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NURBS-based isogeometric analysis for thin plate problems

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Abstract. An isogeometric approach is presented for static analysis of thin plate problems of various geometries. Non-Uniform Rational *B*-Splines (NURBS) basis function is applied for approximation of the thin plate deflection, as for description of the geometry. The governing equation based on Kirchhoff plate theory, is discretized using the standard Galerkin method. The essential boundary conditions are enforced by the Lagrange multiplier method. Several typical examples of thin plate and thin plate on elastic foundation are solved and compared with the theoretical solutions and other numerical methods. The numerical results show the robustness and efficiency of the proposed approach.

Keywords: isogeometric analysis; NURBS; kirchhoff plate; lagrange multiplier method; elastic foundation

1. Introduction

Isogeometric analysis (IGA) is a newly developed computational approach (Hughes *et al.* 2005) with the aim of integrating computer aided design (CAD) into structural analysis. Isogeometric approach employs the same shape functions used for geometry description as basis functions for the analysis. Using CAD basis functions directly in the finite element analysis (FEA), leads to eliminate the existing time-consuming data conversion process among CAD systems and finite element packages in engineering problems. This is one main idea behind developing isogeometric analysis and vastly simplifies mesh refinement of complex industrial geometries (Hughes *et al.* 2005). Moreover, exact representation of common engineering shapes even at the coarsest level of discretization, simple and systematic refinement strategy, and more accurate modeling of complex geometries make isogeometric analysis a very robust approach.

Since Non-Uniform Rational *B*-splines (NURBS) are the most common basis functions in the CAD systems, they are the first candidate as a basis for the isogeometric analysis. NURBS-based isogeometric analysis benefits from three refinement strategies including classical *h*- and *p*-refinements analogue, and a new *k*-refinement strategy that increases the inter-element continuity beyond the standard C^0-continuity of conventional finite elements and presents superior accuracy and efficiency compared with the classical *p*-refinement (Hughes *et al.* 2005). These IGA features attracted many research works in various engineering applications including fluid mechanics

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(Bazilevs *et al.* 2008a), fluid-structure interaction problems (Bazilevs *et al.* 2006, 2008b), structural shape optimization (Wall *et al.* 2008, Qian 2010), shell analysis (Benson *et al.* 2010, Kiendl *et al.* 2009, Uhm *et al.* 2009), damage and fracture mechanics (Verhoosel *et al*. 2011a, b, Ghorashi *et al*. 2011), etc. In this paper, we investigate the performance of NURBS-based isogeometric analysis for thin plate problems with various geometries.

Among many plate theories that have been developed up to now, two are widely accepted and used in finite element methods. These are the Kirchhoff theory of plates (classical plate theory) and the Reissner-Mindlin theory of plates (first-order shear plate theory). From the perspective of finite element analysis, for Reissner-Mindlin theory, the elements with only C^0-inter-element continuity are required, but the Kirchhoff plate formulation needs elements with at least C^1-inter-element continuity. Hence, Kirchhoff theory needs higher order elements in comparison with Reissner-Mindlin theory which is easier to implement by standard polynomial basis functions but suffers from shear locking effects. Using *k*-refinement strategy in isogeometric analysis, higher order NURBS basis functions with increased inter-element continuity can be obtained easily. This feature simplifies the formulation of Kirchhoff elements in IGA. Therefore, the Kirchhoff plate theory is used in this study for formulation of thin plate problems.

In isogeometric analysis, NURBS basis functions are not interpolatory, which implies that the shape functions do not satisfy the Kronecker-delta property. Therefore, imposition of essential boundary conditions in IGA has some difficulties. To impose the essential boundary conditions in numerical methods with non-interpolatory basis function, some techniques have been proposed such as Lagrange multipliers (Belytschko *et al*. 1994), penalty method (Zhu and Atluri 1998) and the Nitsche's method (Nitsche 1971). In the present paper, Lagrange multiplier method, originated from (Belytschko *et al.* 1994, Krysl and Belytschko 1995) is used to enforce essential boundary conditions.

The paper is organized as follow: First the NURBS basis function is introduced in Section 2. According to Kirchhoff plate theory, the isogeometric formulation of thin plate is derived in Section 3. Section 4 gives some numerical examples of thin plate and thin plate on elastic foundation with different boundary conditions and various geometries to verify the method proposed in this paper. Finally, ending remarks are given in Section 5.

2. NURBS basis function

In this section, the *B*-spline and NURBS functions are briefly reviewed. For more details on NURBS, readers are referred to (Piegl and Tiller 1997).

The non-uniform rational *B*-spline (NURBS) curve of order *p* is given by

$$C(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi) B_i \tag{1}$$

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi) w_i}{\sum_{j=1}^{n} N_{j,p}(\xi) w_j} \tag{2}$$

where $R_{i,p}$ is the NURBS basis function, $B_i = (x_{i_1}, x_{i_2})$, $i = 1, 2, \ldots, n$ represents the positions of a set of control points, w_i is the weight corresponding with each control point that must be non-negative and $N_{i,p}$ is the B-spline basis function of order p which is defined in a parametric space on a so-called knot vector Ξ.

