T_{im}\frac{z_m - z_i}{L_{im}} + F_{zi} = 0 \tag{1}$$

Where T_{ij} is the tensile force and L_{ij} is the length of the cable element between the nodes *i* and *j*.

By introducing the following vectors and matrices, Eq. (1) can be set into a matrix form:

- $\mathbf{x}_S$, $\mathbf{y}_S$, $\mathbf{x}_S$, $[n_S \times 1]$, coordinates of the nodes. By numbering the set of the fixed nodes after that of the free ones, the three vectors are partitioned into the following subvectors: $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $[n \times 1]$, coordinates of the free nodes; $\mathbf{x}_f$, $\mathbf{y}_f$, $\mathbf{z}_f$, $[n \times 1]$, coordinates of the fixed nodes;
- $\mathbf{f}_x$, $\mathbf{f}_y$, $\mathbf{f}_z$, $[n \times 1]$, nodal forces;
- $\mathbf{l}$, $[m \times 1]$, length of the elements. $\mathbf{L} = diag(\mathbf{l})$;
- $\mathbf{t}$, $[m \times 1]$, tensile forces in the elements.
- connectivity matrix $\mathbf{C}_S$, having dimensions $[m \times n_S]$, whose terms are

$$c_S(e) = \begin{cases} +1 & \text{if} \quad i = 1 \\ -1 & \text{if} \quad i = 2 \\ 0 & \text{in other cases} \end{cases} \tag{2}$$

The difference between the couples of coordinates in the three directions *x*, *y*, *z*, are

$$\mathbf{u} = \mathbf{C}_S\mathbf{x}_S, \quad \mathbf{v} = \mathbf{C}_S\mathbf{y}_S, \quad \mathbf{w} = \mathbf{C}_S\mathbf{z}_S \tag{3}$$

In this equation, by partitioning the matrix $\mathbf{C}_S$, we can put in evidence separately the coordinates of the free nodes and those of the fixed nodes, as follows

$$\mathbf{u} = \mathbf{C}_S\mathbf{x}_S = \mathbf{C}\mathbf{x} + \mathbf{C}_f\mathbf{x}_f, \quad \mathbf{v} = \mathbf{C}_S\mathbf{y}_S = \mathbf{C}\mathbf{y} + \mathbf{C}_f\mathbf{y}_f, \quad \mathbf{w} = \mathbf{C}_S\mathbf{z}_S = \mathbf{C}\mathbf{z} + \mathbf{C}_f\mathbf{z}_f \tag{4}$$

By introducing the diagonal matrices $\mathbf{U} = \text{diag}(\mathbf{u})$, $\mathbf{V} = \text{diag}(\mathbf{v})$, $\mathbf{W} = \text{diag}(\mathbf{w})$, $\mathbf{L} = \text{diag}(\mathbf{l})$, the equilibrium equations are expressed by the system

$$\begin{cases} \mathbf{C}^T\mathbf{U}\mathbf{L}^{-1}\mathbf{t} = \mathbf{f}_x \\ \mathbf{C}^T\mathbf{V}\mathbf{L}^{-1}\mathbf{t} = \mathbf{f}_y \\ \mathbf{C}^T\mathbf{W}\mathbf{L}^{-1}\mathbf{t} = \mathbf{f}_z \end{cases} \tag{5}$$

Now, if we introduce the concept of force density $q = T/L$, in matrix form we obtain

$$\mathbf{q} = \mathbf{L}^{-1}\mathbf{t} \tag{6}$$

Through this transformation, the equations of the system (5) become linear and uncoupled in the three cartesian directions

$$\mathbf{C}^T\mathbf{U}\mathbf{q} = \mathbf{f}_x, \quad \mathbf{C}^T\mathbf{V}\mathbf{q} = \mathbf{f}_v, \quad \mathbf{C}^T\mathbf{W}\mathbf{q} = \mathbf{f}_z \tag{7}$$

By introducing the diagonal matrix $\mathbf{Q}$ = diag($\mathbf{q}$), the following identities hold

$$\mathbf{U}\mathbf{q} = \mathbf{Q}\mathbf{u}, \quad \mathbf{V}\mathbf{q} = \mathbf{Q}\mathbf{v}, \quad \mathbf{W}\mathbf{q} = \mathbf{Q}\mathbf{w} \tag{8}$$

and substituting Eq. (4) and Eq. (8) into Eq. (7), we obtain the following relationships

$$\mathbf{D}\mathbf{x} = \mathbf{f}_x - \mathbf{D}_f\mathbf{x}_f, \quad \mathbf{D}\mathbf{y} = \mathbf{f}_v - \mathbf{D}_f\mathbf{y}_f, \quad \mathbf{D}\mathbf{z} = \mathbf{f}_z - \mathbf{D}_f\mathbf{z}_f \tag{9}$$

whose solution is

$$\mathbf{x} = \mathbf{D}^{-1}(\mathbf{f}_x - \mathbf{D}_f\mathbf{x}_f), \quad \mathbf{y} = \mathbf{D}^{-1}(\mathbf{f}_v - \mathbf{D}_f\mathbf{y}_f), \quad \mathbf{z} = \mathbf{D}^{-1}(\mathbf{f}_z - \mathbf{D}_f\mathbf{z}_f) \tag{10}$$

Being $\mathbf{Q}$ diagonal and ($\mathbf{Q}^T = \mathbf{Q}$), the matrix $\mathbf{D}$ is symmetric and, for pretensioned nets, positive defined. Given a net topology and assumed a vector $\mathbf{q}$ of force densities, Eq. (10) allows us to find the unique equilibrium configuration of the system.

2.1 Influence of the force density choices on the net configuration

We consider a square net made of $21 \times 21 = 441$ nodes and 840 elements. The net has 4 fixed nodes (Fig. 2). Fig. 3 shows some equilibrium configurations generated by varying the force density only, according to the combinations between force densities on the internal and on the border elements showed in Table 1. An increase of the force densities on the border elements with respect to the internal ones, widens the net (cases 3(a) and 3(b)). Different values of the force densities, having the same ratio between internal and border elements, lead to the same equilibrium configuration (3(c) and 3(d)). For a given net, its shape depends from the ratio of the internal and border densities only and not from their actual values. An increase of the force density in the internal elements shrinks the net and increases the curvature of the borders. The actual value of the force density distributions influences the stiffness of the net (Linkwitz 1999, Grnding 2000). This is important when we want to control the deformation of a net under assigned loads, as in Fig. 3(f). An increase of the force densities increases the pretensioning state and makes the structure stiffer.

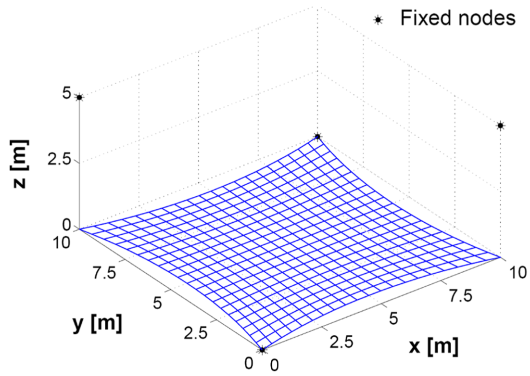


Fig. 2 A cable network made of 21 × 21 = 421 nodes (4 fixed) and 840 elements

Table 1 Force densities for internal and border elements of the square net of Fig. 2

	(a)	(b)	(c)	(d)	(e)	(f*)
$q_{\text{internal elements}}$	1	1	5	10	5	1
$q_{\text{border elements}}$	1	10	20	20	1	1

(f*) with a concentrated force at the centre.

