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Natural stiffness matrix for beams on Winkler foundation: exact force-based derivation

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Abstract. This paper presents an alternative way to derive the exact element stiffness matrix for a beam on Winkler foundation and the fixed-end force vector due to a linearly distributed load. The element flexibility matrix is derived first and forms the core of the exact element stiffness matrix. The governing differential compatibility of the problem is derived using the virtual force principle and solved to obtain the exact moment interpolation functions. The matrix virtual force equation is employed to obtain the exact element flexibility matrix using the exact moment interpolation functions. The so-called "natural" element stiffness matrix is obtained by inverting the exact element flexibility matrix. Two numerical examples are used to verify the accuracy and the efficiency of the natural beam element on Winkler foundation.

Keywords: beam elements; winkler foundation; finite element; flexibility-based formulation; virtual force principle; soil-structure interaction; natural stiffness matrix

1. Introduction

Several problems in structural, geotechnical, highway, railroad, and mechanical engineering can be formulated and solved using the concept of a beam on elastic foundation. Numerous analytical methods have been proposed in the research community to study the problems of beams on elastic foundation. These models range from comparatively simple approaches in which the foundation is represented by a set of continuous springs to rigorous continuum approaches in which the foundation is considered as a homogeneous semi-infinite elastic body (Selvadurai 1979). Due to the complexity of the elastic continuum foundation model (Mindlin 1936), the continuous-spring model has been used extensively by several researchers and practicing engineers as a compromising approach to represent the foundation (e.g., Hetenyi 1946, Yankelevsky *et al*. 1988, Gendy and Saleeb 1999, Morfidis and Avramidis 2002, Maheshwari *et al*. 2004, Celep and Demir 2007, Allotey and El Naggar 2008, Zhang *et al*. 2009, Civalek and Ozturk 2010). The famous continuous-

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spring foundation model is attributed to Winkler (1867). Therefore, this foundation model is called "Winkler foundation". In the Winkler foundation model, the foundation medium is replaced by a set of continuously distributed non-interconnected springs. Hetenyi (1946) comprehensively studied the problems of beams on Winkler foundation and provided closed-form solutions to the problems under a variety of loading conditions. In addition to the analytical studies by Hetenyi (1946), several numerical techniques have been used to solve the problems. The finite difference method was used by several researchers (Matlock and Reese 1960, Bowles 1968, Beaufait and Hoadley 1980) to solve for the numerical solutions of the problems. However, this numerical technique had been diminished some decades ago due to the emergence of the finite element method.

The stiffness-based finite element method has been widely used as a numerical tool to solve the problems of beams on Winkler foundation. Bowles (1974) formulated the beam-foundation model by combining the conventional beam element with discrete foundation springs attached at the beam ends. However, the model accuracy was hampered by the replacement of continuous foundation springs with discrete ones. To overcome this drawback, the continuous nature of foundation springs had been preserved by considering the beam and the foundation as a whole system. Tong and Rossettos (1977) used the principle of minimum potential energy to formulate the beam-foundation element using Hermite displacement interpolation functions. This model was adapted by Limkatanyu and Spacone (2006) and Limkatanyu *et al*. (2009) to study the nonlinear behaviors of soil-pile systems. Due to the assumed nature of displacement interpolation functions, the model accuracy was still limited. Alternatively, the "*exact*" displacement interpolation functions can be obtained by solving the governing differential equilibrium equation of the problem. Following this approach, the "*exact*" stiffness matrix for the beam-foundation element was derived by several researchers (e.g., Avramidis and Avramidou 1979, Avramidis and Golm 1980, Zhaohua and Cook 1983, Ting and Mockry 1984, Eisenberger and Yankelevsky 1985, etc.). It was shown by Zhaohua and Cook (1983) that in some case, eighty beam-foundation elements derived based on the assumed Hermite displacement interpolation functions were needed to obtain results as accurate as those obtained with one beam-foundation element derived based on the exact displacement interpolation functions. Besides the conventional stiffness-based finite element formulation, mixed and flexibility-based finite element formulations have been used in recent years to develop the beam-foundation elements (Mullapudi and Ayoub 2010, Erguven and Gediki 2003, and Limkatanyu and Spacone 2006). Even though the element accuracies are greatly enhanced by these unconventional finite element formulations, only approximate element stiffness matrices are obtained due to the assumed nature of displacement and force interpolation functions.

The main objective of this paper is to alternatively derive the exact beam-Winkler foundation stiffness matrix based on the exact beam-Winkler foundation flexibility matrix. The governing differential equilibrium equations and constitutive relations of the beam on Winkler foundation are first presented. Next, the governing differential compatibility equation and the associated end-boundary compatibility conditions are derived based on the virtual force principle and solved analytically to obtain the exact moment interpolation functions. The matrix virtual force equation is employed to obtain the exact element flexibility matrix using the exact moment interpolation functions. It is noted that the element flexibility matrix presented in this paper is different from that presented in Limkatanyu and Spacone (2006) in that the foundation force distribution in Limkatanyu and Spacone (2006) has to be assumed, thus resulting in the approximate moment interpolation functions and the approximate element flexibility matrix. The so-called "natural" element stiffness matrix is obtained by inverting the exact element flexibility matrix. Two numerical examples are used to verify the accuracy and the efficiency of the natural beam element on Winkler foundation. All symbolic calculations throughout this paper are performed using the computer software Mathematica (Wolfram 1992) and the resulting beam-foundation model is implemented in the general-purpose finite element platform FEAP (Taylor 2000).