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\} \quad \xi_i \le \xi_{i+1} \qquad (3)$$
$$i = 1, 2, \ldots, n+p$$

In isogeometric analysis, open knot vectors are used to satisfy the Kronecker-delta property at boundary points (Roh and Cho 2004). A knot vector is said to be open if the end knots are repeated $p+1$ times. With a given knot vector, B-spline basis functions are defined by the following Cox-de Boor recursive formula (Piegl and Tiller 1997)

$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi \le \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

and

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \quad p = 1, 2, 3, \ldots \qquad (5)$$

B-spline basis functions which are built from open knot vectors are interpolatory at the ends of parametric space. Fig. 1 shows the quartic B-spline basis functions with the interpolation feature at the ends of parametric space.

A NURBS surface of order p in ξ_1 direction and order q in ξ_1 direction is

$$S(\xi_1, \xi_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} R_{i,j}^{p,q}(\xi_1, \xi_2) B_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{N_{i,p}(\xi_1) M_{j,q}(\xi_2) w_{i,j}}{\sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m} N_{\hat{i},p}(\xi_1) M_{\hat{j},q}(\xi_2) w_{\hat{i},\hat{j}}} B_{i,j} \quad 0 \le \xi_1, \xi_2 \le 1 \qquad (6)$$

where $B_{i,j}$ is a control net of $n \times m$ control points, $w_{i,j}$ are the corresponding weights, $N_{i,p}$ and $M_{j,q}$ are the B-spline basis functions defined on Ξ_1 and Ξ_2 knot vectors, respectively.

NURBS basis functions have the following important properties:

(1) Partition of unity, $\forall \xi, \sum_{i=1}^{n} R_{i,p}(\xi) = 1$

(2) Non-negativity, $\forall \xi, R_{i,p}(\xi) \ge 0$

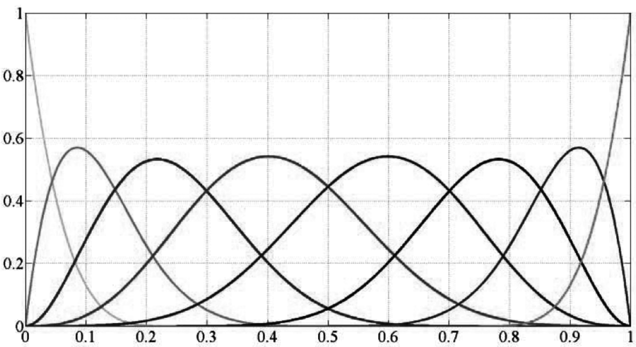

Fig. 1 Quartic basis functions for an open knot vector Ξ = {0, 0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1, 1}

(3) Basis functions of order p have $p-m_i$ continual derivatives across knot ξ_i, where m_i is the multiplicity of the value of ξ_i in the knot vector.

(4) Local support, i.e., the support of $R_{i,p}(\xi)$ is compact and contained in the interval $[\xi_i, \xi_{i+p+1}]$. In two dimensional cases, the support of a specified bivariate basis function $R_{i,j}^{p,q}(\xi_1, \xi_2)$ is $[\xi_i, \xi_{i+p+1}] \times [\eta_j, \eta_{j+p+1}]$. In other words, there are only $(p+1)\times(q+1)$ number of non-zero basis functions for a given knot span (element). Therefore, the total number of control points per element is $n_{en} = (p+1)\times(q+1)$.

3. Isogeometric formulation of Kirchhoff plate problem

Based on the assumptions of Kirchhoff theory of plates, the strong form of governing equation for thin plate problem can be written as

Equilibrium equation: $$\nabla^4 w = \frac{q}{D_0} \quad \text{in} \quad \Omega \tag{7a}$$

Natural boundary condition: $$\boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_t \tag{7b}$$

Essential boundary condition: $$\tilde{\mathbf{w}} = \overline{\mathbf{w}} \quad \text{on} \quad \Gamma_u \tag{7c}$$

where w is the deflection of the plate, q is uniform load, Ω denotes the domain of the plate problem. $D_0 = Et^3/12(1-\nu^2)$ is the flexural stiffness matrix, where E, ν and t are Young's modulus, Poisson ratio and the thickness of the plate, respectively. $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{n}$ is the unit outward normal vector on the boundary of the domain, $\bar{\mathbf{t}}$ is the prescribed boundary forces on the natural boundary and $\overline{\mathbf{w}}$ is the prescribed displacement on the essential boundary.

$\tilde{\mathbf{w}}$ is a vector defined as $\tilde{\mathbf{w}} = \mathbf{Q}w$, where $\mathbf{Q} = \{1 \ \ \partial^2/\partial n^2\}^T$ for simply supported boundary and $\mathbf{Q} = \{1 \ \ \partial/\partial n\}^T$ for clamped boundary.

