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Free vibration of a rectangular plate with an attached three-degree-of-freedom spring-mass system

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Abstract. The present paper studies the variation of the natural frequencies and mode shapes of rectangular plates carrying a three degree-of-freedom spring-mass system (subsystem), when the subsystem changes (stiffness, mass, moment of inertia, location). An analytical approach based on Lagrange multipliers as well as a finite element formulation are employed and compared. Numerically reliable results are presented for the first time, illustrating the convenience of using the present analytical method which requires only the solution of a linear eigenvalue problem. Results obtained through the variation of the mass, stiffness and moment of inertia of the 3-DOF system can be understood under the effective mass concept or Rayleigh's statement. The analysis of frequency values of the whole system, when the 3-DOF system approaches or moves away from the center, shows that the variations depend on each particular mode of vibration. When the 3-DOF system is placed in the center of the plate, "new" modes are found to be a combination of the subsystem's modes (two rotations, traslation) and the bare plate's modes that possess the same symmetry. This situation no longer exists as the 3-DOF system moves away from the center of the plate, since different bare plate's modes enable distinct motions of the 3-DOF system contributing differently to the "new' modes as its location is modified. Also the natural frequencies of the compound system are nearly uncoupled have been calculated by means of a first order eigenvalue perturbation analysis.

Keywords: 3-DOF spring-mass system; rectangular plate; frequencies; mode shapes; linear eigenvalue problem

1. Introduction

Investigations into natural frequencies and modal shapes of beams and plates carrying a spring-mass system have been done by several researchers. Among them, the following works can be mentioned: Jen and Magrab (1993) presented the exact solution for the natural frequencies and mode shapes of beams with two degree of freedom spring-mass system attached; Low (2003) studied the frequencies of beams carrying multiple masses; Cha (2007) studied the free vibration of a uniform beam with multiple elastically mounted two-degree-of-freedom systems; Wu (2002) presented an alternative approach for the free vibration of beams carrying a number of two-degree of freedom spring-mass systems; Vera *et al.* (2005) analyzed the vibrations of plates with an attached two degree of freedom

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system, Avalos, Larrondo and Laura (1994) obtained the first six, axisymmetric natural frequencies of vibration of circular plates with an elastically mounted centered mass; Mermerta and Gürgoze (2004), analysed the preservation of the fundamental natural frequencies of rectangular plates with mass and spring modification; Wu (2003), proposed the use of effective stiffness matrix for the free vibration analyses of a non-uniform cantilever beam carrying multiple two degree-of-freedom spring-damper-mass systems; Gürgoze (2005) presented the representation of a cantilevered beam carrying a tip mass by an equivalent spring-mass system; Li and Daniels (2002) studied the vibrations of elastically restrained plates arbitrary loaded with springs and masses with a Fourier series method; Bambill, Felix and Rossit (2006) presented and approximate solution for the natural frequencies of thin, rectangular plates with holes or orthotropic "patches" carrying an elastically mounted mass; Dowell (1979) studied the vibration of beams carrying elastically mounted systems. Rossit and Laura (2001) analyzed the free vibration of a cantilever beam with a spring-mass system attached to the free end and Wu and Whittaker (1999) studied the natural frequencies and mode shapes of a uniform cantilever beam with multiple two degree of freedom spring-mass systems. Rossit and Ciancio (2008) studied the natural frequencies and normal modes of vibration of rectangular anisotropic plates supported by different combinations of the classical boundary conditions and with additional complexities, holes and attached concentrated masses, using the Ritz method. A survey of the literature reveals that the analysis of three degree-of-freedom systems attached to plates has not apparently been studied, except for the solution obtained by Wu in his recent papers (Wu 2005, Wu 2006).

In the present work, an analytical approach for the study of the vibration of plates carrying three-degree-of-freedom (3-DOF) systems is presented. The problem is of considerable technological importance in view of the great variety of circumstances in which it can be found. In certain situations, the natural frequencies of this compound system are required to determine its response when used as a component of structural systems (like machinery elastically mounted on plates in building structure or elements added to printed wiring boards for electronic systems). In other cases, the whole system can be used as a vibration absorber to attenuate the vibration amplitude of the plate. From these points of view, it is possible to affirm that the problem is not only of particular interest in respect to basic knowledge of the mechanical behaviour of physical systems but, also of technological importance. The paper is organized as follows. Section 2 presents the mathematical formulation of the problem within the Lagrangian formalism. The natural frequencies and the normal mode shapes of the compound system for different values of stiffness, mass, moment of inertia and location of the 3-DOF system is presented in Section 3. Finally, concluding remarks are presented and discussed in Section 4.

2. Lagrange's formulation

In the next lines, the equations of motion are derived using Lagrangian formalism. The advantage of this approach resides in that a physical interpretation of Lagrange multipliers as the forces that the 3-DOF system exerts on the bare plate (plate without the spring-mass system) is feasible.