2. Governing equations of beams on winkler foundation

2.1 Differential equilibrium equations: direct approach

The governing differential equilibrium equations of a beam on Winkler foundation shown in Fig. 1(a) are derived in a direct manner. A differential element *dx* taken from the beam on Winkler foundation is shown in Fig. 1(b). Following the small-displacement hypothesis, all of the equilibrium equations are considered in the undeformed configuration. Considering vertical equilibrium of the infinitesimal segment *dx* leads to

$$\frac{dV(x)}{dx} - p_y(x) + D_s(x) = 0 \tag{1}$$

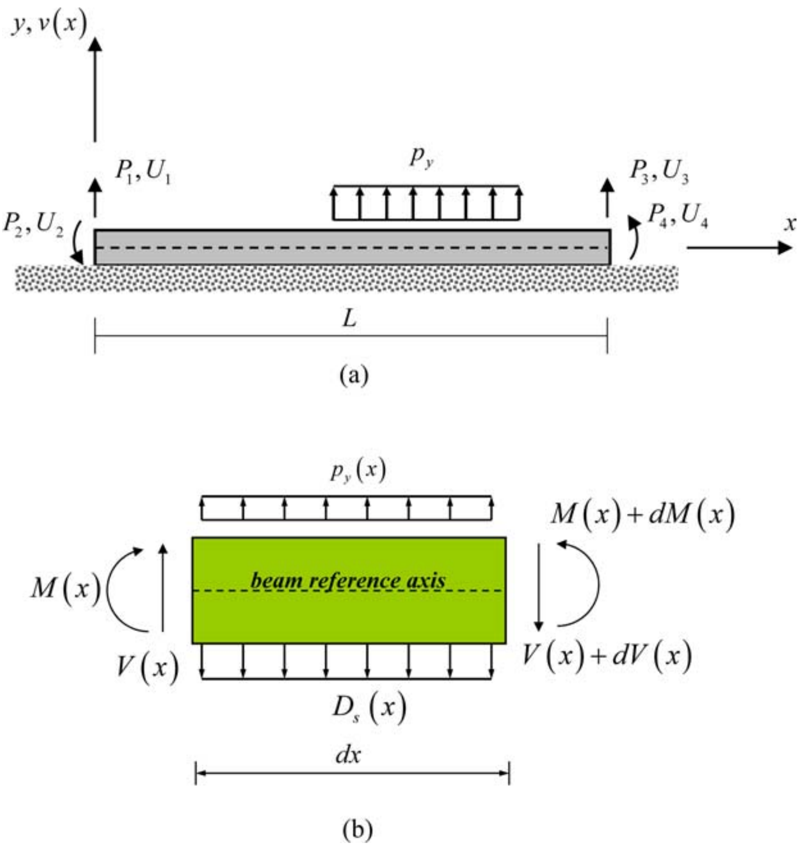


Fig. 1 (a) A beam on Winkler foundation, (b) a differential segment cut from the beam where $V(x)$ is the beam-section shear force; $p_y(x)$ is the transverse distributed load; and $D_s(x)$ is the foundation-force at the bottom face of the beam. Moment equilibrium results in

$$\frac{dM(x)}{dx} - V(x) = 0 \tag{2}$$

where $M(x)$ is the beam-section bending moment. Based on the Euler-Bernoulli beam theory, only flexural responses are considered in the paper. The shear force $V(x)$ is eliminated by substituting Eqs. (1) into (2), leading to

$$\frac{d^2M(x)}{dx^2} - p_y(x) + D_s(x) = 0 \tag{3}$$

It is worthwhile to note that at any beam section there are 2 internal force unknowns, $M(x)$ and $D_s(x)$ while only one equilibrium equation is available. Consequently, this system is internally statically indeterminate and the internal forces cannot be determined merely by equilibrium conditions.

2.2 Force-deformation relations

The system sectional forces are related to their conjugate-work deformations as follows

$$M(x) = IE\kappa(x) \quad \text{and} \quad D_s(x) = k_s v_s(x) \tag{4}$$

where $\kappa(x)$ is the beam-section curvature; $v_s(x)$ is the foundation deformation; IE is the flexural rigidity; and k_s is the foundation modulus known as subgrade-reaction coefficient (Terzaghi 1955).

2.3 Differential compatibility equations: the virtual force principle

As an alternative way to express the system compatibility equations, the virtual force equation is written in the general form as

$$\delta W^* = \delta W^*_{int} + \delta W^*_{ext} = 0 \tag{5}$$

where δW^* is the system total complementary virtual work; δW^*_{int} is the system internal complementary virtual work; and δW^*_{ext} is the system external complementary virtual work.

In the case of th beam on Winkler foundation, δW^*_{int} and δW^*_{ext} can be expressed as

$$\delta W^*_{int} = \int_L \delta M(x)\kappa(x)dx + \int_L \delta D_s(x)v_s(x)dx \tag{6}$$

$$\delta W^*_{ext} = -\int_L \delta p_y(x)v(x)dx - \delta \mathbf{P}^T\mathbf{U} \tag{7}$$

where the vector $\mathbf{P} = \{P_1 \; P_2 \; P_3 \; P_4\}^T$ contains shear forces and moments acting at beam ends and the vector $\mathbf{U} = \{U_1 \; U_2 \; U_3 \; U_4\}^T$ contains their conjugate-work displacements and rotations. At the moment, external force quantities, $\delta p_y(x)$ is arbitrarily chosen to be zero. Thus, Eq. (5) becomes

$$\delta W^* = \int_L \delta M(x)\kappa(x)dx + \int_L \delta D_s(x)v_s(x)dx - \delta\mathbf{P}^T\mathbf{U} = 0 \tag{8}$$