Because of non-interpolatory nature of NURBS basis functions, they do not satisfy the kronecker delta property. For this reason, the essential boundary conditions need to be imposed by an appropriate approach. In this study, the Lagrange multiplier method is employed as a scheme for imposition of essential boundary conditions. In this method, using the Lagrange multiplier $\boldsymbol{\lambda}$, the essential boundary condition (Eq. (7c)) is converted into an integral form

$$\int_{\Gamma_u} \boldsymbol{\lambda}^T(\tilde{\mathbf{w}} - \overline{\mathbf{w}})d\Gamma \tag{8}$$

Therefore, two additional boundary condition terms are added to the weak form of the static elastic equilibrium equation of thin plate. This modified weak form can be written as

$$\int_{\Omega} \delta\boldsymbol{\varepsilon}_p^T \boldsymbol{\sigma}_p d\Omega - \int_{\Omega} \delta w q d\Omega - \int_{\Gamma_t} \delta\mathbf{u}^T \bar{\mathbf{t}} d\Gamma - \int_{\Gamma_u} \delta\boldsymbol{\lambda}^T(\tilde{\mathbf{w}} - \overline{\mathbf{w}})d\Gamma - \int_{\Gamma_u} \delta\tilde{\mathbf{w}}^T \boldsymbol{\lambda} d\Gamma = 0 \tag{9}$$

where $\mathbf{u}$ denotes the displacement fields of the Kirchhoff plate, $\boldsymbol{\varepsilon}_p$ and $\boldsymbol{\sigma}_p$ are the pseudo-strain and the pseudo-stress, respectively. The displacement fields in a Kirchhoff plate (Timoshenko and Woinowsky-Krieger 1995) are

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} -z\frac{\partial}{\partial x} & -z\frac{\partial}{\partial y} & 1 \end{Bmatrix}^T w = \mathbf{L}_u w \tag{10}$$

The pseudo-strains of the plate are defined as

$$\boldsymbol{\varepsilon}_p = \begin{Bmatrix} -\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial y^2} & -2\frac{\partial^2}{\partial x \partial y} \end{Bmatrix}^T w = \mathbf{L}w \tag{11}$$

The pseudo-stresses or moments (per unit length) of the plate are also defined as

$$\boldsymbol{\sigma}_p = \{M_x \;\; M_y \;\; M_{xy}\}^T = D_0 \mathbf{D} \boldsymbol{\varepsilon}_p \tag{12}$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{13}$$

To solve the Kirchhoff plate problem in an isogeometric framework, the two dimensional NURBS basis functions are applied to approximate the geometry and deflection of the plate as

$$\mathbf{x}(\xi_1, \xi_2) = \mathbf{RB} \qquad 0 \le \xi_1, \xi_2 \le 1 \tag{14}$$

$$w^h(\xi_1, \xi_2) = \hat{\mathbf{R}}\mathbf{w} \qquad 0 \le \xi_1, \xi_2 \le 1 \tag{15}$$

where $\hat{\mathbf{R}}$ is a vector of NURBS basis functions that has the form of $(R_{1,1}, R_{2,1}, \ldots, R_{n,m})$. $\mathbf{R}$ is the matrix of NURBS basis functions, $\mathbf{B}$ is the vector of control point positions and $\mathbf{w}$ is the vector of control variables (the plate deflection at the control points) which are defined as

$$\mathbf{R} = \begin{bmatrix} R_{1,1} & 0 & R_{2,1} & 0 & \ldots & R_{n,m} & 0 \\ 0 & R_{1,1} & 0 & R_{2,1} & \ldots & 0 & R_{n,m} \end{bmatrix} \tag{16}$$

$$\mathbf{B} = (\mathbf{B}^x_{1,1}, \mathbf{B}^y_{1,1}, \mathbf{B}^x_{2,1}, \mathbf{B}^y_{2,1}, \ldots, \mathbf{B}^y_{n,m})^T \tag{17}$$

$$\mathbf{w} = (w_{1,1}, w_{2,1}, \ldots, w_{n,m})^T \tag{18}$$

The matrix-form of $R_{i,j}$ and $B_{i,j}$ can be also changed into vector-form by the following mapping

$$k = i + (j-1)n \quad \text{with} \quad k = 1, 2, \ldots, n, m \tag{19}$$

To obtain the discretized equation from the Eq. (9), Lagrange multipliers must be discretized. In this study, the Lagrange shape functions are considered as the interpolation space for the Lagrange multipliers. The Lagrange multipliers are defined at several essential boundary points. An appropriate choice for these desired essential boundary points is the *Greville abscissas* (Hoschek and Lasser 1993), defined as

$$s_i^p = \frac{\xi_{i+1} + \ldots + \xi_{i+p}}{p}, \quad i = 1, \ldots, n \tag{20}$$

where p is the NURBS order and n is the number of control points.