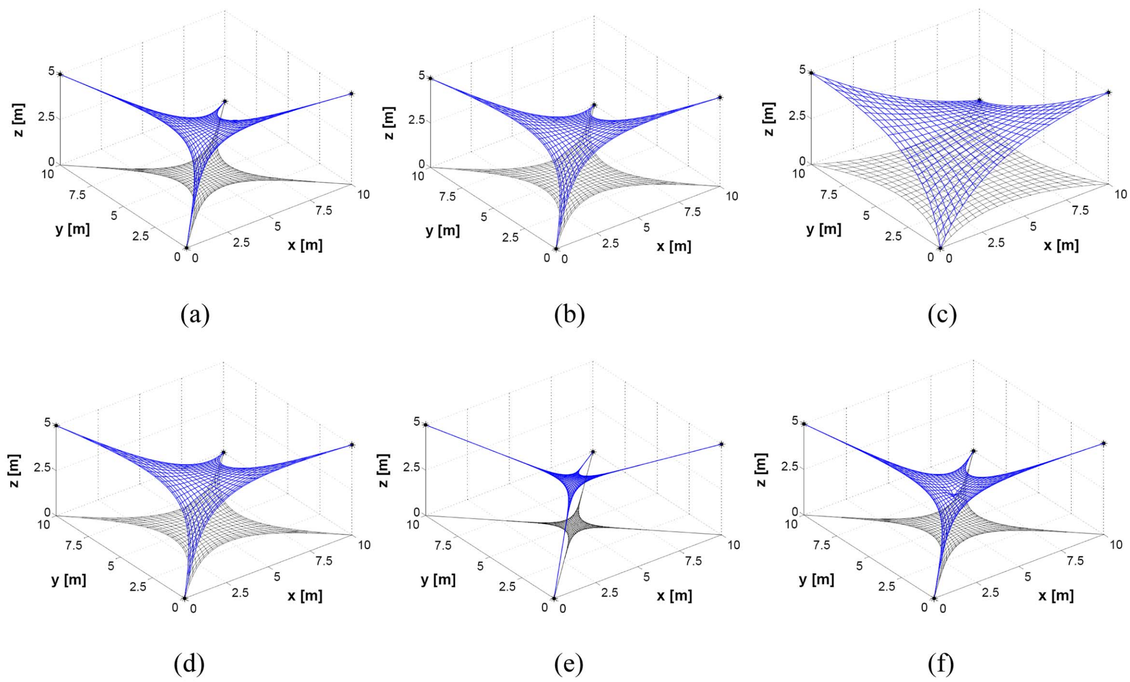


Fig. 3 Equilibrium configurations for different ratios between q_{border} and q_{internal} on a cable network made of 21 × 21 = 421 nodes (4 fixed) and 840 elements

3. The non-linear force density method (NLFDM)

The linear formulation of the force density method allows us to find all the possible equilibrium configurations of a net with a certain given connectivity and with given boundary conditions of the nodes. Each singular configuration corresponds to an assumed force density distribution. The possibility of imposing some further additional constraints should help us to find shapes not only equilibrated, but also technologically sound. The possibility of imposing assigned relative distances among the nodes, the tensile level in the elements and/or their initial undeformed length, was once again introduced in Schek (1974). If we suppose that all these conditions are function of the nodal coordinates and of the force densities, the generic additional condition assumes the following form

$$g_i(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{q}) = 0 \quad (i = 1{:}r;\ r < m) \tag{11}$$

For all the r conditions introduced, we have

$$\mathbf{g}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{q}) = \mathbf{g}(\mathbf{x}(\mathbf{q}), \mathbf{y}(\mathbf{q}), \mathbf{z}(\mathbf{q}), \mathbf{q}) = \mathbf{g}(\mathbf{q}) = \mathbf{0} \tag{12}$$

We choose an initial force density vector $\mathbf{q}^{(0)}$. For this assumed force density state, Eq. (12) is in general not satisfied. Hence, the solution is searched in an iterative form. We adopt the Newton method and search for a vector $\Delta\mathbf{q}$ which satisfies the following linearized condition

$$\mathbf{g}(\mathbf{q}^{(0)}) + \frac{\partial \mathbf{g}(\mathbf{q}^{(0)})}{\partial \mathbf{q}} \Delta\mathbf{q} = 0 \tag{13}$$

By calling

$$\mathbf{G}^T = \frac{\partial \mathbf{g}(\mathbf{q}^{(0)})}{\partial \mathbf{q}} \quad \text{and} \quad \mathbf{r} = -\mathbf{g}(\mathbf{q}^{(0)}) \tag{14}$$

Eq. (14) becomes

$$\mathbf{G}^T \Delta\mathbf{q} = \mathbf{r} \tag{15}$$

Being $m > r$, the system (15) is underdetermined and admits ∞^{m-r} solutions. Among the infinite solutions we search that one having minimum norm. In other words, among all the vectors which satisfy the system (15) we search the solution $\Delta\mathbf{q}$ which satisfy also the equation

$$\Delta\mathbf{q} = \arg\min \|\Delta\mathbf{q}\|_2^2 \tag{16}$$

Eqs. (15) and (16) form a problem of constrained optimisation, consisting in the search for the minimum of the function

$$f(\Delta\mathbf{q}) = \Delta\mathbf{q}^T \Delta\mathbf{q}\,, \quad \text{with the constraints} \quad \mathbf{G}^T \Delta\mathbf{q} = \mathbf{r} \tag{17}$$

By appling the Lagrange multipliers method we have

$$\Delta\mathbf{q} = \mathbf{G}(\mathbf{G}^T\mathbf{G})^{-1}\mathbf{r} \tag{18}$$

Being the initial conditions approximated through the linearization given by the Eq. (14), the solution is reached in an iterative way. At the beginning of each iteration we assume

$$\mathbf{q}^{(k+1)} = \mathbf{q}^{(k)} + \Delta\mathbf{q}^{(k)} \tag{19}$$

Then, after the updating of the corresponding matrix $\mathbf{G}^T$ and of the vector $\mathbf{r}$ and, we compute through Eq. (18) the vector $\Delta\mathbf{q}$. The iterative process is stopped, when we obtain, with a given small tolerance

$$\mathbf{g}(\mathbf{q}^{(k)}) = -\mathbf{r}(\mathbf{q}^{(k)}) \tag{20}$$

3.1 Jacobian matrix

The iterative solution involves an efficient formulation of the Jacobian matrix $\mathbf{G}^T$. By expanding $\mathbf{G}^T$ through the chain rule derivation we obtain

$$\mathbf{G}^R = \frac{\partial\mathbf{g}}{\partial\mathbf{q}} = \frac{\partial\mathbf{g}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{y}}\frac{\partial\mathbf{y}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{z}}\frac{\partial\mathbf{z}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{q}} \tag{21}$$