Using the deformation-force relations, Eq. (8) becomes

$$\delta W^* = \int_L \delta M(x)\frac{M(x)}{IE}dx + \int_L \delta D_s(x)\frac{D_s(x)}{k_s}dx - \delta\mathbf{P}^T\mathbf{U} = 0 \tag{9}$$

The foundation forces $D_s(x)$ and $\delta D_s(x)$ can be eliminated through the differential equilibrium condition of Eq. (3). Thus, Eq. (9) becomes

$$\int_L \delta M(x)\frac{M(x)}{IE}dx + \int_L \frac{d^2\delta M(x)}{dx^2}\left(\frac{1}{k_s}\right)\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)dx - \delta\mathbf{P}^T\mathbf{U} = 0 \tag{10}$$

In order to move all differential operators to the bending moment $M(x)$, integration by parts is applied twice to the second term of Eq. (10), hence leading to the following expression

$$\int_L \delta M(x)\left(\frac{M(x)}{IE} + \frac{1}{k_s}\left(\frac{d^4M(x)}{dx^4} - \frac{d^2p_y(x)}{dx^2}\right)\right)dx + \left[\frac{d\delta M(x)}{dx}\frac{1}{k_s}\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)\right]_0^L +$$

$$\left[\delta M(x)\frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3M(x)}{dx^3}\right)\right]_0^L - \delta\mathbf{P}^T\mathbf{U} = 0 \tag{11}$$

Considering the shear-force definition of Eq. (2) and following the Cartesian sign convention, Eq. (11) can be written as

$$\int_L \delta M(x)\left(\frac{M(x)}{IE} + \frac{1}{k_s}\left(\frac{d^4M(x)}{dx^4} - \frac{d^2p_y(x)}{dx^2}\right)\right)dx - \delta P_1\left(U_1 - \frac{1}{k_s}\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)_{x=0}\right)$$

$$-\delta P_2\left(U_2 - \frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3M(x)}{dx^3}\right)_{x=0}\right) - \delta P_3\left(U_3 - \frac{1}{k_s}\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)_{x=L}\right)$$

$$-\delta P_4\left(U_4 - \frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3M(x)}{dx^3}\right)_{x=L}\right) = 0 \tag{12}$$

Due to the arbitrariness of $\delta M(x)$, the governing differential compatibility equation is obtained as

$$\frac{d^4M(x)}{dx^4} + 4\lambda^4 M(x) - \frac{d^2p_y(x)}{dx^2} = 0 \quad \text{for} \quad x \in (0, L) \tag{13}$$

where $\lambda = \sqrt[4]{k_s/4IE}$. Additionally, the end-boundary compatibility conditions are obtained due to the arbitrariness of $\delta\mathbf{P}$ as

$$U_1 = \frac{1}{k_s}\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)_{x=0} \quad ; \quad U_2 = \frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3M(x)}{dx^3}\right)_{x=0}$$

$$U_3 = \frac{1}{k_s}\left(\frac{d^2M(x)}{dx^2} - p_y(x)\right)_{x=L} \quad ; \quad U_4 = \frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3M(x)}{dx^3}\right)_{x=L} \tag{14}$$

3. "NATURAL" element stiffness matrix via element flexibility matrix

The "*exact*" moment interpolation functions are obtained by solving the governing differential compatibility equation of Eq. (13). For the sake of simplicity, the applied distributed load $p_y(x)$ is assumed to be a linear function and to act along the whole length of the beam. It is imperative to mention that the applied distributed load $p_y(x)$ in Eq. (13) does not influence the exact moment interpolation functions as long as it varies linearly along the whole length of the beam. This finding renders the proposed flexibility-based model attractive since the analytical solution to the governing differential compatibility equation of Eq. (13) requires only the homogeneous part. Unfortunately, this beneficial effect is not available in the exact stiffness-based models published in the literature (e.g., Zhaohua and Cook 1983, Ting and Mockry 1984, Eisenberger and Yankelevsky 1985, etc.) since the analytical solution to governing differential equilibrium equation requires both homogeneous and particular parts with the presence of the applied distributed load $p_y(x)$. Therefore, the derivation of the exact displacement interpolation functions becomes more involved.

The homogeneous solution to Eq. (13) is

$$M(x) = e^{-\lambda x}(c_1\cos\lambda x + c_2\sin\lambda x) + e^{\lambda x}(c_3\cos\lambda x + c_4\sin\lambda x) \tag{15}$$

where c_1, c_2, c_3 and c_4 are constants of integration to be determined by imposing force boundary conditions. These force boundary conditions are

$$\left.\frac{dM}{dx}\right|_{x=0} = P_1; \quad -M(0) = P_2; \quad -\left.\frac{dM}{dx}\right|_{x=L} = P_3; \quad M(L) = P_4 \tag{16}$$

By imposing force boundary conditions of Eq. (16), the moment interpolation relation can be expressed as

$$M(x) = \mathbf{N}_{BB}(x)\mathbf{P} \tag{17}$$

where $\mathbf{N}_{BB}(x) = [N_{BB1}(x)\ \ N_{BB2}(x)\ \ N_{BB3}(x)\ \ N_{BB4}(x)]$ is an array containing the moment interpolation functions. The expression of each moment interpolation function is given in Appendix A. Imposing the differential equilibrium equation of Eq. (3), the foundation force $D_s(x)$ can be written in terms of $\mathbf{P}$ as

$$D_s(x) = \mathbf{N}_{sB}(x)\mathbf{P} \tag{18}$$

where $\mathbf{N}_{sB}(x) = [N_{sB1}(x)\ \ N_{sB2}(x)\ \ N_{sB3}(x)\ \ N_{sB4}(x)]$ is an array containing the foundation-force interpolation functions. The expression of each foundation-force interpolation function is given in Appendix A.