Interpolating the Lagrange multipliers using 1D Lagrange shape functions on physical boundary points corresponding to Greville abscissas, leads to

$$\lambda(u) = \sum_{i=1}^{n_\lambda} S_i(u)\lambda_i \tag{21}$$

where $S_i(u)$ is the 1D Lagrange basis function and n_λ is the number of the essential boundary points applied for this interpolation.

Substituting the deflection of the plate w of form (15) into the weak form (34) leads to the following discretized equations

$$\begin{pmatrix} \mathbf{K} & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{q} \end{pmatrix} \tag{22}$$

where $\mathbf{K}$ and $\mathbf{f}$ are given in Eqs. (23) and (24), respectively.

$$\mathbf{K}_{ij} = \int_\Omega \mathbf{B}_i^T D_0 \mathbf{D} \mathbf{B}_j d\Omega \tag{23}$$

$$\mathbf{f}_i = \int_\Omega R_i q d\Omega + \int_{\Gamma_t} \mathbf{F}_i^T \bar{\mathbf{t}} d\Gamma \tag{24}$$

$$\mathbf{B}_i = \begin{Bmatrix} -R_{i,xx} \\ -R_{i,yy} \\ -2R_{i,xy} \end{Bmatrix}, \quad \mathbf{F}_i = \begin{Bmatrix} -zR_{i,x} \\ -zR_{i,y} \\ R_i \end{Bmatrix} \tag{25}$$

where $R_j(\xi, \eta)$ is the 2D NURBS basis function and z is the transverse direction. The nodal matrix $\mathbf{G}_{ij}$ and the vector $\mathbf{q}_i$ are defined as

$$\mathbf{G}_{ij} = -\int_{\Gamma_u} \mathbf{S}_i(u) \mathbf{\Psi}_j(\xi, \eta) d\Gamma \tag{26}$$

$$\mathbf{q}_i = -\int_{\Gamma_u} \mathbf{S}_i(u) \bar{\mathbf{w}} d\Gamma \tag{27}$$

$$\mathbf{S}_i = \begin{bmatrix} S_i & 0 \\ 0 & S_i \end{bmatrix} \tag{28}$$

For clamped boundary, $\quad \mathbf{\Psi}_j(\xi, \eta) = \begin{bmatrix} R_j(\xi, \eta) \\ R_{j,n}(\xi, \eta) \end{bmatrix}$

For simply supported boundary, $\quad \mathbf{\Psi}_j(\xi, \eta) = \begin{bmatrix} R_j(\xi, \eta) \\ R_{j,nn}(\xi, \eta) \end{bmatrix}$ (29)

where n is the unit normal to the essential boundary Γ_u.

4. Numerical simulations

To validate the proposed approach, several numerical examples of various geometric shapes with different boundary conditions are solved and the results are compared with available analytical or other numerical solutions. In all examples, the material properties are taken as Young's modulus $E = 2.0 \times 10^8$ N/m^2, Poisson ratio $\nu = 0.3$, unless specified otherwise. For the sake of simplicity, the fully simply supported boundary conditions have been enforced using the direct method (Hughes *et al.* 2005). The standard Gauss quadrature over each knot span is applied for numerical integration. The number of Gauss points in each direction is considered $\max(p+1, 4)$, where p is the order of NURBS basis function. For the convergence study, h-refinement strategy has been applied. In each refinement step, knots are added to the middle of knot spans.

4.1 Square plate uniformly loaded

For the first example, a thin square plate under uniform load is modeled with different boundary conditions: (SSSS) simply supported boundary conditions on all sides, (CCCC) clamped boundary conditions on all sides, (SCSC) simply supported boundary conditions on two opposite sides and clamped boundary conditions on the other sides, (CSSS) clamped boundary condition on one side and simply supported conditions on the other sides. For this example, the following parameters are considered: length $L_x = L_y = 1$ m, thickness $t = 0.01$ m, uniform lateral load $p_0 = -100$ N.

This problem has been computed with different mesh refinements (16, 64, 256 and 1024 elements) and different polynomial orders ($p = 2, 3, 4, 5$). Tables 1 and 2 show the results of dimensionless central deflections, obtained by NURBS-based Isogeometric analysis, for fully clamped and simply supported boundary conditions, respectively. The relative error of central deflection for these boundary conditions are also plotted in Figs. 2 and 3. For all polynomial orders,

Table 1 Dimensionless central deflections of CCCC square plate under uniform load

Number of elements	Order of NURBS basis function				Exact solution (Timoshenko and Woinowsky 1995)
	2	3	4	5	
16	0.0009875	0.0012611	0.0012665	0.0012650	0.0012653
64	0.0011950	0.0012651	0.0012653	0.0012653	
256	0.0012477	0.0012653	0.0012653	0.0012653	
1024	0.0012609	0.0012653	0.0012653	0.0012653	