The derivatives ($\partial\mathbf{x}/\partial\mathbf{q}, \partial\mathbf{y}/\partial\mathbf{q}, \partial\mathbf{z}/\partial\mathbf{q}$) are independent from Eq. (12) and can be expressed in terms of known quantities as follows

$$\frac{\partial\mathbf{x}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\mathbf{C}^T\mathbf{U}, \quad \frac{\partial\mathbf{y}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\mathbf{C}^T\mathbf{V}, \quad \frac{\partial\mathbf{z}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\mathbf{C}^T\mathbf{W} \tag{22}$$

Instead the derivatives $\partial\mathbf{g}/\partial\mathbf{x}, \partial\mathbf{g}/\partial\mathbf{y}, \partial\mathbf{g}/\partial\mathbf{z}$ and $\partial\mathbf{g}/\partial\mathbf{q}$ depend on the additional conditions set by Eq. (12) and, hence, on the assumed additional conditions. Explicit forms of these derivatives have been done to impose constraints on the distances between the end nodes, or on the forces acting in the elements or of the cutting lengths (Schek 1974).

3.2 The choice of the initial force densities

A typical question arising in form finding methods is the way the force densities are initially chosen. The solution of this constrained problem (Eq. (17)) is not unique and so the Newton method finds that more near to the initial distribution of force densities.

The non-linear force density method is now used in form finding of cable nets, having elements with assigned length between the end nodes. Such introductory application concerns the cable towers (Tibert 1999). For an assigned cable net topology, we want to search for that particular form associated with assigned values of the radius of the inferior ring (R_{Inf}), the radius of the superior compressed ring (R_{Sup}), the level of the superior compressed ring (H_T) and the height of the antenna (H_P), as shown in Fig. 4. For the sake of the example, we assume that the 16 nodes at the basis and the node at the top of the antenna are fixed and that the antenna has infinite axial stiffness. The general dimensions are: R_{Sup} = 6.00 m, R_{Inf} = 9.00 m, H_P = 25.00 m, H_T = 21.00 m. The upper ring is divided in n_P = 16 elements. The lateral envelope is made of a rectangular mesh of 400 elements and 209 nodes. The cable net form depends on the initial configuration assigned to the cable net, that is on the given set of the force densities assigned to the hangers (q_p), to the compressed ring (q_r), to the vertical meridian cables ($q_{c,v}$) and to the circumferential parallel cables ($q_{c,c}$). The

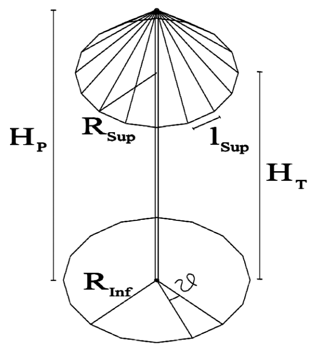
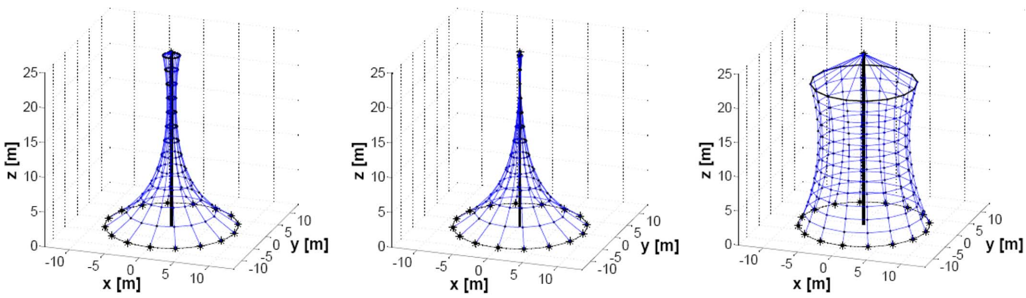
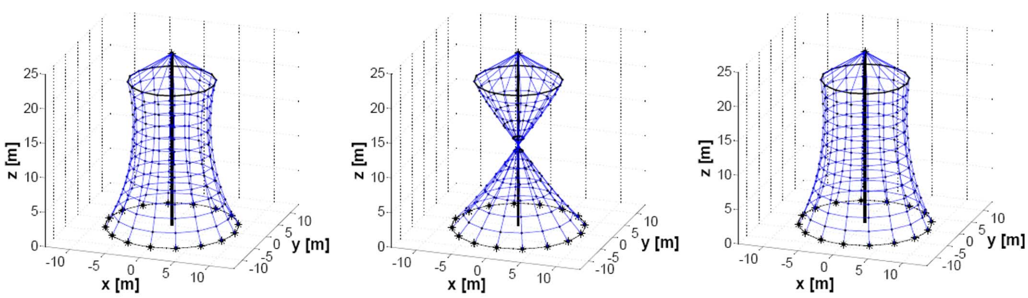


Fig. 4 Cable tower

Table 2 Influence of the initial choice of the force densities

	(a)	(b)	(c)
q_p	400	400	400
q_r	-2800	-3000	-3000
$q_{c,v}$	100	100	1000
$q_{c,c}$	100	100	100
Initial Configuration			
Final Configuration			

numerical values assumed for these force densities are summarized in Table 2.

From these given data, fixed values of the distances between certain couples of nodes follow and, in particular we obtain: the length of the hangers: $l_p^2 = (H_t - H_p)^2 + R_{sup}^2$; the length of the segments that form the upper compressed ring $l_{sup} = 2R_{sup}\cos(\vartheta/2)$, with $\vartheta = 2\pi/n_p$, (n_p = number of hangers).

These values can be obtained by using the nonlinear FDM, specifically by imposing the constraint on the distance between nodes. The results of the form finding process dealt with NLFDM are shown in Table 2. We can observe that the configurations (a) and (b) differ only in the force densities given to the compressed ring $(-2800 \Rightarrow -3000)$, with an increase of +7%. Both the solutions fit the imposed conditions, but the case (b) leads to results bereft of practical significance.

In order to avoid to go blindly on the initial choice of the force densities, a specific attention to the static role of the different groups of elements must be paid. This allows us to exclude degenerate solutions. In this case, for instance, it is easy to observe that the vertical cables tend to open the net, while the circumferential ones tend to close it. This distinction between the roles of the two sets of cables can guide us in the choice of the given force densities. In fact, for the same values of q_p, q_r, $q_{c,c}$ and for $q'_{c,v} = 10 \times q_{c,v} = 1000$, we obtain the final configuration shown in Table 2, column (c), which can be adopted for practical applications.

4. An extension of the force density method

As shown, the non-linear force density method allows us to deal with constraints concerning imposed relative distances among the nodes, the tensile level in the elements and/or their initial undeformed length. A contribution which extends the capabilities of the method consists in posing conditions in terms of given fixed nodal reactions or, in other words, to fix the positions of a certain number of nodes and, at the same time, to impose the intensity of the reaction force.

