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Parametrically excited viscoelastic beam-spring systems: nonlinear dynamics and stability

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Abstract. The aim of the investigation described in this paper is to study the nonlinear parametric vibrations and stability of a simply-supported viscoelastic beam with an intra-span spring. Taking into account a time-dependent tension inside the beam as the main source of parametric excitations, as well as employing a two-parameter rheological model, the equations of motion are derived using Newton's second law of motion. These equations are then solved via a perturbation technique which yields approximate analytical expressions for the frequency-response curves. Regarding the main parametric resonance case, the local stability of limit cycles is analyzed. Moreover, some numerical examples are provided in the last section.

Keywords: parametric vibrations; stability; viscoelastic materials; perturbation techniques

1. Introduction

Beams subjected to several adornments (springs, concentrated masses, dashpots, non-ideal supports, and so on) at several locations along their lengths (Ghayesh and Païdoussis 2010a, b, Ghayesh *et al*. 2011a, b, c, d, Darabi *et al*. 2011), are widely used in civil, automotive, industrial, and aerospace applications. Due to this, several investigations on the vibrations and stability of these systems have been carried out for many years and are still of interest today.

Many contributions on the vibrations and stability of plain beams (i.e., without adornments) can be found in the literature (Nayfeh and Mook 1979). A fundamental work is by Eisley (1964) who studied the nonlinear vibrations of beams and rectangular plates, specifically by considering the influence of initial membrane stress and employing a single mode approximation. In a paper by Srinivasan (1965), a Ritz-Galerkin method was employed to obtain an approximate solution for the large amplitude vibration response of beams. Wrenn and Meyers (1970) examined the vibrations of solid and sandwich beams including the effects of transverse shear, rotating inertia, and variable midplane stretching. In a paper by Hu and Kirmser (1971), the Duffing and Ritz-Kantorovich methods were used to reduce the nonlinear partial differential equation of motion to a nonlinear ordinary one, which was solved via perturbation and shooting techniques. These investigations were then extended for a buckled beam by Tseng and Dugundji (1971).

The above-mentioned analyses were extended for beams with intra-span adornments such as springs,

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masse, and so on, for example, by Dowell (1980), Birman (1986), and Szemplinksa-Stupnicka (1990). These analyses involved the inclusion of a nonlinear spring-mass system (Dowell 1980), nonlinear foundation (Birman 1986), and flexible boundary conditions (Szemplinska-Stupnicka 1990). The work of Dowell (1980) was later pursued further to study the stretching effect (Pakdemirli and Nayfeh 1994, Pakdemirli and Boyaci 2003). Additional extensions have also been made in (Özkaya *et al*. 1997) to deal with the system with different boundary conditions, and in (Karlik *et al*. 1998) by employing the artificial neural networks technique.

All the references cited above considered only elastic models of the structure; however, in some applications such as artificial muscles (Cohen 2001) and some vibration absorbers, the role of viscosity is essential (Marynowski and Kapitaniak 2002, 2007, Marynowski 2004, 2006, 2010, Chen and Yang 2005a, b, 2006, Chen *et al*. 2002, 2010, Zhang and Chen 2005, Zhang 2008). A two-parameter rheological model therefore, is considered in this paper in order to model the energy dissipation mechanism.

Another important issue worthy of consideration is that generally the real mechanisms are *inherently* imperfect. For example, the supports are not firmly fixed and as a result, the tension inside the beam is not constant even when it is intended to be (Marynowski 2004, Mockensturm and Guo 2005, Ghayesh and Moradian 2011). Since imperfections, generated by dynamical sources, fluctuate with time, the tension inside the beam is modeled here as a harmonically varying function of time. This time dependence is the cause of parametric resonance, and thus is important to be investigated.

With this study, I present a systematic, analytical approach for the prediction of vibration response of a simply-supported, parametrically excited, viscoelastic beam, additionally supported by a nonlinear spring. Specifically, (i) Section 2 presents the mathematical model of the system; (ii) the equations of order one and epsilon as well as the linear natural frequencies and mode functions are determined in Section 3; (iii) closed-form solutions for the amplitude of the principal parametric resonance are obtained in Section 4; (iv) the stability analysis of the near-resonance case is conducted in Section 5; (iv) the numerical results are given in Section 6.

2. Equations of motion

A parametrically excited, Kelvin-Voigt viscoelastic beam with an intra-span spring is shown in Fig. 1. This system consists of a beam of length L, density ρ, cross-sectional area A, viscosity coefficient η, and Young's modulus E. There is a nonlinear spring, attached at a distance $x = x_s$ from the left end of the beam; α and γ are respectively the linear and nonlinear stiffness coefficients of the spring.

In the following, the equations of motion are obtained by considering the beam as a two-part system; i.e. the spans before and after the spring and thus the effect of spring is taken into account as an internal boundary condition. $\hat{w}_1$ and $\hat{w}_2$ represent respectively the transverse displacement fields for the spans $0 < x < x_s$ and $x_s < x < \mathrm{L}$.