Table 2 Dimensionless central deflections of SSSS square plate under uniform load

Number of elements	Order of NURBS basis function				Exact solution (Timoshenko and Woinowsky 1995)
	2	3	4	5	
16	0.0039800	0.0040644	0.0040632	0.0040622	0.0040624
64	0.0040433	0.0040625	0.0040624	0.0040624	
256	0.0040577	0.0040624	0.0040624	0.0040624	
1024	0.0040612	0.0040624	0.0040624	0.0040624	

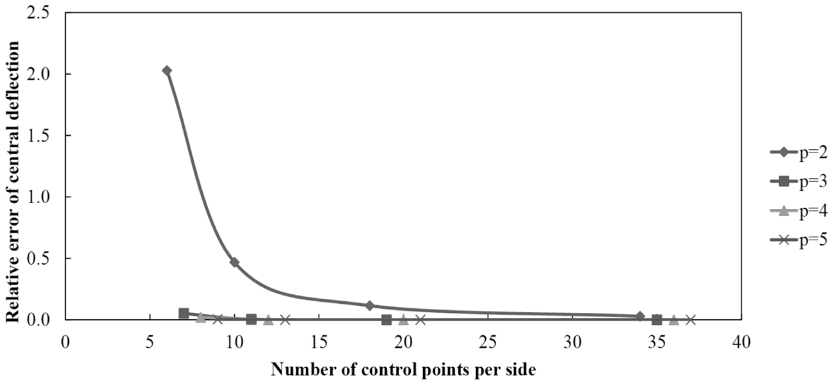

Fig. 2 The relative error of central deflection for a fully simply supported square plate under uniform load

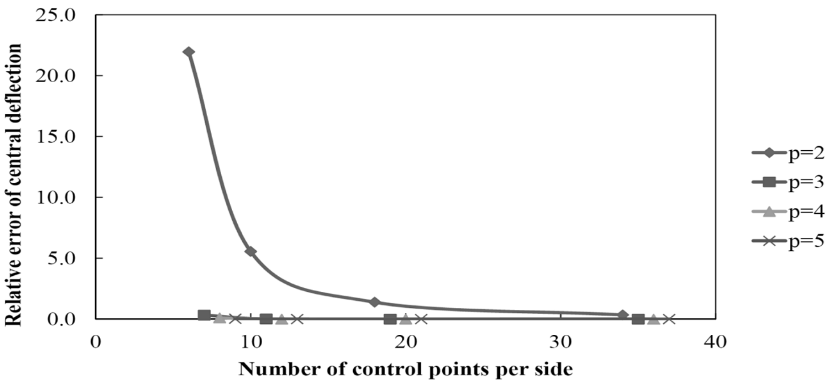

Fig. 3 The relative error of central deflection for a fully clamped square plate under uniform load

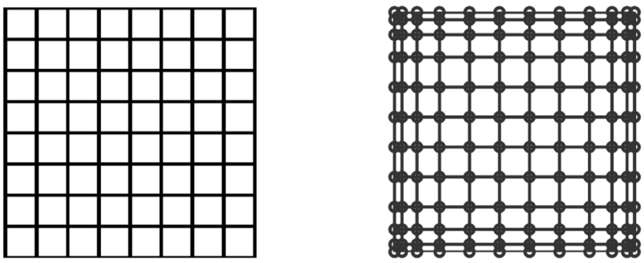

Fig. 4 Physical mesh with 64 elements (left) and the corresponding control mesh with 144 control points (right) for the quartic NURBS basis function

the convergence to exact solution is quite fast. For quartic and quintic NURBS basis functions, the second refinement step yields the converged solution. In Table 3, dimensionless central deflections and moments of uniformly loaded square plate with various boundary conditions are presented. These results are obtained by quartic NURBS basis function with 64 elements and 144 control points. The physical and control meshes for this problem are shown in Figure 4. It can be observed from Table 3 that the results obtained by isogeometric analysis are in good agreement with the closed-form solutions.

Table 3 Dimensionless central deflections and moments of square plate under uniform load with various boundary conditions ($p = 4$, 64 elements, 144 control points)

B.C.	Method	w	M_x	M_y
CCCC	IGA	0.0012653	0.22925	0.22925
	Exact (Timoshenko and Woinowsky 1995)	0.0012653	0.22905	0.22905
	Error (%)	0	0.0873	0.0873
SSSS	IGA	0.0040624	0.47895	0.47895
	Exact (Timoshenko and Woinowsky 1995)	0.0040624	0.4789	0.4789
	Error (%)	0	0.0001	0.0001
SCSC	IGA	0.0019172	0.24399	0.33262
	Exact (Timoshenko and Woinowsky 1995)	0.00192	0.244	0.332
	Error (%)	0.1458	0.0041	0.1867
CSSS	IGA	0.0027855	0.33897	0.39192
	Exact (Timoshenko and Woinowsky 1995)	0.0028	0.34	0.39
	Error (%)	0.5179	0.3029	0.4923