Applying the virtual force expression of Eq. (9), substituting Eqs. (17) and (18), and accounting for the arbitrariness of $\delta\mathbf{P}$ yield the following element flexibility equation

$$\mathbf{FP} = \mathbf{U} + \mathbf{U}_{p_y} \tag{19}$$

where $\mathbf{F}$ is the element flexibility matrix defined as

$$\mathbf{F} = \mathbf{F}_{BB} + \mathbf{F}_{ss} \tag{20}$$

where $\mathbf{F}_{BB}$ and $\mathbf{F}_{ss}$ are beam and foundation contributions to the element flexibility matrix, respectively.

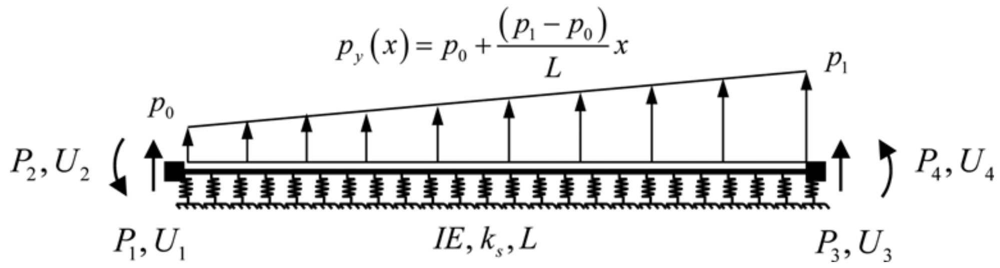


Fig. 2 Natural beam element on winkler foundation

$$\mathbf{F}_{BB} = \int_L \mathbf{N}_{BB}^T \left(\frac{1}{IE}\right) \mathbf{N}_{BB} dx \quad \text{and} \quad \mathbf{F}_{ss} = \int_L \mathbf{N}_{sB}^T \left(\frac{1}{k_s}\right) \mathbf{N}_{sB} dx \tag{21}$$

The explicit expressions of $\mathbf{F}_{BB}$ and $\mathbf{F}_{ss}$ are given in Appendix B. It is noted that element end displacements $\mathbf{U}_{p_y}$ due to the applied load $p_y(x)$ is supplemented into Eq. (19). In the case of linear variation of $p_y(x)$, $\mathbf{U}_{p_y}$ can be written in a simple expression as given in Appendix A.

Due to the supporting foundation, the beam does not experience any rigid- body motion (neither rigid-body translation nor rigid-body rotation). Therefore, the complete element stiffness equation can be obtained simply by inverting the element flexibility equation as

$$\mathbf{P} = \mathbf{K}_N \mathbf{U} + \mathbf{P}_{p_y}^{FE} \tag{22}$$

where the complete element stiffness matrix $\mathbf{K}_N$ is $\mathbf{F}^{-1}$ and the fixed-end force vector due to $p_y(x)$ is simply computed as $\mathbf{K}_N \mathbf{U}_{p_y}$. It is observed that the effect of the applied load $p_y(x)$ can be easily accounted for in the proposed flexibility-based model unlike the stiffness-based models available in the literature. The fixed-end force vectors due to other loading types can be derived in similar manner. The element stiffness matrix obtained in this manner is known as the "natural" element stiffness matrix (Argyris and Kelsey 1960). The configuration of the natural beam element on Winkler foundation is shown in Fig. 2.

As opposed to the stiffness-based formulation, there is no displacement interpolation function to describe the vertical displacement and rotational fields. However, the following compatibilities can be used to retrieve the vertical displacement and rotational fields once the internal force distributions are obtained.

$$v(x) = \frac{1}{k_s}\left(p_y(x) - \frac{d^2 M(x)}{dx^2}\right) \tag{22}$$

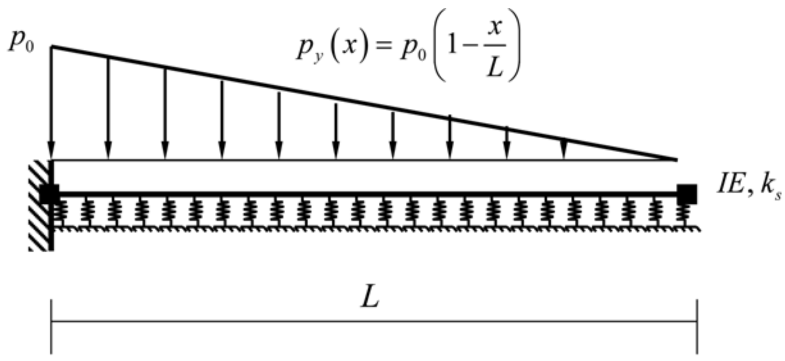
$$\theta(x) = \frac{dv(x)}{dx} = \frac{1}{k_s}\left(\frac{dp_y(x)}{dx} - \frac{d^3 M(x)}{dx^3}\right) \tag{23}$$

4. Numerical examples

Two numerical examples are used to verify the accuracy and show the efficiency of the natural beam element on Winkler foundation. The first numerical example is a cantilever beam-foundation system subjected to a triangular distributed load along the whole length. The second numerical example is a free-free beam on Winkler foundation subjected to concentrated loads along its length.