For a Kelvin-Voigt rheological model, which is a two-parameter energy dissipation mechanism, considering only the transverse displacement, the constitutive relation and the bending moment are given by

$$\sigma_i = E\varepsilon_i + \eta\frac{\partial \varepsilon_i}{\partial t},\ i = 1,2 \tag{1}$$

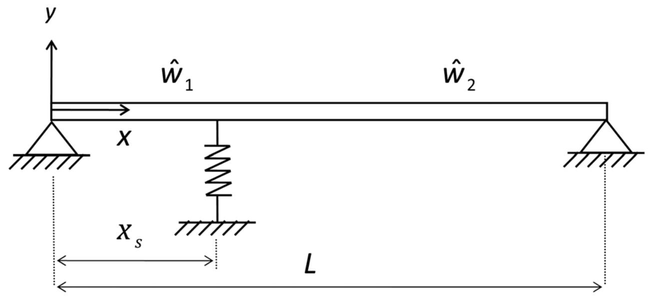

Fig. 1 A parametrically excited viscoelastic beam-spring system. The spring is nonlinear and α and γ represent respectively the linear and nonlinear stiffness coefficients of the spring

$$M_i = EI\frac{\partial^2 \hat{w}_i}{\partial x^2} + \eta I\frac{\partial^3 w_i}{\partial x^2 \partial t}, \ i = 1,2 \tag{2}$$

where properties before the spring are noted with subscript 1, and after the spring with subscript 2.

The tension inside the beam is assumed to involve a constant mean value p_0 and a harmonic variation $-\varepsilon p_1 \cos(\Omega t)$, where Ω is the frequency of tension fluctuations, and $\varepsilon \ll 1$, showing that the amplitude of tension variations is much less than the mean value of it. Considering this and Eqs. (1) and (2), assuming equal tensions in both the spans, as well as applying Newton's second law of motion to an element of each span of the simply-supported beam in the transverse direction results in the following dimensionless equations of motion and the corresponding (internal and external) boundary conditions

$$\frac{\partial^2 w_i}{\partial \tau^2} + \frac{\partial^4 w_i}{\partial \xi^4} + \varepsilon\varphi\frac{\partial^5 w_i}{\partial \xi^4 \partial \tau} - \gamma_1\frac{\partial^2 w_i}{\partial \xi^2} + \varepsilon\gamma_2\cos(\omega\tau)\frac{\partial^2 w_i}{\partial \xi^2} =$$

$$\varepsilon\varphi\left[\frac{\partial^3 w_i}{\partial \xi^2 \partial \tau}\left(\frac{\partial w_i}{\partial \xi}\right)^2 + 2\frac{\partial^2 w_i}{\partial \xi^2}\frac{\partial w_i}{\partial \xi}\frac{\partial^2 w_i}{\partial \xi \partial \tau}\right] + \frac{3}{2}\frac{\partial^2 w_i}{\partial \xi^2}\left(\frac{\partial w_i}{\partial \xi}\right)^2, \quad i = 1,2 \tag{3a}$$

$$\text{at } \xi = 0: w_1 = 0, \ \frac{\partial^2 w_1}{\partial \xi^2} = 0 \tag{3b}$$

$$\text{at } \xi = 1: w_2 = 0, \ \frac{\partial^2 w_2}{\partial \xi^2} = 0 \tag{3c}$$

$$\text{at } \xi = \xi_s: w_1 = w_2, \ \frac{\partial w_1}{\partial \xi} = \frac{\partial w_2}{\partial \xi}, \ \frac{\partial^2 w_1}{\partial \xi^2} + \varepsilon\varphi\frac{\partial^3 w_1}{\partial \xi^2 \partial \tau} = \frac{\partial^2 w_2}{\partial \xi^2} + \varepsilon\varphi\frac{\partial^3 w_2}{\partial \xi^2 \partial \tau}$$

$$\frac{\partial^3 w_1}{\partial \xi^3} - \frac{\partial^3 w_2}{\partial \xi^3} + \varepsilon\varphi\left[\frac{\partial^4 w_1}{\partial \xi^3 \partial \tau} - \frac{\partial^4 w_2}{\partial \xi^3 \partial \tau}\right] + \varepsilon\varphi\frac{\partial w_1}{\partial \xi}\left[\frac{\partial^2 w_2}{\partial \xi \partial \tau}\frac{\partial w_2}{\partial \xi} - \frac{\partial^2 w_1}{\partial \xi \partial \tau}\frac{\partial w_1}{\partial \xi}\right]$$

$$-k_1 w_1 - k_2 w_1^3 = 0 \tag{3d}$$

where

$$\xi = \frac{x}{L},\ \xi_s = \frac{x_s}{L},\ w_1 = \frac{\hat{w}_1}{r},\ w_2 = \frac{w_2}{r},\ \tau = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}}$$

$$\varphi = \sqrt{\frac{I\eta^2}{\rho AEL^4\varepsilon^2}},\ \gamma_1 = \frac{p_0L^2}{EI},\ \gamma_2 = \frac{p_1L^2}{EI},\ k_1 = \frac{\alpha L^3}{EI},\ k_2 = \frac{\gamma L^3}{EA} \tag{4}$$

are the nondimensional quantities. Here, r is the radius of gyration of the cross-sectional area. Eq. 3(a-d) forms a set of partial differential equations with nonlinear and time-dependent coupling terms; the nonlinear relation between the strain of the mid-plane of the beam and the displacement field, as well as the nonlinear force due to the spring cause the nonlinear terms in Eq. 3 (a-d).

3. The application of the method of multiple timescales

In this Section, the method of multiple timescales (Ghayesh 2008, 2009, 2010, Ghayesh and Balar 2008, 2010, Ghayesh *et al*. 2010), an asymptotic technique, is applied directly to the equations of motion – no spatial mode function is assumed. In this method, solutions in the form of uniformly valid expansions is sought, i.e. (Nayfeh 1993, Nayfeh and Mook 1979)

$$w_i(\xi,\tau;\varepsilon) = w_{i0}(\xi,T_0,T_1) + \varepsilon w_{i1}(\xi,T_0,T_1) + \mathrm{O}(\varepsilon^2),\quad i = 1,2 \tag{5}$$

where w_{10}, w_{20}, w_{11} and w_{21} are the functions of order of magnitude one, O(1); O(ε^n) denotes terms of order of magnitude ε^n and smaller; $T_0 = \tau$ is the fast timescale and $T_1 = \varepsilon\tau$ the slow one; $\varepsilon \ll 1$.