4.2 Simply supported square plate subjected to sinusoidal distributed load

In this example, a simply supported square plate under sinusoidal distributed load $p = p_0\sin(\pi x/L_x) \times \sin(\pi y/L_y)$ is considered, where length $L_x = L_y = 1$ m, uniform lateral load $p_0 = -100$ N and the thickness $t = 0.01$ m. Table 4 shows the isogeometric results of dimensionless central deflections with quadratic to quintic NURBS basis functions and different mesh refinements. The relative error of central deflection is also shown in Fig. 5. Similar to previous example, the convergence to exact solution is quite fast for all polynomial orders. Table 5 presents dimensionless central deflection, moments and corner torque of this example for cubic NURBS basis functions with 64 and 1024 elements (121 and 1225 control points, respectively). The results are also compared with BSWI element (121DOFs) (Xiang *et al.* 2007, Zhang *et al.* 2010), SHELL63 (38400DOFs) (Zhang *et al.* 2010) and the exact solution (Timoshenko and Woinowsky 1995). Through comparison with other numerical methods, it can be observed that the results of isogeometric analysis are exactly the same as BSWI element (with 121 DOFs for both methods). As expected, performing h-refinement

Table 4 Dimensionless central deflections of simply supported square plate under distributed load

Number of elements	Order of NURBS basis function				Exact Solution (Timoshenko and Woinowsky 1995)
	2	3	4	5	
16	0.0024904	0.0025688	0.0025661	0.0025666	0.0025665
64	0.0025494	0.0025666	0.0025665	0.0025665	
256	0.0025623	0.0025665	0.0025665	0.0025665	
1024	0.0025655	0.0025665	0.0025665	0.0025665	

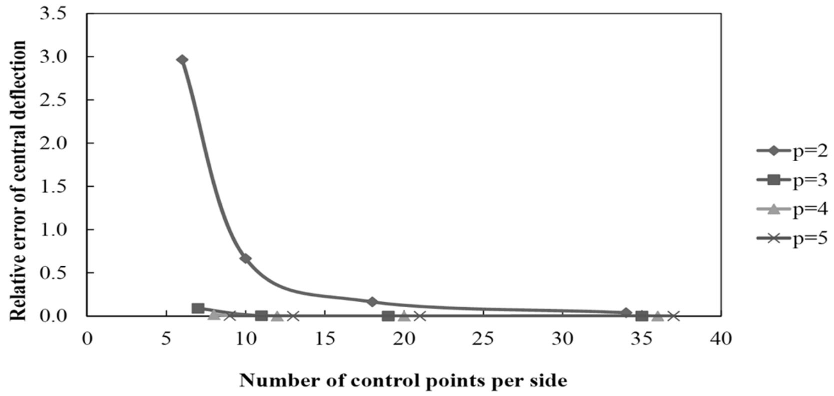

Fig. 5 The relative error of central deflection for a fully simply supported square plate under sinusoidal distributed load

Table 5 Comparison of dimensionless central deflection, moments and corner torque of simply supported square plate under sinusoidal distributed load

Method (DOF)	w	M_x	M_y	M_{xy} (corner)
IGA (121)	0.0025666	0.33357	0.33357	0.17732
IGA (1225)	0.0025665	0.32956	0.32956	0.17731
BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0025666	0.33357	0.33357	0.17732
SHELL63 (38400) (Zhang *et al.* 2010)	0.0025652	0.32881	0.32881	0.17816
Exact (Timoshenko and Woinowsky 1995)	0.0025665	0.32930	0.32930	0.18041

strategy, improves the results of isogeometric analysis for central deflection and moments, but corner torque is not affected and it is observed that the IGA result for corner torque with 64 elements is approximately the same as the IGA solution with 1024 elements.

These results show the isogeometric analysis robustness in static plate analysis with variable load condition and the efficiency of higher order NURBS basis functions.

4.3 Clamped trapezoidal plate uniformly loaded

Consider a trapezoidal plate has shown in Fig. 6. The geometrical parameters are as follow, $a = b = 2$ m and $c = 1.4$ m. All sides are clamped and the plate is subjected to uniform load $p_0 = -100$ N. As detailed in Table 6, different order of NURBS basis functions with different number of elements is considered for this problem. Dimensionless central deflections and moments for this problem is also presented in Table 7. Except quadratic NURBS which has slow convergence, the isogeometric analysis converges very fast and very well agreements with other numerical methods with low DOFs are achieved.