4.1 Fixed end reaction computation

Eq. (1) set the equilibrium equations of a free node of the net. The equilibrium of a fixed node is set in an analogous way, by substituting the forces F_i with the end reactions R_i, projected in their three components. Through this substitution the equilibrium equations of the fixed nodes are

$$\begin{cases} \mathbf{C}_f^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t} = \mathbf{r}_x \\ \mathbf{C}_f^T \mathbf{V} \mathbf{L}^{-1} \mathbf{t} = \mathbf{r}_y \\ \mathbf{C}_f^T \mathbf{W} \mathbf{L}^{-1} \mathbf{t} = \mathbf{r}_z \end{cases} \tag{23}$$

The steps to compute the end reactions are: (1) for a given topology of the net and for a given set of force densities, through Eq. (10) the free nodal coordinates are determined; (2) being known the nodal coordinates and the corresponding diagonal matrices **U**, **V**, **W**, Eq. (23) furnishes directly the end reactions.

4.2 Constraints on the end reactions

Through Eq. (23), which allows the end reaction computation, new form finding conditions can be set. The previous conditions were working on sets of r elements. The constraints on the end reactions work on sets of the n_f fixed nodes. We suppose that the constraints are set on a number $s \le n_f$ of the fixed nodes. Each reaction has three components. We treat the reactions in each direction separately and compute the difference between the basic value of the reactions r_{x1} given by Eq. (23) and the value of the reactions we want to impose r_{x1v}.

$$\begin{cases} g_{x1}(\mathbf{x},\mathbf{y},\mathbf{z}) = r_{x1}(\mathbf{x},\mathbf{y},\mathbf{z}) - r_{xv1} = 0 \\ g_{x2}(\mathbf{x},\mathbf{y},\mathbf{z}) = r_{x2}(\mathbf{x},\mathbf{y},\mathbf{z}) - r_{xv2} = 0 \\ \vdots \\ g_{xs}(\mathbf{x},\mathbf{y},\mathbf{z}) = r_{xs}(\mathbf{x},\mathbf{y},\mathbf{z}) - r_{xvs} = 0 \end{cases} \tag{24}$$

Doing the same in the other directions and writing the equations in matrix form, we have

$$\begin{aligned} \mathbf{g}_x &= \bar{\mathbf{r}}_x - \bar{\mathbf{r}}_{xv} = \mathbf{0} \\ \mathbf{g}_y &= \bar{\mathbf{r}}_y - \bar{\mathbf{r}}_{yv} = \mathbf{0} \\ \mathbf{g}_z &= \bar{\mathbf{r}}_z - \bar{\mathbf{r}}_{zv} = \mathbf{0} \end{aligned} \tag{25}$$

The vectors $\bar{\mathbf{r}}_{(x,y,z)}$ and $\bar{\mathbf{r}}_{(x,y,z)_v}$ have dimensions $[s \times 1]$ and contain respectively the basic values of the end reactions and the prescribed values to be imposed. They are obtained by partitioning the vectors $\bar{\mathbf{r}}_{(x,y,z)}$ as follows

$$\begin{aligned} \bar{\mathbf{r}}_x &= \overline{\mathbf{C}}_f^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t} \\ \bar{\mathbf{r}}_y &= \overline{\mathbf{C}}_f^T \mathbf{V} \mathbf{L}^{-1} \mathbf{t} \\ \bar{\mathbf{r}}_z &= \overline{\mathbf{C}}_f^T \mathbf{W} \mathbf{L}^{-1} \mathbf{t} \end{aligned} \tag{26}$$

Matrix $\overline{\mathbf{C}}_f^T$ has dimensions $[s \times m]$, as it can be verified by the inspection of matrices and vectors in Eq. (26). This matrix derives from matrix $\mathbf{C}_f^T$, by extracting the row corresponding to the nodes to be constrained. It must be pointed out that, working on the nodes, and not on the elements, all the elements and all the terms of the matrices $\mathbf{U}, \mathbf{V}, \mathbf{W}, \mathbf{L}^{-1}$ and of the vector $\mathbf{t}$ are involved in the computation.

4.3 Jacobian matrix

With reference to Eq. (21), the derivatives of the nodal coordinates with respect to the force densities $\partial\mathbf{x}/\partial\mathbf{q}$, $\partial\mathbf{y}/\partial\mathbf{q}$ $\partial\mathbf{z}/\partial\mathbf{q}$ should be computed as before, while $\partial\mathbf{g}/\partial\mathbf{x}$, $\partial\mathbf{g}/\partial\mathbf{y}$, $\partial\mathbf{g}/\partial\mathbf{z}$ and $\partial\mathbf{g}/\partial\mathbf{q}$, depend on the new conditions to be imposed. We consider the vector $\mathbf{g}_x$. The vectors $\mathbf{g}_y$ and $\mathbf{g}_z$ should be treated in an analogous manner. Being $\bar{\mathbf{r}}_{x_v}$ a constant vector, we can write that

$$\frac{\partial \mathbf{g}_x}{\partial \mathbf{x}} = \frac{\partial \bar{\mathbf{r}}_x}{\partial \mathbf{x}} \tag{27}$$

The dimensions of $\bar{\mathbf{r}}_x, \mathbf{x}$ and $\partial\mathbf{g}_x/\partial\mathbf{x}$ are respectively $[s \times 1][n \times 1][s \times n]$. By remembering Eq. (4), Eq. (6) and Eq. (8) after some manipulation we obtain

$$\frac{\partial \bar{\mathbf{r}}_x}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}}(\overline{\mathbf{C}}_f^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t}) = \overline{\mathbf{C}}_f^T \frac{\partial}{\partial \mathbf{x}}(\mathbf{U} \mathbf{L}^{-1} \mathbf{t}) = \overline{\mathbf{C}}_f^T \frac{\partial}{\partial \mathbf{x}}(\mathbf{U}\mathbf{q}) = \overline{\mathbf{C}}_f^T \frac{\partial}{\partial \mathbf{x}}(\mathbf{Q}\mathbf{u}) = \overline{\mathbf{C}}_f^T \mathbf{Q} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \overline{\mathbf{C}}_f^T \mathbf{Q} \mathbf{C} \tag{28}$$

Since $\mathbf{u} = \mathbf{u}(\mathbf{x})$, the derivatives $\partial\mathbf{g}_x/\partial\mathbf{y}$ and $\partial\mathbf{g}_z/\partial\mathbf{z}$ are equal to zero.