First, consider the system of Fig. 1 which shows a 3-DOF spring-mass system (subsystem) attached to a plate. The intervening parameters m_e, I_{ex}, I_{ey} are respectively, the lumped mass and mass moment of inertia about the x and y axes; k_1, k_2, k_3 and k_4 are the spring constants and a_1, a_2, a_3 and a_4 are the distances between the barycentre and the sides of the rigid mass of the 3-DOF spring-mass system (see Fig. 2).

The total kinetic and strain energies of the entire system are

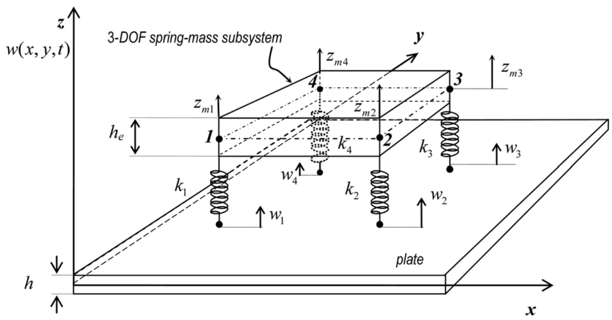


Fig. 1 Rectangular plate with a three degree of freedom spring-mass subsystem attached to it

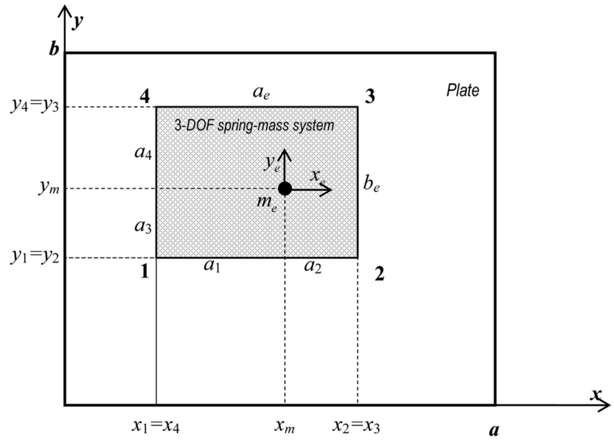


Fig. 2 Coordinates and dimensions

$$T = \frac{1}{2}\sum_{i,j}^{m,n'} m_{ij}\dot{c}_{ij}^2 + \frac{1}{2}m_e\left(\frac{\dot{z}_{m1}a_2}{a_e} + \frac{\dot{z}_{m2}(a_1a_4 - a_2a_3)}{a_eb_e} + \frac{\dot{z}_{m3}a_3}{b_e}\right)^2 + \frac{1}{2}I_{ey}\left(\frac{\dot{z}_{m2} - \dot{z}_{m1}}{a_e}\right)^2 + \frac{1}{2}I_{ex}\left(\frac{\dot{z}_{m3} - \dot{z}_{m2}}{b_e}\right)^2 \quad (1)$$

with $a_e = a_1 + a_2$; $b_e = a_3 + a_4$

$$V = \frac{1}{2}\sum_{i,j}^{n,n'} m_{ij}\omega_{ij}^2 c_{ij}^2 + \frac{1}{2}k_1(z_{m1} - w_1)^2 + \frac{1}{2}k_2(z_{m2} - w_2)^2 + \frac{1}{2}k_3(z_{m3} - w_3)^2 + \frac{1}{2}k_4(z_{m4} - w_4)^2 \quad (2)$$

where c_{ij}'s are the time dependent generalized coordinates, ω_{ij} are the eigenfrequencies of the bare plate and the m_{ij} 's are given by

$$\rho h \int_A \phi_{ij}\phi_{mn} dA = \delta_{im}\delta_{jn}m_{ij} \quad (3)$$

Here, δ_{ij} represents Kronecker's delta, ρ is the plate's mass density and h its thickness. The transverse displacement of the plate is represented by

$$w(x,y,t) = \sum_{i,j}^{n,n'} c_{ij}(t)\phi_{ij}(x,y) \tag{4}$$

where the $\phi_{ij}(x, y)$'s represent the normal mode shapes of the bare plate (Appendix A). The summation is carried out up to the $n \times n'$ normal mode where the first N modes are considered.

Fig. 2 illustrates the connection between the position of a point in the drawing plane and its coordinates.

For the mathematical model four restriction functions f_1's are imposed

$$f_1 = \sum_{i,j}^{n,n'} c_{ij}(t)\phi_{ij}(x_l, y_l) - w_1(t) = 0; \;\; l = 1,\ldots,4 \tag{5a,b,c,d}$$

and another one is imposed, because the mass was considered as a rigid body

$$f_5 = z_{m4} - (z_{m1} + z_{m3} - z_{m2}) \tag{6}$$

This gives a total of $\gamma = 5$ restriction functions

As it was previously expressed, the equations of motion are obtained by means of Lagrange's equations. The equations are

$$\frac{d}{dx}\left(\frac{\partial L}{\partial s_p}\right) - \frac{\partial L}{\partial s_p} = \sum_{l=1}^{\gamma} \lambda_l \frac{\partial f_l}{\partial s_p}; \qquad p = 1, 2, \ldots.., N+m \tag{7}$$

and γ constraint equations, Eq. (5) and Eq. 6).