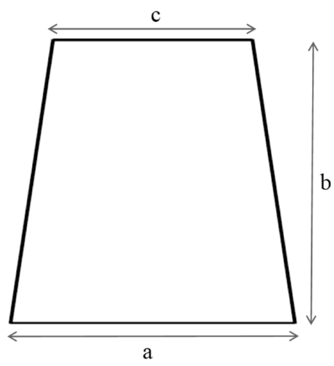

Fig. 6 Trapezoidal plate

Table 6 Details of isogeometric meshes used for trapezoidal problem

Method	Order of NURBS	Number of elements	Number of control points
IGA-p2-t1	2	64	100
IGA-p2-t2	2	256	324
IGA-p3-t1	3	64	121
IGA-p3-t2	3	256	361
IGA-p4-t1	4	64	144
IGA-p4-t2	4	256	400

Table 7 Comparison of dimensionless central deflections and moments of fully clamped trapezoidal plate under uniform load

Method	w	M_x	M_y
IGA-p2-t1	0.0008119	0.02018	0.01674
IGA-p3-t1	0.0008588	0.02113	0.01668
IGA-p4-t1	0.0008590	0.02076	0.01661
IGA-p2-t2	0.0008472	0.02060	0.01661
IGA-p3-t2	0.0008590	0.02084	0.01660
IGA-p4-t2	0.0008590	0.02074	0.01659
Meshfree method (Liu *et al.* 2006)	0.0008583	0.02055	0.01595
Differential cubature method (Liu and Liew 1998)	0.000860	0.02090	0.01634
FEM (Liu and Liew 1998)	0.000844	0.02061	0.01650

4.4 Simply supported triangular plate uniformly loaded

In this example, a simply supported equilateral triangular plate subjected to uniform load $p_0 = -100$ N is investigated. All the edges are assumed to be 2 m. Table 8 presents the dimensionless maximum deflections with different mesh refinements and quadratic to quartic NURBS basis functions. The results are in well agreement with theoretical solution.

Table 8 Dimensionless maximum deflections of simply supported equilateral triangular plate under uniform load

Num. of elements	Order of NURBS basis function			Exact solution (Timoshenko and Woinowsky 1995)
	2	3	4	
16	0.0005205	0.0005792	0.0005786	0.0005787
64	0.0005667	0.0005786	0.0005787	
256	0.0005757	0.0005787	0.0005787	
1024	0.0005780	0.0005787	0.0005787	

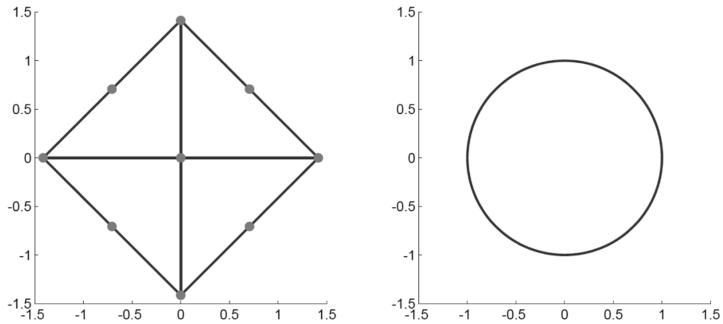

Fig. 7 Initial discretized geometry: (left) control mesh; (right) physical mesh

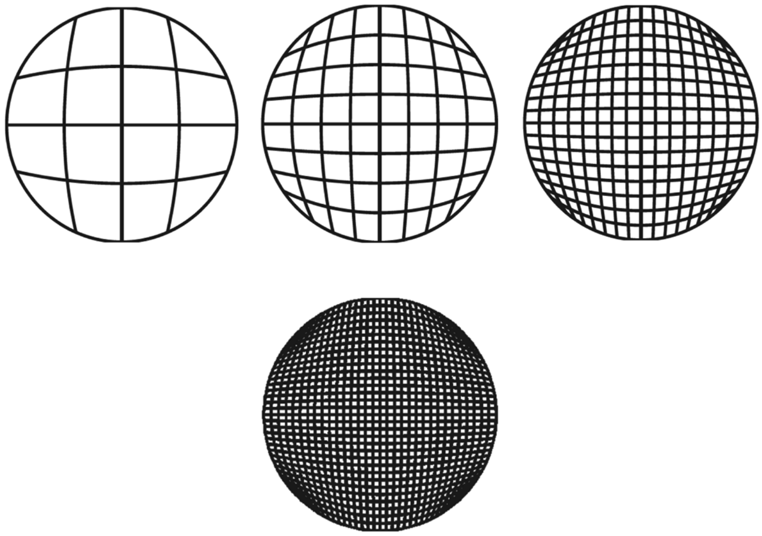

Fig. 8 Physical meshes with 16, 64, 256 and 1024 elements for the convergence study

4.5 Clamped circular plate uniformly loaded

To investigate the exact representation of common engineering shapes in isogeometric analysis, a clamped circular plate subjected to uniform load $p_0 = -100$ N is considered. The radius of the plate is assumed to be $R = 1$ m. The geometry is constructed by a NURBS surface of order $p = 2$ for

Table 9 Comparison of dimensionless central displacements of clamped circular plate subjected to uniform load

Order of NURBS	Num. of elements	Num. of control points	Central deflection
2	16	36	0.013085
	64	100	0.015076
	256	324	0.015493
	1024	1156	0.015592
Exact solution (Ventsel and Krauthammer 2001)			0.015625