The last term of the Jacobian matrix **G**, that is $\partial\mathbf{g}/\partial\mathbf{q}$, is obtained as follows

$$\frac{\partial \mathbf{g}_x}{\partial \mathbf{q}} = \frac{\partial \bar{\mathbf{r}}_x}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}}(\overline{\mathbf{C}}_f^T \mathbf{U} \mathbf{L}^{-1} \mathbf{t}) = \overline{\mathbf{C}}_f^T \frac{\partial}{\partial \mathbf{q}}(\mathbf{U}\mathbf{q}) = \overline{\mathbf{C}}_f^T \mathbf{U} \frac{\partial \mathbf{q}}{\partial \mathbf{q}} = \overline{\mathbf{C}}_f^T \mathbf{U} \tag{29}$$

In its final form, the Jacobian matrix $\mathbf{G}$ has dimensions $[s \times m]$ and is given by

$$\mathbf{G}_{Rx}^T = \overline{\mathbf{C}}_f^T \mathbf{U} - \overline{\mathbf{C}}_f^T \mathbf{Q} \mathbf{C} \mathbf{D}^{-1} \mathbf{C}^T \mathbf{U}$$

$$\mathbf{G}_{Ry}^T = \overline{\mathbf{C}}_f^T \mathbf{V} - \overline{\mathbf{C}}_f^T \mathbf{Q} \mathbf{C} \mathbf{D}^{-1} \mathbf{C}^T \mathbf{V}$$

$$\mathbf{G}_{Rz}^T = \overline{\mathbf{C}}_f^T \mathbf{W} - \overline{\mathbf{C}}_f^T \mathbf{Q} \mathbf{C} \mathbf{D}^{-1} \mathbf{C}^T \mathbf{W} \tag{30}$$

With these equations we can solve the problem of finding the geometry of a net for which, in certain fixed nodes, the end reactions assume prescribed values in the three directions of the reference system.

4.4 Multiple constraints

We suppose to assign end reaction forces with arbitrary intensities and directions. This involves a generalization of the method with the setting of multiple conditions. Let n_{vx} and n_{vy} the number of the constrained nodes respectively in x and y directions. The $[n_{vx} + n_{vy}]$ additive conditions are

$$\mathbf{g}_x = \bar{\mathbf{r}}_x - \bar{\mathbf{r}}_{xv} = \mathbf{0} \quad [n_{vx} \times 1]$$

$$\mathbf{g}_y = \bar{\mathbf{r}}_y - \bar{\mathbf{r}}_{yv} = \mathbf{0} \quad [n_{vy} \times 1] \tag{31}$$

and the form finding problem, under given end reaction forces, can be solved through the following steps. By letting the constraint conditions in matrix form, we obtain the non linear system $\mathbf{g}(\mathbf{q}) = \mathbf{g}(\mathbf{x}(\mathbf{q}), \mathbf{y}(\mathbf{q}), \mathbf{z}(\mathbf{q}), \mathbf{q}) = \mathbf{0}$ which can be linearized as $\mathbf{G}^T \Delta\mathbf{q} = \mathbf{r}$ (Eq. (15)). The solution $\mathbf{q}^{(k+1)} = \mathbf{q}^{(k)} + \Delta\mathbf{q}^{(k)}$ is iterated up to convergence, by minimizing the residuals $\mathbf{r}$ below a prefixed tolerance. At each step, the vector $\Delta\mathbf{q}$ has to satisfy both the conditions on x and y, which are given by

$$\begin{aligned} \mathbf{G}_{Rx}^T \Delta\mathbf{q} &= \mathbf{r}_x = -\mathbf{g}_x \\ \mathbf{G}_{Ry}^T \Delta\mathbf{q} &= \mathbf{r}_y = -\mathbf{g}_y \end{aligned} \quad \text{or, in compact form} \quad \begin{bmatrix} \mathbf{G}_{Rx}^T \\ \mathbf{G}_{Ry}^T \end{bmatrix} \Delta\mathbf{q} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_y \end{bmatrix} \tag{32}$$

By letting

$$\mathbf{G}_R^T = \begin{bmatrix} \mathbf{G}_{Rx}^T \\ \mathbf{G}_{Ry}^T \end{bmatrix} \qquad \mathbf{r}_{xy} = \begin{bmatrix} \mathbf{r}_x \\ \mathbf{r}_y \end{bmatrix} \tag{33}$$

we have this final compact equation

$$\mathbf{G}_R^T \Delta\mathbf{q} = \mathbf{r}_{xy} \tag{34}$$

Eq. (34) is analogous to Eq. (15). Only the dimensions of vectors and matrices change: now the matrix $\mathbf{G}_R^T$ and the vector $\mathbf{r}_{xy}$ have respectively dimensions $[(n_{vx}+n_{vy})\times m]$ and $[(n_{vx}+n_{vy})\times 1]$, while $\Delta\mathbf{q}$ maintains the dimension $[m\times 1]$. From the computational point of view, it is sufficient to introduce and compile the set of relations listed in Eq. (33). In the following, this procedure will be called Extended Force Density Method (EFDM).

5. Dealing with mixed systems

The Extended Force Density Method (EFDM) allows us to deal with form findings problems of cable nets subjected to given constraints and in particular with mixed structures, made of cables and struts, as shown through the following topical examples.

5.1 The rombic system

We consider the simple structure shown in Fig. 5. It is a mixed system, made of four cable and a strut. The strut opens the net and generates two opposite curvature (Tibert 1999).

In absence of singularities, by the system of Eq. (10) we obtain

$$\mathbf{x}^T = [b \quad b], \quad \mathbf{z}^T = [0 \quad 0] \tag{35}$$

According to this solution, the length of the strut results zero, regardless to the force density values assigned to the strut and hence, despite the presence of the strut, the structure degenerates into a straight line, passing through the fixed ends. Such a problem can be overcame adopting the Singular Value Decomposition technique (SVD, Pellegrino (1993)), or by EFDM.

5.1.1 Solution through the Singular Value Decomposition (SVD)

For the rombic frame shown in Fig. 5, let a = 0.5 and b = 1. We compile the equilibrium matrix **A**. By the SVD operator we obtain three matrix so that

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T \tag{36}$$

By calling r = rank(**A**), the $s = m - r$ right columns of the vector **V** are the bases of the s states of self-stresses. All these states satisfy the equilibrium condition $\mathbf{At} = \mathbf{0}$.

For our structure is r = 1. So, from the last column of **V**, we directly obtain the intensity of the prestressing and then we compute all the force densities. The results are summarized in Table 3.

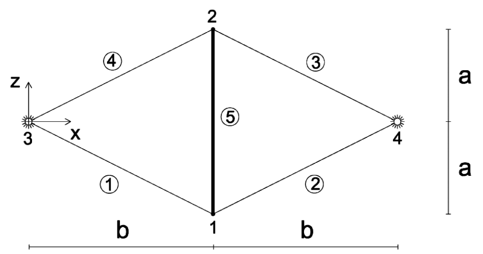


Fig. 5 A simple mixed system

Table 3 Solution obtained by SVD

	Elem. no.				
	1	2	3	4	5
Prestress	0.46	0.46	0.46	0.46	-0.41
Force densities	0.41	0.41	0.41	0.41	-0.41