4.1 Example I

The cantilever beam on Winkler foundation subjected to the triangular distributed load along its whole length is shown in Fig. 3. To verify the accuracy and efficiency of the proposed beam element, only one element is used to discretize the entire cantilever beam-foundation system. Given data are: beam length $L = 5$ m; flexural rigidity $IE = 45 \times 10^3$ kN-m^2; foundation stiffness $k_s = 10^3$ kN/m^2; and distributed load parameter $p_0 = 100$ kN/m. The vertical displacement, rotation, shear force, and bending moment diagrams obtained with one natural beam-foundation element are shown in Fig. 4 and Fig. 5. On the same diagrams, the exact responses given by Hetenyi (1946) are also presented for comparison. All responses obtained from the two solutions match exactly, thus confirming the accuracy and efficiency of the proposed beam-foundation element. It is worth mentioning that only one "*exact*" stiffness-based beam-foundation element proposed by Eisenberger and Yankelevsky (1985) could also be used to accurately represent this cantilever beam-foundation system.



$L = 5\ m, IE = 45 \times 10^3\ kN - m^2; k_s = 10^3\ kN/m^2, p_0 = 100\ kN/m$

Fig. 3 Example I: cantilever beam subjected to triangular distributed loading along its whole span

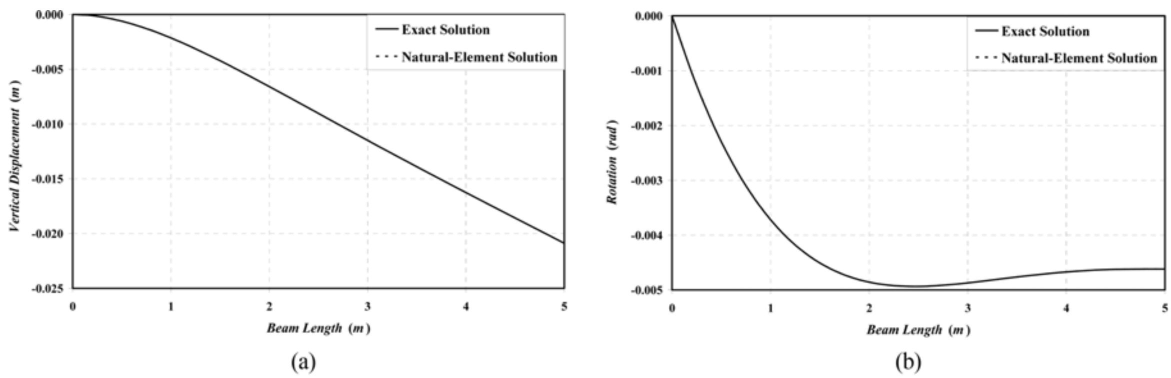
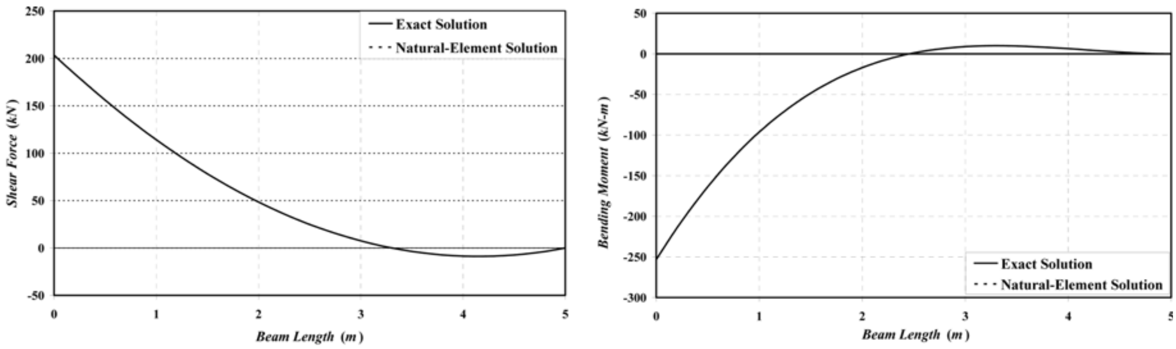


Fig. 4 Diagrams for vertical displacement and rotation for Example I



(a) (b)

Fig. 5 Diagrams for shear force and bending moment for Example I

4.2 Example II

The free-free beam on Winkler foundation subjected to concentrated loads along its length is shown in Fig. 6. This beam-foundation system is employed to demonstrate the application of the derived beam-foundation element to study the structure-foundation interaction problem. Given data are: beam length $L = 20$ m; flexural rigidity $IE = 200 \times 10^3$ kN-m^2; and foundation stiffness $k_s = 30 \times 10^3$ kN/m^2. Four natural beam-foundation elements (elements *AB*, *BC*, *CD*, and *DE*) are used to discretize the system, thus resulting in eight nodal unknowns. The obtained vertical displacement, rotation, shear force, and bending moment diagrams are shown in Fig. 7 and Fig. 8. On the same diagrams, the exact responses given by Hetenyi (1946) are also shown for comparison. Clearly, the natural beam-foundation model is capable of representing exact displacement and force responses using only one element for each free span. In fact, only one natural element is sufficient to model the whole beam-foundation system if the end displacements and rotations due to intermediate concentrated loads are consistently derived. It is worth mentioning that this beam-foundation example is also analyzed by Pilkey (2007) using the stiffness-based beam-foundation model with cubic displacement interpolation functions proposed by Tong and Rossettos (1977). To obtain satisfactory nodal-displacement values, ten stiffness-based elements are required, thus resulting in twenty two nodal unknowns. Furthermore, twenty one stiffness-based elements (forty four nodal unknowns) are needed to satisfactorily represent the shear and bending moment variations along the beam length.