Table 10 Comparison of dimensionless central displacements of clamped circular plate subjected to uniform load using p-refinement strategy

Number of elements	Order of NURBS	Number of control points	Central deflection
16	2	36	0.013085
	3	100	0.015681
	4	196	0.015631
	5	324	0.015632
64	2	100	0.015076
	3	324	0.015629
	4	676	0.015625
	5	1156	0.015625
Exact solution (Ventsel and Krauthammer 2001)			0.015625

both directions. Initial discretized geometry is shown in Fig. 7. To study the convergence of the numerical approach, h-refinement strategy is employed and meshes with 16, 64, 256 and 1024 elements are investigated (see Fig. 8). The p-refinement strategy is also employed and meshes with 16 and 64 elements with quadratic to quintic orders are investigated. The results for quadratic NURBS and the p-refinement strategy are given in Tables 9 and 10. It is observed that the quadratic NURBS elements are converged slowly, but using the p-refinement strategy, the higher order NURBS elements converges very fast to the analytical solution. This example also proves the efficiency of the p-refinement strategy in improving the IGA results.

4.6 Uniformly loaded square plate on elastic foundation

For the last example, a square plate on elastic foundation subjected to uniform load $p_0 = -100$ N is investigated. Table 11 shows the dimensionless central deflections and moments of an uniformly loaded square plate on elastic foundation with different boundary conditions. The isogeometric results are obtained by cubic NURBS basis functions with 64 and 1024 elements (121 and 1225 control points, respectively). The results are also compared with BSWI element (121DOFs) (Xiang *et al.* 2007, Zhang *et al.* 2010), Spline FEM (225DOFs) (Shen 1991) and SHELL63 (38400DOFs) (Zhang *et al.* 2010). Through comparison, we can see that the results of IGA (121DOFs), BSWI element (121DOFs) and Spline FEM (225DOFs) have almost the same accuracy. It seems the IGA (1225DOFs) has the best performance among these methods. With fewer DOFs, the results of

Table 11 Comparison of dimensionless central deflections and moments of square plate on elastic foundation with different boundary conditions subjected to uniform load

B.C.	Winkler constant	Method (DOFs)	w	M_x	M_y
SSSS	k = 5	IGA (121)	0.0040098	0.047455	0.047455
		IGA (1225)	0.0040097	0.047229	0.047229
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0040098	0.047455	0.047455
		Spline FEM (225) (Shen 1991)	0.0040098	0.047455	0.047455
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0040091	0.047185	0.047185
	k = 100	IGA (121)	0.0032138	0.037168	0.037168
		IGA (1225)	0.0032137	0.037060	0.037060
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0032138	0.037167	0.037167
		Spline FEM (225) (Shen 1991)	0.0032138	0.037168	0.037168
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0032136	0.037035	0.037035
SCSC	k = 5	IGA (121)	0.0019052	0.024344	0.033665
		IGA (1225)	0.0019053	0.024225	0.033061
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0019052	0.024355	0.033668
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0019055	0.024201	0.032993
	k = 100	IGA (121)	0.0017050	0.021377	0.029730
		IGA (1225)	0.0017050	0.021307	0.029230
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0017050	0.021388	0.029732
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0017052	0.021290	0.029174
CSSS	k = 5	IGA (121)	0.0027602	0.033710	0.039265
		IGA (1225)	0.0027602	0.033557	0.038823
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0027605	0.033722	0.039703
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0027601	0.033525	0.038765
	k = 100	IGA (121)	0.0023522	0.028132	0.032917
		IGA (1225)	0.0023522	0.028054	0.032593
		BSWI FEM (121) (Xiang *et al.* 2007, Zhang *et al.* 2010)	0.0023524	0.028144	0.032921
		Spline FEM (225) (Shen 1991)	0.0023522	0.028113	-
		SHELL63 (38400) (Zhang *et al.* 2010)	0.0023522	0.028034	0.032551

isogeometric method are very close to those characterized by high precision likes that of SHELL63 element with much more DOFs. In this example only cubic NURBS basis functions are used. It's obvious that like previous examples, higher order NURBS elements can be applied to get higher accuracy isogeometric results.

5. Conclusions

In this paper, the NURBS-based isogeometric analysis is successfully applied to solve Kirchhoff plate problems. Using NURBS basis functions, the necessity for C^1-inter-element continuity in Kirchhoff formulation, can be easily satisfied. Because of non-interpolating nature of NURBS, the Lagrange multiplier method has been adapted to impose essential boundary conditions. Several numerical simulations of thin plates with various shapes and different boundary conditions are presented to validate the proposed approach. The problem of thin plate on elastic foundation is also investigated. The numerical examples are also compared with available analytical and numerical methods. The results demonstrate the robustness and efficiency of isogeometric analysis for thin plate problems and its high accuracy and fast convergence rate.

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