Substituting Eq.(5) into weak form of Eq. 3(a-d), which are obtained by inserting $w_i \leftrightarrow \sqrt{\varepsilon}w_i$ ($i = 1,2$), employing the chain rule for differentiation, and equating the coefficients of like powers of ε to zero yields

$$0(\varepsilon^0): \frac{\partial^2 w_{i0}}{\partial T_0^2} + \frac{\partial^4 w_{i0}}{\partial\xi^4} - \gamma_1\frac{\partial^2 w_{i0}}{\partial\xi^2} = 0,\quad i = 1,2 \tag{6a}$$

$$\text{at } \xi = 0:\ w_{10} = \frac{\partial^2 w_{10}}{\partial\xi^2} = 0 \tag{6b}$$

$$\text{at } \xi = 1:\ w_{20} = \frac{\partial^2 w_{20}}{\partial\xi^2} = 0 \tag{6c}$$

$$\text{at } \xi = \xi_s:\ w_{10} = w_{20},\ \frac{\partial w_{10}}{\partial\xi} = \frac{\partial w_{20}}{\partial\xi},\ \frac{\partial^2 w_{10}}{\partial\xi^2} = \frac{\partial^2 w_{20}}{\partial\xi^2},\ \frac{\partial^3 w_{10}}{\partial\xi^3} - \frac{\partial^3 w_{20}}{\partial\xi^3} - k_1w_{10} = 0 \tag{6d}$$

and

$$0(\varepsilon^1): \frac{\partial^2 w_{i1}}{\partial T_0^2} + \frac{\partial^4 w_{i1}}{\partial\xi^4} - \gamma_1\frac{\partial^2 w_{i1}}{\partial\xi^2} = -2\frac{\partial^2 w_{i0}}{\partial T_0\partial T_1} - \varphi\frac{\partial^5 w_{i0}}{\partial\xi^4\partial T_0}$$

$$-\gamma_2\cos(\omega\tau)\frac{\partial^2 w_{i0}}{\partial\xi^2} + \frac{3}{2}\frac{\partial^2 w_{i0}}{\partial\xi^2}\left(\frac{\partial w_{i0}}{\partial\xi}\right)^2,\quad i = 1,2 \tag{7a}$$

$$\text{at } \xi = 0: w_{11} = \frac{\partial^2 w_{11}}{\partial \xi^2} = 0 \tag{7b}$$

$$\text{at } \xi = 1: w_{21} = \frac{\partial^2 w_{21}}{\partial \xi^2} = 0 \tag{7c}$$

$$\text{at } \xi = \xi_s: w_{11} = w_{21}$$

$$\frac{\partial w_{11}}{\partial \xi} = \frac{\partial w_{21}}{\partial \xi}, \quad \frac{\partial^2 w_{11}}{\partial \xi^2} - \frac{\partial^2 w_{21}}{\partial \xi^2} + \varphi\left[\frac{\partial^3 w_{10}}{\partial \xi^2 \partial T_0} - \frac{\partial^3 w_{20}}{\partial \xi^2 \partial T_0}\right] = 0$$

$$\frac{\partial^3 w_{11}}{\partial \xi^3} - \frac{\partial^3 w_{21}}{\partial \xi^3} + \varphi\left[\frac{\partial^4 w_{10}}{\partial \xi^3 \partial T_0} - \frac{\partial^4 w_{20}}{\partial \xi^3 \partial T_0}\right] - k_1 w_{11} - k_2 w_{10}^3 = 0 \tag{7d}$$

The solution of Eq. 6(a-d) takes the form

$$w_{i0}(\xi, T_0, T_1) = \sum_{n=1}^{\infty} [\mathrm{X}_n(T_1) e^{i\omega_n T_0} + \overline{\mathrm{X}}_n(T_1) e^{-i\omega_n T_0}] Z_{in}(\xi), \quad i = 1,2 \tag{8}$$

where $X_n(T_1)$ is an unknown, complex-valued function of the slow timescale T_1, and the overbar is used to denote the complex conjugate.

Inserting Eq. (8) into Eq. 6(a-d) gives

$$\frac{d^4 Z_{in}}{d\xi^4} - \gamma_1 \frac{d^2 Z_{in}}{d\xi^2} - \omega_n^{\,2} Z_{in} = 0, \quad i = 1,2 \tag{9a}$$

$$\text{at } \xi = 0: Z_{1n} = \frac{d^2 Z_{1n}}{\partial \xi^2} = 0 \tag{9b}$$

$$\text{at } \xi = 1: Z_{2n} = \frac{d^2 Z_{2n}}{d\xi^2} = 0 \tag{9c}$$

$$\text{at } \xi = \xi_s: Z_{1n} = Z_{2n}, \quad \frac{dZ_{1n}}{d\xi} = \frac{dZ_{2n}}{d\xi}, \quad \frac{d^2 Z_{1n}}{\partial \xi^2} = \frac{d^2 Z_{2n}}{\partial \xi^2}, \quad \frac{d^3 Z_{1n}}{\partial \xi^3} - \frac{\partial^3 Z_{2n}}{\partial \xi^3} - k_1 Z_{1n} = 0 \tag{9d}$$