5.1.2 Solution through the Extended Force Density Method (EFDM)

Previously it was shown that, in presence of aligned constraints, the rombic structure degenerates on the line joining their ends. A solution which does not became flat can be obtained by fixing the nodes of the strut by auxiliary constraints. In this way, however, we alter the static scheme of the structure, because the nodes remain fixed thanks to reactions which do not exist in the real structure. By Eq. (34) is possible to search for the particular distribution of $\mathbf{q}$ such that the reactions in the auxiliary constraints are null. In this case we have to obtain null reactions in two directions (x-z) and hence multiple conditions must be set. We proceed according to EFDM, as described in Section 4.4, through the following steps: (1) the ends of the struts are fixed with auxiliary constraints; (2) an initial distribution of force densities is assumed and by Eq. (26) the reaction forces provided by the auxiliary constraints are computed; (3) through the EFDM, the reactions of the auxiliary constraints are set null. In this way the additional constraints allow to open the net without alter the overall equilibrium. We assume as set of initial force densities the vector $\mathbf{q} = [2\ 2\ 2\ 2\ -1]^T$. We obtain that at the nodes 1 and 2 the reactions R_x are zero (by symmetry), while the reactions R_z are equal to -1 and 1 respectively. The additional constraints open the net, but the assumed configuration do not fulfill the objective of the searching procedure, because the free body equilibrium is altered. Through the EFDM, we set null the reactions of the auxiliary constraints and we obtain the tensile forces in the elements and the force densities listed in Table 4. One can observe that the ratio among the forces is again the ratio found by the SVD and shown in Table 3.

The EFDM removes some limitations of the NLFDM and is suitable to solve the force finding problems for mixed system. In the following, more complex applications, like cable domes structures, are presented.

5.2 Cable domes

A typical cable dome consists of ridge cables, diagonal cables, hoop cables, vertical struts, an inner tension ring and an outer compression ring. Cables work in tension and individual struts work in compression. The rigidity of the dome is a result of self-stress equilibrium between cables and struts (Yuan 2007). We consider here the topologies and the prestressing systems defined by Geiger

Table 4 Solution obtained by EFDM

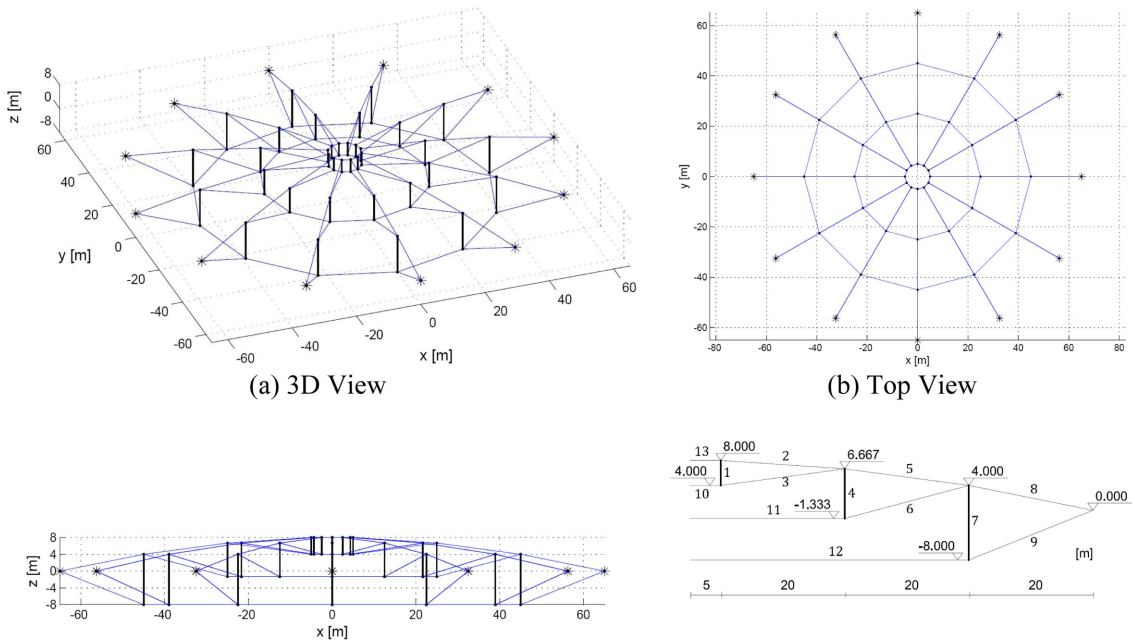
	Elem. no.				
	1	2	3	4	5
Prestress	2.01	2.01	2.01	2.01	-1.80
Force densities	1.80	1.80	1.80	1.80	-1.80

(1986), Levy (1994), and Kiewitt (1960). Some variations of the Geiger scheme is also studied. In all the following examples only prestress is considered. A simple approach for force finding analysis of circular Geiger domes considering self-weight is proposed in (Wang 2010). Once the initial equilibrium problem has been solved, the structural behaviour under other loads can be dealt throught some tool of analysis like, for instance, the FEM (Zong 2009).

5.2.1 Geiger dome

In a Geiger dome, the ridge cables are radially oriented, and the roof is composed by wedge-shaped basic units in plan, cyclically distributed around the center. We consider a Geiger dome, defined by a total number of nodes n = 84 and connected by m = 156 elements, as shown in Fig. 6. The structure is composed by 36 struts and 120 cables. The 12 external nodes are fixed. The symmetry of the domes allows us to subdivide the elements into 13 groups, as show in Fig. 6(d). Given the connectivity and the fixed nodes, we search for that equilibrated form, which puts cables in tension and struts in compression. Dealing with the SVD technique, we have a single eigenvector which respects this condition and the solution, shown in Table 5, is immediate. The same problem is solved through the EFDM. In giving the vector of the initial force densities, we adopt the simple criterion to assign q = 1 to the cables and $q = -1$ to the struts. The results are the same and agree, in this case, with those reported by Yuan (2007).

It is easy to verify the vertical equilibrium in a central node of the top ring. With reference to the Cable No. 2, the inclination is $\alpha = tg^{-1}[(8-6.667)/20] = 3.81^{\circ}$ and the compressive force in Strut No. 1 is: $T_2 \cdot \sin\alpha = 0.437 \cdot 0.066 = 0.029 = T_1$.