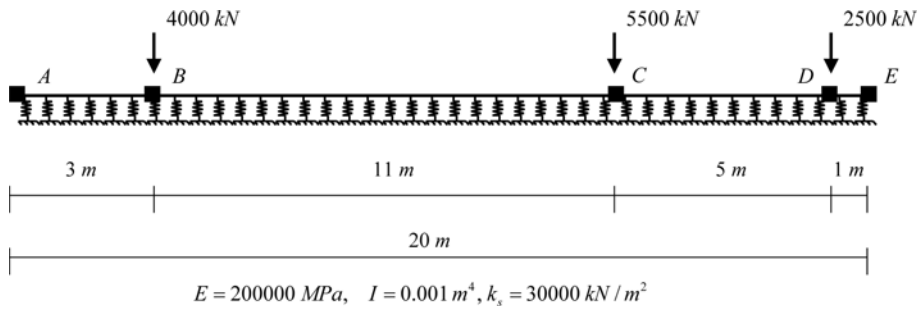


Fig. 6 Example II: free-free beam on Winkler foundation subjected to concentrated loads along its length

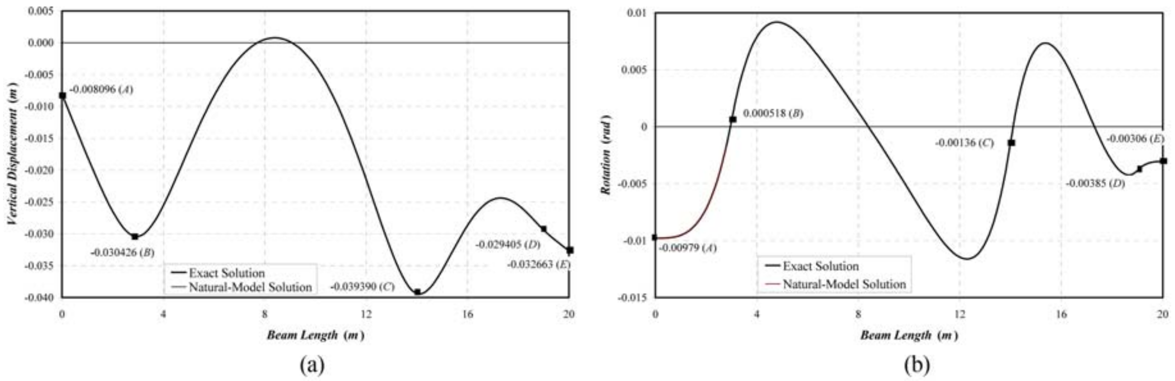


Fig. 7 Diagrams for vertical displacement and rotation for Example II

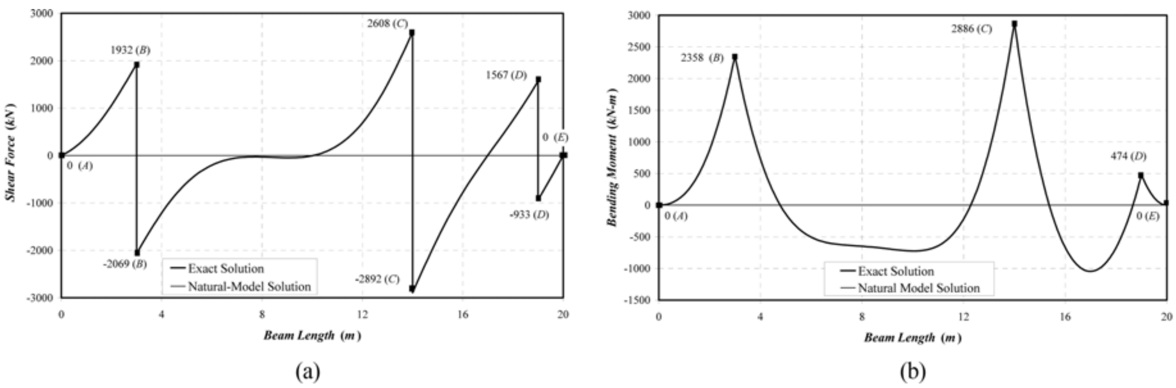


Fig. 8 Diagrams for shear force and bending moment for Example II

5. Conclusions

The "*natural*" element stiffness matrix and fixed-end force vector for a beam on Winkler foundation subjected to a linearly distributed load are derived in this paper. The element flexibility matrix forms the core of the natural element stiffness matrix and is derived based on the virtual force principle using the "*exact*" moment interpolation functions. The exact moment interpolation functions are obtained by solving analytically the governing differential compatibility equation. Compared to the stiffness-based models published in the literature, the effect of the applied element load can readily be included in the proposed model. Two numerical examples are employed to verify the accuracy and efficiency of the natural beam-foundation model. These two numerical examples demonstrate that the natural beam-foundation element is capable of giving exact solutions for vertical displacements, rotations, shear forces, and bending moments. Therefore, the exactness of the proposed element obviates the requirement for discretizing the beam into several elements between loading points. The number of elements needed in analysis of a beam-foundation system is largely governed by the proper way of representing loadings (concentrated or linearly distributed loads). The next steps in this research direction are the inclusion of the nonlinearities into both beam and foundation and applying the extended model to represent several engineering systems (e.g., shallow foundation-structure system, pavement-soil system, carbon nanotube-Van der Waals system, etc.).