Eq. 9(a-d) forms a set of linear ordinary differential equations, which can be solved using classical methods. The result is as follows

$$Z_{1n}(\xi) = c_{1n}\cosh(\beta_{1n}\xi) + c_{2n}\sinh(\beta_{1n}\xi) + c_{3n}\sin(\beta_{2n}\xi) + c_{4n}\cos(\beta_{2n}\xi) \tag{10a}$$

$$Z_{2n}(\xi) = d_{1n}\cosh(\beta_{1n}\xi) + d_{2n}\sinh(\beta_{1n}\xi) + d_{3n}\sin(\beta_{2n}\xi) + d_{4n}\cos(\beta_{2n}\xi) \tag{10b}$$

where $\beta_{1n} = \sqrt{0.5(\gamma_1 + \sqrt{\gamma_1^2 + 4\omega_n^2})}$ and $\beta_{2n} = \sqrt{0.5(-\gamma_1 + \sqrt{\gamma_1^2 + 4\omega_n^2})}$.

Substituting these into equations of boundary conditions (Eq. 9 (b-d)), the following is obtained

$$[M]_{8\times 8}\,[c_{1n} \;\; c_{2n} \;\; c_{3n} \;\; c_{4n} \;\; d_{1n} \;\; d_{2n} \;\; d_{3n} \;\; d_{4n}]^T = [0]_{8\times 1} \tag{11}$$

where $[M]_{8\times8}$ is called the coefficient matrix.

In order to obtain non-trivial solutions for the constant c_{in} and d_{in} ($i = 1,2,3,4$; $n = 1,2, ..., N$) terms, the determinant of $[M]_{8\times8}$ should be equal to zero. This operation gives an algebraic equation which should be solved for ω_n. Inserting any ω_n, $n = 1,2, ..., N$ in Eq. (11) and employing the elimination process, the vector $[c_{1n}\ c_{2n}\ c_{3n}\ c_{4n}\ d_{1n}\ d_{2n}\ d_{3n}\ d_{4n}]^T$ is determined.

One may proceed with Eq. (8) as is; however, this article is not concerned with internal resonances and it is sufficient to retain only the nth mode. Therefore, one can eliminate the sigma sign in Eq. (8). With this, and expressing the trigonometric functions in exponential forms, Eq. 7(a-d) can be rewritten as

$$\frac{\partial^2 w_{j1}}{\partial T_0^{\,2}} + \frac{\partial^4 w_{j1}}{\partial \xi^4} - \gamma_1 \frac{\partial^2 w_{j1}}{\partial \xi^2} =$$

$$\left[-2i\omega_n \frac{dX_n}{dT_1} Z_{jn} - \varphi i\omega_n X_n \frac{d^4 Z_{jn}}{d\xi^4} + \frac{9}{2} X_n^2 \overline{X}_n \frac{d^2 Z_{jn}}{d\xi^2}\left(\frac{dZ_{jn}}{d\xi}\right)^2\right] e^{i\omega_n T_0}$$

$$-\frac{1}{2}\gamma_2 X_n \frac{d^2 Z_{jn}}{d\xi^2} e^{i(\omega+\omega_n)T_0} - \frac{1}{2}\gamma_2 \overline{X}_n \frac{d^2 Z_{jn}}{d\xi^2} e^{i(\omega-\omega_n)T_0} + cc + NST, \quad j = 1,2 \tag{12a}$$

$$\text{at } \xi = 0: w_{11} = \frac{\partial^2 w_{11}}{\partial \xi^2} = 0 \tag{12b}$$

$$\text{at } \xi = 1: w_{21} = \frac{\partial^2 w_{21}}{\partial \xi^2} = 0 \tag{12c}$$

$$\text{at } \xi = \xi_s: w_{11} = w_{21}$$

$$\frac{\partial w_{11}}{\partial \xi} = \frac{\partial w_{21}}{\partial \xi}, \quad \frac{\partial^2 w_{11}}{\partial \xi^2} - \frac{\partial^2 w_{21}}{\partial \xi^2} = \varphi i\omega_n X_n \left(\frac{d^2 Z_{2n}}{d\xi^2} - \frac{d^2 Z_{1n}}{d\xi^2}\right) e^{i\omega_n T_0} + cc$$

$$\frac{\partial^3 w_{11}}{\partial \xi^3} - \frac{\partial^3 w_{21}}{\partial \xi^3} - k_1 w_{11} = \varphi i\omega_n X_n \left(\frac{d^3 Z_{2n}}{d\xi^3} - \frac{d^3 Z_{1n}}{d\xi^3}\right) e^{i\omega_n T_0}$$

$$+3k_2 X_n^2 \overline{X}_n Z_{1n}^3 e^{i\omega_n T_0} + cc + NST \tag{12d}$$

where cc denotes the complex conjugate of the preceding terms on the right-hand side and NST stands for the non-secular terms.