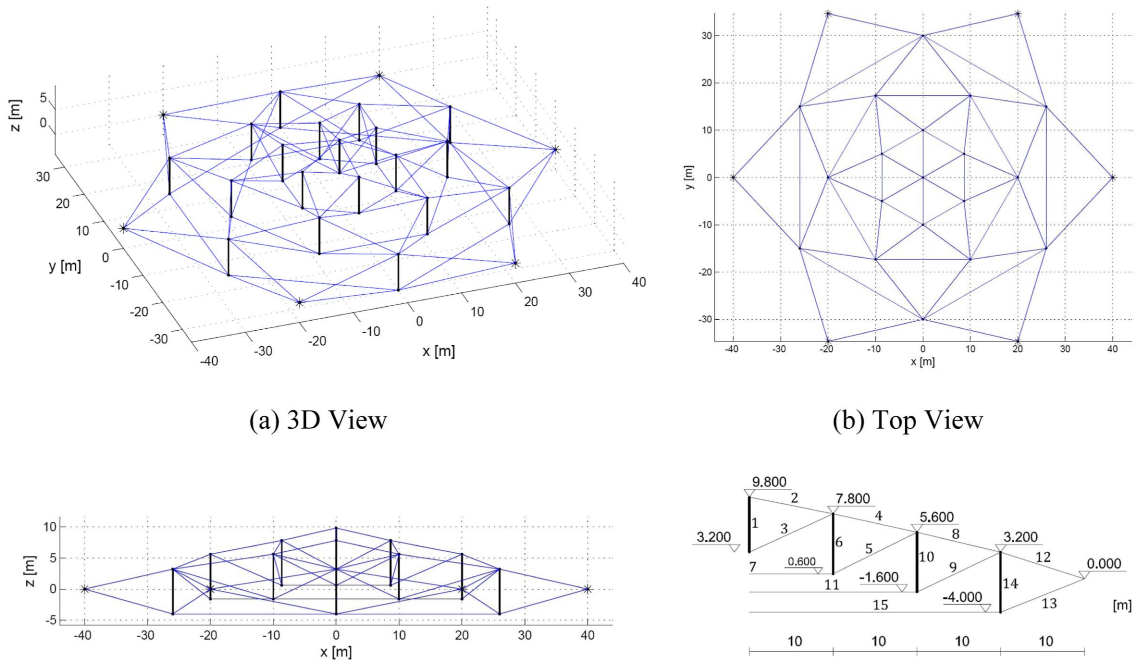
(a) 3D View (b) Top View

(c) Side View (d) Geometry and groups

Fig. 6 Geiger dome

Table 5 Prestress in the Geiger dome, by SVD and EFDM

	Group no.							
	1	2	3	4	5	6	7	8
Prestress	**-0.029**	**0.437**	0.220	-0.087	0.659	0.338	-0.196	1
	9	10	11	12	13			
Prestress	0.528	0.421	0.631	0.947	0.842			



(a) 3D View (b) Top View

(c) Side View (d) Geometry and groups

Fig. 7 Levy dome

5.2.2 Levy dome

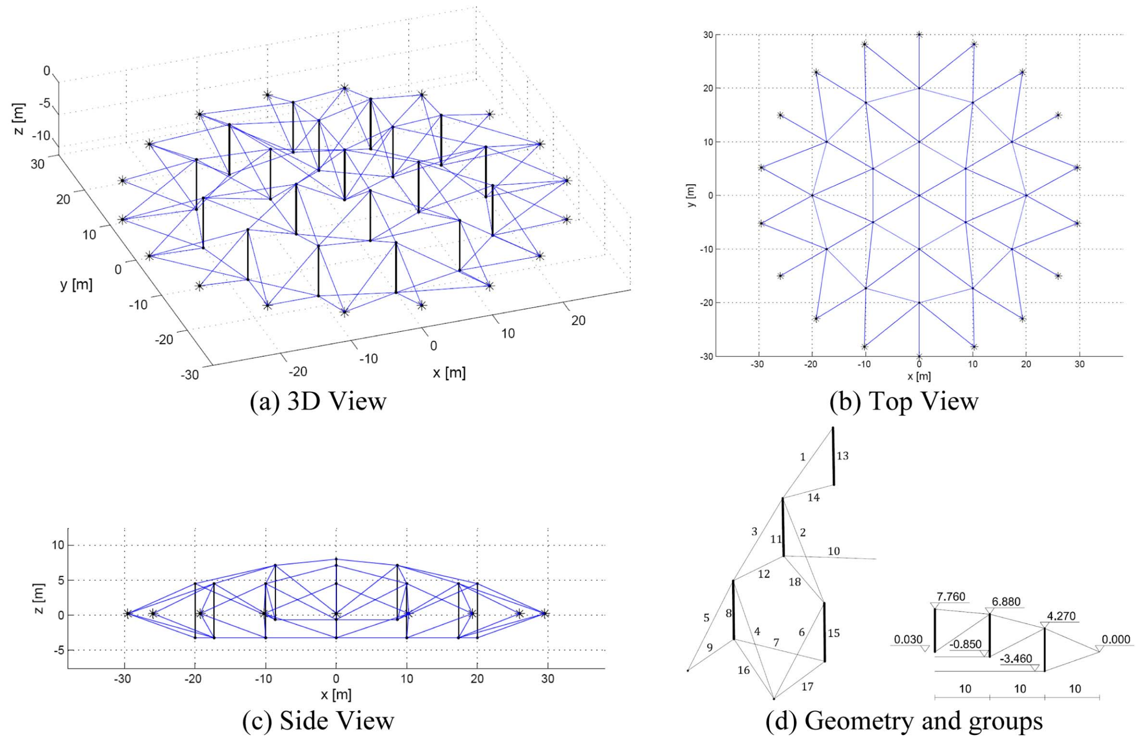
In the Levy form, the ridge cables form a triangular pattern. The roof surface is then primarily comprised of quadrilateral saddle shaped (anticlastic) membrane panels, connected at ridge cables. Fig. 7 shows a Levy dome, defined by a total number of nodes n = 44, connected by m = 121 elements. The structure is composed by 19 struts and 102 cables.

The equilibrium matrix **A** has rank r = 114 and so we have $s = m - r$ =121 – 114 = 7 independent self-equilibrated modes. The DSVD technique (Yuan 2007) allows us to find the sole eigenvector combination, which satisfy the condition of tensioned cables and compressed struts. The results are listed in Table 6.

In applying the EFDM, we choose the initial force densities as in the previous case and we obtain the same results given by DSVD. As a check we write the vertical equilibrium at the top node of the central strut. The inclination of the six cables of group 2 is $\alpha = tg^{-1}[(9.8-7.8)/10] = 11.31^{o}$ and we obtain $6 \cdot T_2 \cdot \sin\alpha = 6 \cdot 0.850 \cdot 0.196 = 1 = T_1$.

Table 6 Prestress in the Levy dome, by DSVD and EFDM

	Group no.							
	1	2	3	4	5	6	7	8
Prestress	**-1**	**0.850**	0.400	1.028	0.480	-0.359	0.526	3.640
	9	10	11	12	13	14	15	
Prestress	1.878	-1.070	1.334	18.987	15.290	-5.848	6.785	



(a) 3D View (b) Top View

(c) Side View (d) Geometry and groups

Fig. 8 Kiewitt dome

5.2.3 Kiewitt dome

This dome is show in Fig. 8. It is defined by a total number of nodes n = 56, connected by m = 145 elements, and it is composed by 19 struts and 126 cables. The 18 external nodes are fixed. The symmetry of the domes allows us to subdivide the elements into 18 groups, as show in Fig. 8(d). In this case the SVD operator applied to the equilibrium matrix, identify 31 independent self-stress modes. By proceeding through the DSVD, we can get four solutions (integral prestress modes (Yuan 2003)) which do not satisfy the condition of tensioned cables and compressed struts. Yuan (2007) search for a solution by means of an optimization technique and shows that a linear combination of these solutions can be found through an optimum cable prestressing design. The present solution, based on EFDM, allows us to remain in the frame of a pure equilibrium problem and to obtain directly a feasible prestress system which satisfies the condition of tensioned cables and compressed struts. Once again, for the first step, we have assigned q = 1 to the cables and

Table 7 Prestress in the Kiewitt dome, by EFDM

	Group no.								
	1	2	3	4	5	6	7	8	9
Prestress	**0.415**	0.176	0.272	0.179	0.227	0.274	1	-0.213	0.086
	10	11	12	13	14	15	16	17	18
Prestress	0.244	-0.141	0.178	**-0.218**	0.064	-0.188	0.361	0.309	0.079

$q = -1$ to the struts. Also this choice introduces an a priori criteria, but in this case it directly conditions the mechanic of the dome towards the static function attributed to tensioned and compressed elements. The results is listed in Table 7.