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Appendix A: Moment interpolation functions, foundation- force interpolation functions, and nodal displacements due to $p_y(x)$

The moment interpolation functions may be written as

$$N_{BB1}(x) = \frac{\phi_3 e^{2\lambda L}(\cos(2L-x)\lambda - \cos\lambda x) + \phi_4(e^{2\lambda L}-1)\sin\lambda x}{2\lambda\phi_1}$$

$$N_{BB2}(x) = \frac{-\phi_5 e^{2\lambda L}\cos(2L-x)\lambda - \phi_6\cos\lambda x + \phi_3 e^{2\lambda L}\sin(2L-x)\lambda - \phi_7\sin\lambda x}{2\phi_1}$$

$$N_{BB3}(x) = \frac{\phi_4\left(\cos(L+x)\lambda - \cos(L-x)\lambda\right) + \phi_3(e^{2\lambda L}-1)\sin(L-x)\lambda}{\lambda\phi_2}$$

$$N_{BB4}(x) = \frac{(\phi_8 - 2\phi_9)\cos(L-x)\lambda + \phi_9\cos(L+x)\lambda + \phi_{10}\sin(L-x)\lambda - \phi_4\sin(L+x)\lambda}{\phi_2}$$

The foundation-force interpolation functions may be written as

$$N_{sB1}(x) = \frac{\lambda(\phi_9(e^{2\lambda L}-1)\cos\lambda x - \phi_5 e^{2\lambda L}(\sin(2L-x)\lambda + \sin\lambda x))}{\phi_1}$$

$$N_{sB2}(x) = \frac{\lambda^2(\phi_5 e^{2\lambda L}\cos(2L-x)\lambda - \phi_{11}\cos\lambda x + \phi_3 e^{2\lambda L}\sin(2L-x)\lambda + \phi_{12}\sin\lambda x)}{\phi_1}$$

$$N_{sB3}(x) = \frac{2\lambda(\phi_5(e^{2\lambda L}-1)\cos(L-x)\lambda - 2\phi_9\sin\lambda L\cos\lambda x)}{\phi_2}$$

$$N_{sB4}(x) = \frac{2\lambda^2(\phi_8\cos(L-x)\lambda - \phi_9\cos(L+x)\lambda - \phi_{13}\sin(L-x)\lambda - \phi_4\sin(L+x)\lambda)}{\phi_2}$$

where

$$\phi_1 = \frac{-2+\cos 2\lambda L + \cosh 2\lambda L}{e^{-(x+2L)\lambda}};\quad \phi_2 = \frac{1+e^{4\lambda L} + 2e^{2\lambda L}(\cos 2\lambda L - 2)}{e^{(L-x)\lambda}};\quad \phi_3 = e^{2\lambda x} - 1;$$

$$\phi_4 = e^{2\lambda L} - e^{2\lambda x};\quad \phi_5 = e^{2\lambda x}+1;\ \phi_6 = -2e^{2\lambda L} + e^{4\lambda L} + e^{2\lambda x} - 2e^{2(L+x)\lambda};\ \phi_7 = e^{4\lambda L} - e^{2\lambda x};$$

$$\phi_8 = e^{2(L+x)\lambda}+1;\ \phi_9 = e^{2\lambda L} + e^{2\lambda x};\ \phi_{10} = e^{2(L+x)\lambda} - 1;\ \phi_{11} = e^{4\lambda L} + e^{2\lambda x};$$

$$\phi_{12} = -2e^{2\lambda L} + e^{4\lambda L} - e^{2\lambda x} + 2e^{2(L+x)\lambda};\ \text{and}\ \phi_{13} = -1 + 2e^{2\lambda L} - 2e^{2\lambda x} + e^{2(L+x)\lambda}$$

The nodal displacements due to the linearly distributed load $p_y(x) = p_0 + \frac{(p_1-p_0)}{L}x$ may be written as

$$U_{1p_y} = \frac{p_0}{k_s};\quad U_{2p_y} = \frac{p_1-p_0}{k_s L};\quad U_{3p_y} = \frac{p_1}{k_s};\quad \text{and}\quad U_{4p_y} = \frac{p_1-p_0}{k_s L}$$

Appendix B: Beam on winkler foundation flexibility matrix

The beam contribution to the element flexibility matrix may be written as

$$\mathbf{F}_{BB} = \begin{bmatrix} F_{11}^{BB} & F_{12}^{BB} & F_{13}^{BB} & F_{14}^{BB} \\ & F_{22}^{BB} & F_{23}^{BB} & F_{24}^{BB} \\ & & F_{33}^{BB} & F_{34}^{BB} \\ Symm. & & & F_{44}^{BB} \end{bmatrix}$$

$$F_{11}^{BB} = \frac{8\beta\cos(2\beta) + 4\sin(2\beta) - 2\psi_2\cosh(2\beta) - \sin(4\beta) + 2\psi_3\sinh(2\beta) + \sinh(4\beta)}{16\psi_1^2\lambda^3 IE}$$

$$F_{12}^{BB} = \frac{-4\cos(2\beta) + \cos(4\beta) + 4\cosh(2\beta) - \cosh(4\beta) - 8\beta\psi_4}{8\psi_1^2\lambda^2 IE}$$

$$F_{13}^{BB} = -\frac{\psi_5\cosh(3\beta) + \psi_6\cosh(\beta) - \left(\cos(3\beta) + 16\beta\sin(\beta) + \psi_7\cos(\beta)\right)\sinh(\beta)}{8\psi_1^2\lambda^3 IE}$$

$$F_{14}^{BB} = \frac{\beta\cosh(3\beta)\sin(\beta) - \beta\psi_8\cosh(\beta) + \psi_9\sinh(\beta) - \psi_{10}\sinh(3\beta)}{4\psi_1^2\lambda^2 IE}$$

$$F_{22}^{BB} = \frac{-32\beta\sin^2(\beta)\sinh^2(\beta) + 6\psi_{11}\psi_{12}}{8\psi_1^2\lambda IE}$$

$$F_{23}^{BB} = -F_{14}^{BB}$$

$$F_{24}^{BB} = \frac{6\psi_1\cos(\beta)\sinh(\beta) + \psi_{13}\sinh(3\beta) - \psi_{14}\sin(\beta)}{4\psi_1^2\lambda IE}$$

$$F_{33}^{BB} = F_{11}^{BB}$$

$$F_{34}^{BB} = -F_{12}^{BB}$$

$$F_{44}^{BB} = F_{22}^{BB}$$

in which $\beta = \lambda L$; $\psi_1 = \cosh(2\beta) + \cos(2\beta) - 2$; $\psi_2 = 4\beta + \sin(2\beta)$;