4. The near-resonant case

It is clear from Eq. 12(a) that the nth principal parametric resonance occurs when $\omega \approx 2\omega_n$. Here ω is considered to take the form $\omega = 2\omega_n + \varepsilon\sigma$, where σ has been introduced to quantify the nearness of the parametric excitation to the nth linear, unperturbed frequency, ω_n. Considering this, Eq. 12(a-d) can be rewritten as

$$\frac{\partial^2 w_{j1}}{\partial T_0^{\ 2}}+\frac{\partial^4 w_{j1}}{\partial \xi^4}-\gamma_1\frac{\partial^2 w_{j1}}{\partial \xi^2}=$$

$$\left[-2i\omega_n\frac{d\mathrm{X}_n}{d\mathrm{T}_1}\mathrm{Z}_{jn}-\varphi i\omega_n\mathrm{X}_n\frac{d^4\mathrm{Z}_{jn}}{d\xi^4}+\frac{9}{2}\mathrm{X}_n^2\overline{\mathrm{X}}_n\frac{d^2\mathrm{Z}_{jn}}{d\xi^2}\left(\frac{d\mathrm{Z}_{jn}}{d\xi}\right)^2-\frac{1}{2}\gamma_2\overline{\mathrm{X}}_n\frac{d^2\mathrm{Z}_{jn}}{d\xi^2}e^{i\sigma T_1}\right]e^{i\omega_n T_0}+cc+NST$$

$$j=1,2 \tag{13a}$$

$$\text{at }\ \xi=0:\ w_{11}=\frac{\partial^2 w_{11}}{\partial \xi^2}=0 \tag{13b}$$

$$\text{at }\ \xi=1:\ w_{21}=\frac{\partial^2 w_{21}}{\partial \xi^2}=0 \tag{13c}$$

$$\text{at }\ \xi=\xi_s:\ w_{11}=w_{21}$$

$$\frac{\partial w_{11}}{\partial \xi}=\frac{\partial w_{21}}{\partial \xi},\quad \frac{\partial^2 w_{11}}{\partial \xi^2}-\frac{\partial^2 w_{21}}{\partial \xi^2}=\varphi i\omega_n\mathrm{X}_n\left(\frac{d^2\mathrm{Z}_{2n}}{d\xi^2}-\frac{d^2\mathrm{Z}_{1n}}{d\xi^2}\right)e^{i\omega_n T_0}+cc$$

$$\frac{\partial^3 w_{11}}{\partial \xi^3}-\frac{\partial^3 w_{21}}{\partial \xi^3}-k_1 w_{11}=\varphi i\omega_n\mathrm{X}_n\left(\frac{d^3\mathrm{Z}_{2n}}{d\xi^3}-\frac{d^3\mathrm{Z}_{1n}}{d\xi^3}\right)e^{i\omega_n T_0}$$

$$+3k_2\mathrm{X}_n^2\overline{\mathrm{X}}_n\mathrm{Z}_{1n}^3 e^{i\omega_n T_0}+cc+NST \tag{13d}$$

Fulfillment of the *solvability condition* (Thomsen 2003) for the resonant case (Eq.13(a-d)), which ensures the solutions of w_{11} and w_{21} be free of secular terms, gives

$$\mathrm{X}_n^2\overline{\mathrm{X}}_n+i\gamma_{1n}\frac{d\mathrm{X}_n}{d\mathrm{T}_1}+i\gamma_{2n}\mathrm{X}_n+\frac{1}{2}\gamma_{3n}\overline{\mathrm{X}}_n e^{i\sigma T_1}=0 \tag{14}$$

where γ_{1n}, γ_{2n} and γ_{3n} are given in appendix A.

In order to solve Eq. (14) for the complex-valued function $\mathrm{X}_n(T_1)$, this function may be expressed in the following exponential form

$$\mathrm{X}_n(T_1)=\frac{1}{2}x_n(T_1)e^{i\beta_n(T_1)} \tag{15}$$

where $x_n(T_1)$ and $\beta_n(T_1)$ are real-valued functions of slow timescale. Inserting this into Eq. (14), separating the resulting equation into real and imaginary components, and writing $\theta_n(T_1)=\sigma T_1-2\beta_n(T_1)$ results in

$$\frac{1}{2}x_n^3-\gamma_{1n}x_n\left(\sigma-\frac{d\theta_n}{dT_1}\right)+\gamma_{3n}x_n\cos\theta_n=0 \tag{16a}$$

$$2\gamma_{1n}\frac{dx_n}{dT_1}+2\gamma_{2n}x_n+\gamma_{3n}x_n\sin\theta_n=0 \tag{16b}$$

Satisfying the condition that $dx_n/dT_1 = d\theta_n/dT_1 = 0$ gives the steady-state solution, where x_n and θ_n do not change with time. Applying this condition to Eq. 16(a,b) and solving the resulting equation for σ, one finds that there are two possibilities. The first one is that $x_n = 0$, and the other one determines the following relations

$$\sigma_1 = \frac{1}{\gamma_{1n}}\left[\frac{1}{2}x_n^{\ 2} - \sqrt{(\gamma_{3n})^2 - (2\gamma_{2n})^2}\right] \tag{17a}$$

$$\sigma_2 = \frac{1}{\gamma_{1n}}\left[\frac{1}{2}x_n^{\ 2} + \sqrt{(\gamma_{3n})^2 - (2\gamma_{2n})^2}\right] \tag{17b}$$

5. Stability of the near-resonant case

The steady-state response may be either stable or unstable. Since the system is either in a stable state or on the way toward it, the unstable responses cannot be observed in experiments. However, the method of multiple timescales and some continuation techniques give both the stable and unstable solutions; finite difference method (FDM), for example, does not give unstable solutions. In order to do stability analysis of the system, one may construct the Jacobian of the modulation equations (Eq. 16(a, b)) as follows

$$\mathbf{J}(x_n, \theta_n) = \begin{bmatrix} \frac{\partial}{\partial x_n}\left(\frac{dx_n}{dT_1}\right) & \frac{\partial}{\partial \theta_n}\left(\frac{dx_n}{dT_1}\right) \\ \frac{\partial}{\partial x_n}\left(\frac{d\theta_n}{dT_1}\right) & \frac{\partial}{\partial \theta_n}\left(\frac{d\theta_n}{dT_1}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\gamma_n}(2\gamma_{2n} + \gamma_{3n}\sin\theta_n) & -\frac{\gamma_{3n}}{2\gamma_{1n}}x_n\cos\theta_n \\ -\frac{1}{\gamma_n}x_n & \frac{\gamma_{3n}}{\gamma_{1n}}\sin\theta_n \end{bmatrix} \tag{18}$$