By writing the vertical equilibrium in the top node of the central strut, we correctly obtain: $6 \cdot T_1 \cdot \sin\alpha = 6 \cdot 0.415 \cdot 0.0877 = 0.218 = T_{13}$, with $\alpha = tg^{-1}[(7.76-6.88)/10] = 5.03^{\circ}$.

In all these cases, the EFDM represents another type of solution of the same problem, which is alternative with respect to SVD and DSVD, but without giving evident improvements. The advantages of the proposed method will appear in the following applications.

5.2.4 Cable system with new forms

The cable systems shown in the previous examples mainly stem from human intuitions. Once the geometry is defined, through the methods already shown it is possible to compute the prestress system or, equivalently, the cutting lengths. However, sometime the intuition may be misleading. It may happen, for instance, that all these method should not be able to evaluate a system of forces suitable to balance those corresponding to the assigned geometry, because such an arrangement do not exist. Through the EFDM is possible to give the connectivity and the fixed end nodes, but leaving an unknown geometry. The particular spatial arrangement of the elements with unknown end nodes is suggested by the method itself.

To this purpose, we consider, for sake of simplicity, the same Geiger dome show before with new four elements (14-15-16-17), with unknown location (Fig. 9). The structure is now divided into 17

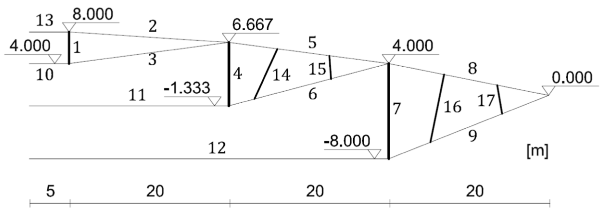
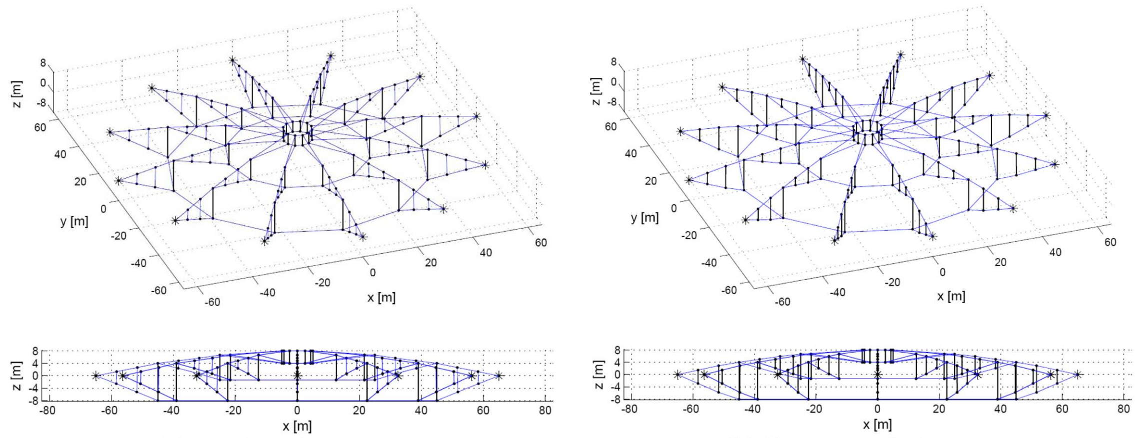


Fig. 9 Geiger dome with other elements

Table 8 Initial force densities

$\mathbf{q}^0$	(a)	(b)
$q_{2,3,5,6,8,9,10,11,12}$	100	1000
$q_{1,4,7}$	-10	-10
$q_{14,15,16,17}$	1	1



(a) Tensioned elements (b) Compressed elements

Fig. 10 A modified Geiger dome

Table 9 Prestress in the dome (a), by EFDM

	Group no.								
	1	2	3	4	5	6	7	8	9
Prestress	-0.029	0.437	0.217	-0.097	0.660	0.341	-0.201	1	0.529
	10	11	12	13	14	15	16	17	
Prestress	0.419	0.632	0.946	0.842	0.013	0.004	0.005	0.005	

Table 10 Prestress in the dome (b), by EFDM

	Group no.								
	1	2	3	4	5	6	7	8	9
Prestress	-0.029	0.440	0.220	-0.074	0.661	0.338	-0.165	1	0.520
	10	11	12	13	14	15	16	17	
Prestress	0.422	0.636	0.953	0.847	-0.019	-0.002	-0.046	-0.005	

groups. The solution obtained depends from the tentative force densities. By specializing the static function of the elements we can explore the two configurations generated by the initial force densities listed in Table 8. The results are shown in Fig. 10. In the case (a), the added elements are tensioned, in the case (b) they are compressed. These two choices may be used to govern the curvatures along the ridges and between the ridges.

6. Conclusions

An Extended version of the Force Density Method (EFDM) has been presented. It contains the basic features of the originals FDM (Force Density Method) and NLFDM (Non Linear Force

Density Method) proposed by Schek (1974). As known, the FDM finds an equilibrated cable net form, given the connectivity, the fixed end nodes and the distribution of the force densities. In addition, the NLFDM allows us to take into account also assigned constraints, that is (a) prescribed distances between the nodes, or (b) prescribed forces in the elements, or (c) prescribed cutting lengths. The Extended Force Density Method (EFDM) is suitable to deal both with nets of cables only and, in addition, with mixed systems, made of cables and struts. From the conceptual point of view, the method derives from the same criteria at the basis of the form finding procedures and, hence, it maintains the problem into the frame of a pure equilibrium methodology. However, its extended features make it suitable to deal also with force finding problems. This means that the method allows us to find, for a given geometry, the set of feasible prestress distributions, with respect to the conditions of tensioned cables and compressed struts. To this purpose, one can observe that the cable systems really built mainly stem from human intuitions. Once the geometry is defined, it is possible to compute the prestress system. This can be done also with others approaches, like the Singular Value Decomposition (SVD). Sometime, however, the intuition may be misleading. It may happen, for instance, that all these methods should not be able to evaluate a system of forces suitable to balance those corresponding to the assigned geometry, because such an arrangement do not exist. Through the EFDM is possible to give the connectivity and the fixed end nodes, but leaving part of the geometry unknown. In this case, by specializing the static function of the elements, we can explore new configurations generated by intuitive sets of the initial force densities. Through these sets, the EFDM allows us to find a particular spatial arrangement of the elements with unknown end nodes which satisfy all the requirements. This can help in the formulation and validation of cable systems with new forms.

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