$\psi_3 = 4\beta\sin(2\beta) + \cos(2\beta) - 2$; $\psi_4 = \sin(2\beta)\sinh^2(\beta) - \sinh(2\beta)\sin^2(\beta)$;

$\psi_5 = 2\beta\cos(\beta) + \sin(\beta)$; $\psi_6 = \sin(3\beta) - 2\beta\cos(3\beta) - 4\sin(\beta)$; $\psi_7 = 2\cosh(2\beta) - 3$;

$\psi_8 = \sin(3\beta) - 2\sin(\beta)$; $\psi_9 = 2\beta\cos(\beta) + \beta\cos(\beta) + 12\sin(\beta) - 2\sin(\beta)$;

$\psi_{10} = \beta\cos(\beta) + 2\sin(\beta)$; $\psi_{11} = 6\cosh(2\beta) + \cos(2\beta) - 2$; $\psi_{12} = \sin(2\beta) + \sinh(2\beta)$;

$\psi_{13} = 3\cosh(\beta) + 2\beta\sinh(\beta)$; $\psi_{14} = 12\cosh(\beta) - 3\cosh(3\beta) + 2\beta\sinh(3\beta)$

The foundation contribution to the element flexibility matrix may be written as

$$\mathbf{F}_{ss} = \begin{bmatrix} F_{11}^{ss} & F_{12}^{ss} & F_{13}^{ss} & F_{14}^{ss} \\ & F_{22}^{ss} & F_{23}^{ss} & F_{24}^{ss} \\ & & F_{33}^{ss} & F_{34}^{ss} \\ Symm. & & & F_{44}^{ss} \end{bmatrix}$$

$$F_{11}^{ss} = \frac{\lambda\left(\zeta_2 \cosh(2\beta) - 8\beta\cos(2\beta) + 12\sin(2\beta) - 3\sin(4\beta) - 2\zeta_3 \sinh(2\beta) + 3\sinh(4\beta)\right)}{4\zeta_1^{\,2}}$$

$$F_{12}^{ss} = \frac{\lambda^2 \zeta_4}{2\zeta_1^{\,2}}$$

$$F_{13}^{ss} = -\frac{\lambda\left(\zeta_5 \cosh(3\beta) + \zeta_6 \cosh(\beta) + \zeta_7 \sinh(\beta) - 3\cos(\beta)\sinh(3\beta)\right)}{2\zeta_1^{\,2}}$$

$$F_{14}^{ss} = \frac{\lambda^2\left(\zeta_8 \beta\cosh(\beta) - \beta\cosh(3\beta)\sin(\beta) - \left(\beta\zeta_9 + 2\zeta_{10}\right)\sinh(\beta) + \zeta_{11}\sinh(3\beta)\right)}{\zeta_1^{\,2}}$$

$$F_{22}^{ss} = \frac{\lambda^3\left(\zeta_{12} + 2(\cos(2\beta) - 2)\sinh(2\beta) + \sinh(4\beta)\right)}{2\zeta_1^{\,2}}$$

$$F_{23}^{ss} = -F_{14}^{ss}$$

$$F_{24}^{ss} = \frac{\lambda^3((e^{2\beta} - 1)\cos(3\beta) + 2e^{\beta}(\zeta_{13} + \zeta_{14}\cos(\beta) + \zeta_{15}\sin(\beta)))}{2e^{\beta}\zeta_1^{\,2}}$$

$$F_{33}^{ss} = F_{11}^{ss}$$

$$F_{34}^{ss} = -F_{12}^{ss}$$

$$F_{44}^{ss} = F_{22}^{ss}$$

in which; $\zeta_1 = \sqrt{k}\psi_1$; $\quad \zeta_2 = 8\beta - 6\sin(2\beta)$; $\quad \zeta_3 = 6 - 3\cos(2\beta) + 4\beta\sin(2\beta)$;

$\zeta_4 = \cos(4\beta) - 4\cos(2\beta) + 4\cosh(2\beta) - \cosh(4\beta) + 8\beta\psi_4$; $\zeta_5 = 3\sin(\beta) - 2\beta\cos(\beta)$;

$\zeta_6 = 2\beta\cos(3\beta) - 3(\psi_8 - 2\sin(\beta))$; $\quad \zeta_7 = 12\cos(\beta) - 3\cos(3\beta) + 16\beta\sin(\beta)$; $\quad$; $\zeta_8 = \psi_8$;

$\zeta_9 = \cos(3\beta) + 2\cos(\beta)$; $\zeta_{10} = \psi_8 - 4\sin(\beta)$; $\zeta_{11} = \beta\cos(\beta) - 2\sin(\beta)$;

$\zeta_{12} = 32\beta\sin^2(\beta)\sinh^2(\beta) + 2\cosh(2\beta)\sin(2\beta) + \sin(4\beta) - 4\sin(2\beta)$;

$\zeta_{13} = \sin(3\beta)\cosh(\beta) - 2\beta\sin(3\beta)\sinh(\beta)$; $\zeta_{14} = \sinh(3\beta) - 4\sinh(\beta)$;

$\zeta_{15} = 2\beta\sinh(3\beta) + \cosh(3\beta) - 4\cosh(\beta)$