The eigenvalues of $\mathbf{J}_n(x_n, \theta_n)$ are determined by

$$|\mathbf{J}(x_n, \theta_n) - \lambda\mathbf{I}| = \lambda^2 + 2\frac{\gamma_{2n}}{\gamma_{1n}}\lambda + \frac{1}{2\gamma_{1n}^2}\left(\frac{1}{2}x_n^4 - \gamma_{1n}\sigma x_n^2\right) = 0 \tag{19}$$

It is obvious from Eq. (19) that by satisfying the following condition, the solution corresponding to the first detuning parameter (σ_1) is stable

$$\frac{\gamma_{2n}}{\gamma_{1n}} > 0 \ \text{ and } \ \left(\frac{\gamma_{2n}}{\gamma_{1n}}\right)^2 - \frac{1}{2}\left(\frac{x_n}{\gamma_{1n}}\right)^2\sqrt{(\gamma_{3n})^2 - (2\gamma_{2n})^2} < 0 \tag{20}$$

As before, the stability condition for the solution corresponding to the second detuning parameter (σ_2) is obtained as

$$\frac{\gamma_{2n}}{\gamma_{1n}} > 0 \ \text{ and } \ \left(\frac{\gamma_{2n}}{\gamma_{1n}}\right)^2 + \frac{1}{2}\left(\frac{x_n}{\gamma_{1n}}\right)^2\sqrt{(\gamma_{3n})^2 - (2\gamma_{2n})^2} < 0 \tag{21}$$

Numerical verifications show that the curve corresponding to σ_1 is always stable and the one for σ_2 is unstable.

In order to investigate the stability condition of the trivial solution of Eq. (14), the following transformation may be considered

$$x_n(T_1) = \frac{1}{2}(\xi_n + i\psi_n)e^{i\sigma T_1/2} \tag{22}$$

Substituting this into Eq. (14) and separating the real and imaginary parts of the resulting equation yields the following

$$\begin{aligned} \frac{d\psi_n}{dT_1} &= -\frac{\sigma}{2}\xi_n + \frac{1}{\gamma_{1n}}\left[\frac{1}{4}(\xi_n^2 - \psi_n^2)\xi_n + \frac{1}{2}\xi_n\psi_n^2 - \gamma_{2n}\psi_n + \frac{1}{2}\gamma_{3n}\xi_n\right] \\ \frac{d\xi_n}{dT_1} &= +\frac{\sigma}{2}\psi_n + \frac{1}{\gamma_{1n}}\left[\frac{1}{4}(\xi_n^2 - \psi_n^2)\psi_n - \frac{1}{2}\xi_n^2\psi_n - \gamma_{2n}\xi_n + \frac{1}{2}\gamma_{3n}\psi_n\right] \end{aligned} \tag{23}$$

Constructing the Jacobian matrix for Eq. (23) and calculating the eigenvalues gives the following conditions for the stable trivial solutions

$$\frac{\gamma_{2n}}{\gamma_{1n}} > 0 \ \text{ and } \ \left(\frac{\gamma_{3n}}{2\gamma_{1n}}\right)^2 - \left(\frac{\sigma}{2}\right)^2 < 0 \tag{24}$$

6. Numerical results

In this section, the influences of system parameters on the linear matural frequencies and

Table 1 The first, second and third natural frequencies of the system as functions of the mean value of the tension; $k_1 = 100$, $\xi_s = 0.3$

γ_1	0	50	100	150	200	250	300	350	400
ω_1	14.553	26.702	34.773	41.280	46.887	51.889	56.448	60.665	64.606
ω_2	41.865	60.988	75.440	87.544	98.169	107.753	116.552	124.732	132.408
ω_3	88.938	111.135	129.585	145.717	160.233	173.540	185.896	197.481	208.422

Table 2 The first, second and third natural frequencies of the system as functions of the linear stiffness coefficient of the spring; $\gamma_1 = 50$, $\xi_s = 0.3$

k_1	10	100	200	300	400	500	600	700	800
ω_1	24.574	26.702	28.612	30.164	31.440	32.502	33.395	34.152	34.799
ω_2	59.587	60.988	62.591	64.224	65.871	67.516	69.147	70.755	72.332
ω_3	111.056	111.135	111.227	111.324	111.426	111.532	111.644	111.762	111.886

Table 3 The first, second and third natural frequencies of the system as functions of the spring location; $\gamma_1 = 50$, $k_1 = 200$

ξ_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ω_1	25.018	26.632	28.612	30.423	31.212	30.423	28.612	26.632	25.018
ω_2	60.542	62.376	62.591	60.690	59.435	60.690	62.591	62.376	60.542
ω_3	112.207	112.731	111.227	111.682	112.876	111.682	111.227	112.731	112.207

frequency-response curves of the system are illustrated through a numerical parametric study.

The first three natural frequencies of the system as a function of the mean tension, linear stiffness coefficient of the spring, and the spring location are given in Tables 1-3, respectively. Increasing either the mean tension or the linear stiffness coefficient of the spring results in an increase of the first three natural frequencies.

The frequency-response curves corresponding to the first and second modes are shown in Fig. 2. As seen in this figure, for $\sigma < \sigma_1$, there is a stable trivial solution. At $\sigma = \sigma_1$, the trivial solution becomes unstable and a stable non-trivial solution bifurcates. At $\sigma = \sigma_2$, the trivial solution starts to

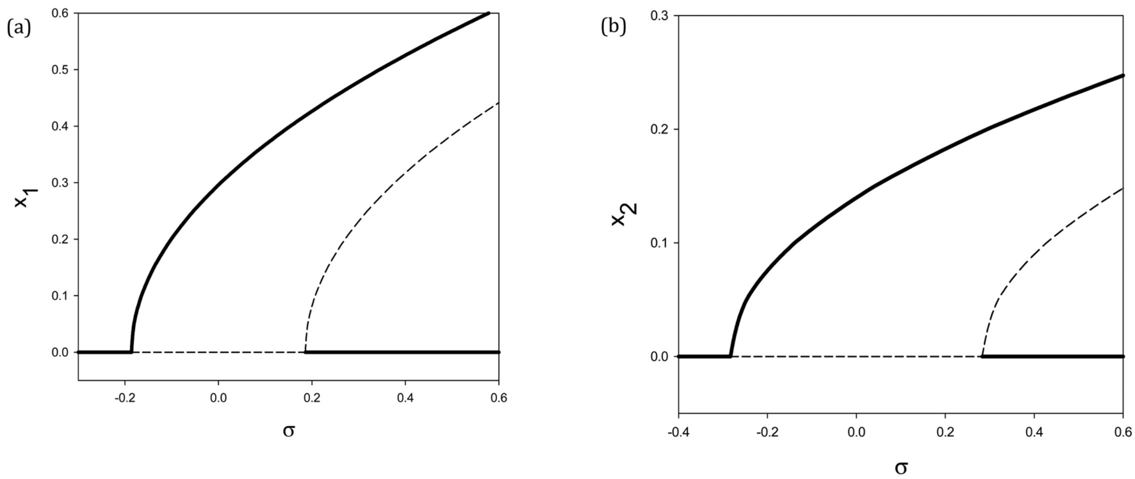

Fig. 2 The first (a) and second (b) mode frequency-response curves of the system; $k_1 = 100$, $k_2 = 1$, $\xi_s = 0.3$, $\gamma_1 = 50$, $\gamma_2 = 1$, $\varphi = 0.0001$

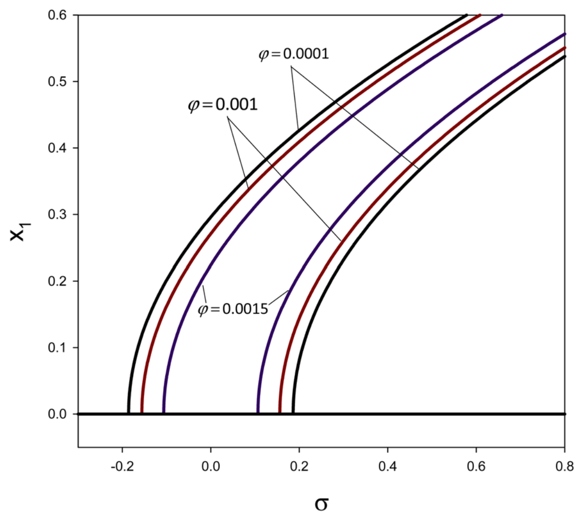

Fig. 3 The first frequency-response curves of the system for several viscosity coefficients; $k_1 = 100$, $k_2 = 1$, $\xi_s = 0.3$, $\gamma_1 = 50$, $\gamma_2 = 1$

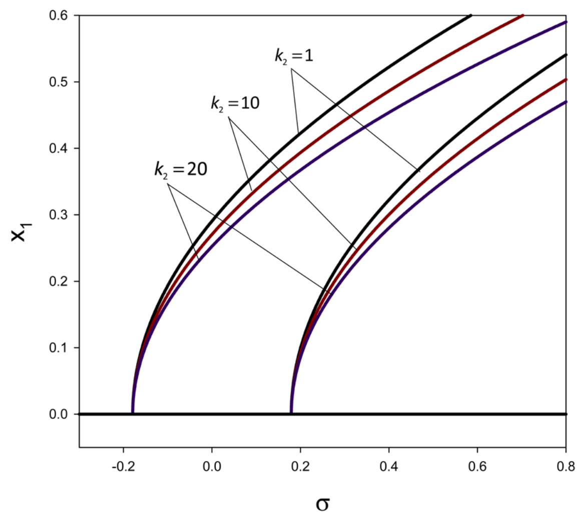

Fig. 4 The first frequency-response curves of the system for several nonlinear spring stiffness coefficients; $k_1 = 100$, $\xi_s = 0.3$, $\gamma_1 = 50$, $\gamma_2 = 1$, $\varphi = 0.0005$

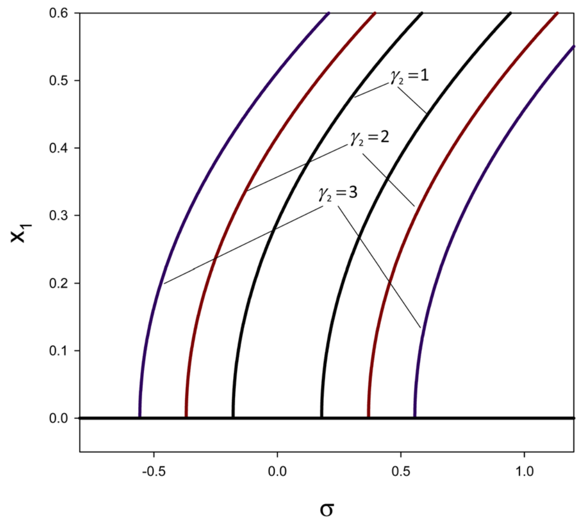

Fig. 5 The first frequency-response curves of the system for several excitation amplitudes; $k_1 = 100$, $k_2 = 1$, $\xi_s = 0.3$, $\gamma_1 = 50$, $\varphi = 0.0005$

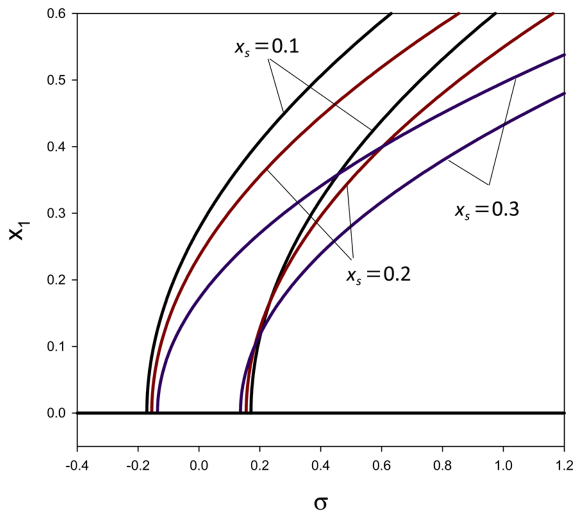

Fig. 6 The first frequency-response curves of the system for several spring-support locations; $k_1 = 200$, $k_2 = 100$, $\gamma_1 = 50$, $\gamma_2 = 1$, $\varphi = 0.001$

be stable again and an unstable nontrivial solution bifurcates. As seen in Fig. 3, the region for the unstable, trivial solution of a system with larger values of φ are smaller than those of a system with smaller φ. As k_2 is increased, the curves bend more to the right, thus showing more nonlinear behavior (Fig. 4). The effect of γ_2 is opposite to the effect of φ, as seen in Fig. 5. Lastly, it is hard to make any general conclusion on the effect of ξ_s on the frequency-response curves of the system, as seen in Fig. 6.

7. Conclusions

Nonlinear parametric vibrations and stability of a simply-supported viscoelastic beam with an intra-span nonlinear spring have been investigated analytically. Kelvin-Voigt viscoelstic mechanism was employed to model the energy dissipation. The method of multiple timescales was employed to the nonlinear resonant response as well as the stability of the system. It was shown that the resonance occurs when the excitation frequency approaches twice any natural frequency of the system. It was determined that there are three solution branches near resonance; two of which are nontrivial and one that is trivial. Also, it was determined that the curve corresponding to the first detuning parameter is stable and the second one unstable. In conclusion, it can be concluded that the present study enlarges the current knowledge concerning the parametric vibrations and stability of beams with additional adornments.

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Appendix A.

$$\gamma_{1n}=\frac{\left[2\omega_n\left(\int_0^{\xi_s}Z_{1n}^2 d\xi+\int_{\xi_s}^1 Z_{2n}^2 d\xi\right)\right]}{\left[3k_2{Z_{1n}}^4(\xi_s)-\frac{9}{2}\left(\int_0^{\xi_s}\frac{d^2Z_{1n}}{d\xi^2}\left(\frac{dZ_{1n}}{d\xi}\right)^2 Z_{1n}d\xi+\int_{\xi_s}^1\frac{d^2Z_{2n}}{d\xi^2}\left(\frac{dZ_{2n}}{d\xi}\right)^2 Z_{2n}d\xi\right)\right]} \tag{A.1}$$

$$\gamma_{2n}=\frac{\varphi\omega_n\left[Z_{1n}(\xi_s)\left(\left.\frac{d^3Z_{2n}}{d\xi^3}\right|_{\xi_s}-\left.\frac{d^3Z_{1n}}{d\xi^3}\right|_{\xi_s}\right)+\left(\int_0^{\xi_s}\frac{d^4Z_{1n}}{d\xi^4}Z_{1n}d\xi+\int_{\xi_s}^1\frac{d^4Z_{2n}}{d\xi^4}Z_{2n}d\xi\right)\right]}{\left[3k_2{Z_{1n}}^4(\xi_s)-\frac{9}{2}\left(\int_0^{\xi_s}\frac{d^2Z_{1n}}{d\xi^2}\left(\frac{dZ_{1n}}{d\xi}\right)^2 Z_{1n}d\xi+\int_{\xi_s}^1\frac{d^2Z_{2n}}{d\xi^2}\left(\frac{dZ_{2n}}{d\xi}\right)^2 Z_{2n}d\xi\right)\right]} \tag{A.2}$$

$$\gamma_{3n}=\frac{\gamma_2\left[\int_0^{\xi_s}\frac{d^2Z_{1n}}{d\xi^2}Z_{1n}d\xi+\int_{\xi_s}^1\frac{d^2Z_{2n}}{d\xi^2}Z_{2n}d\xi\right]}{\left[3k_2{Z_{1n}}^4(\xi_s)-\frac{9}{2}\left(\int_0^{\xi_s}\frac{d^2Z_{1n}}{d\xi^2}\left(\frac{dZ_{1n}}{d\xi}\right)^2 Z_{1n}d\xi+\int_{\xi_s}^1\frac{d^2Z_{2n}}{d\xi^2}\left(\frac{dZ_{2n}}{d\xi}\right)^2 Z_{2n}d\xi\right)\right]} \tag{A3}$$