The system has $N + m$ non-independent degrees of freedom s_p with γ constraints

The complete system summarizes a total of $N + m + \gamma$ equations (Meirovitch 1998) in the variables s_p (p = 1,2,......, $N + m$) and the Lagrange multipliers λ_1(l = 1,....., γ).

Eq. (5) and Eq. (6) are used to remove the λ_1(l = 1,....., γ) variables, and the system of equations becomes in a set of $N + m - \gamma$, (N + 3 in the present case), coupled linear second order differential equations in terms of the independent set of coordinates

$q \equiv [q_1, \ldots q_N, q_{N+1}, q_{N+2}, q_{N+3}] \equiv [c_{11}, \ldots\ldots, c_{n,n'}, z_{m1}, z_{m2}, z_{m3}]$

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = 0 \tag{8}$$

The explicit form of matrices **M** and **K** are given in Appendix B. Finally, in order to calculate the natural frequencies and normal mode shapes of the whole system (plate with 3-DOF system).

Eq. (8) is solved imposing a harmonic motion of $q_p(t) = \bar{q}_p e^{i\omega t}$. It is important to mention that as matrices **M** and **K** contain constant coefficients. This means that the corresponding eigenvalue problem obtained from Eq. (8) is linear.

Once the eigenvalues (frequencies) are obtained, the eigenvectors $\bar{q}_p$ can be calculated. From Eq. (4), it is possible to express every normal mode of the whole system by

$$W^{(i)}(x,y) = \sum_{p=1}^{N} \bar{q}_p^{(i)}\phi_p(x,y) + \sum_{p=N}^{N+3} \bar{q}_p^{(i)}; \qquad i = 1,\ldots..,N+3 \tag{9}$$

where the superscript (i) indicates the plate mode under consideration.

3. Numerical results

Numerical results are presented for the case of a SSSS plate (fully simply supported) and a SFSF plate (simply supported on $x = 0$ and $x = a$; and free at $y = 0$ and $y = b$). The dimensions and the material constants for the plate are: length $a = 2$ m, width $b = 1$ m, thickness $h = 0.005$ m, mass density $\rho = 7.850 \times 10^3$ kg/m^3, plate's mass $m_{plate} = \rho abh = 78.5$ kg, Poisson coefficient $\nu = 0.30$ and Young modulus $E = 2.051 \times 10^{11}$ N/m^2. These values and conditions have been chosen to compare the present results with those given by Wu (2006).

The natural frequencies are shown in radians per second.

Concerning the selection of the number of modes assumed for the displacement amplitude, Eq. (4), it is used $N = 10$ in the expansion. This choice is made since if a major number of modes are considered the changes in all frequency values are negligible (less than 0.05% of the presented values).

With the aim of testing the numerical results (which will be called hereafter LMM model) the authors apply a finite element method approach (FEM) to solve the same problem, implemented with the professional code ALGOR (2007). The plate is modelled considering rectangular elements of 12 degrees of freedom formulated by de Veubeke (1968) under thin elastic plate theory.

Regarding FEM implementation of the plate model, a mesh of 20000 square elements is considered in all cases. This number of elements was previously selected under the result of a convergence test where the number of plate's elements (mesh density) were increased taking 5000, 20000, 80000, 180000 elements at each time. From those results, the authors considered that 20000 elements give an appropriate number to build the plate in all FEM models. As it can be seen in Table 1, if more elements are taken in the FEM formulation no better results are obtained.

The values were obtained with a PC standard: AMD Athlon(tm) 64 Processor 3000+, 1.81 GHz, RAM 448 MB, Microsoft Windows XP Professional.

In the LMM model, the number of degrees of freedom is equal to the number of assumed modes, $N = 10$, (in the present study).

Table 1 Convergence analysis of the FEM model

Model	1	2	3	4
Elements	5000	20000	80000	180000
Nodes	5151	20301	80601	180901
DOF	14849	59699	239399	539099
ω_1	95.421	95.417	95.415	95.415
ω_2	152.689	152.670	152.666	152.665
ω_3	248.135	248.093	248.082	2483081
ω_4	324.436	324.417	324.412	324.412
ω_5	381.760	381.685	381.666	381.663
ω_6	381.760	381.685	381.666	381.663
ω_7	477.301	477.131	477.089	477.081
ω_8	553.564	553.446	553.417	553.411
Elapsed time (h : m : s)	00:00:49	00:05:02	01:04:46	04:34:35

3.1 Comparison between analytical and numerical results; validation

Table 2 presents the first eight natural frequencies of a SSSS plate with a spring-mass system, in one case a 3-DOF system of reduce dimensions, and in another case a 1-DOF system.

The 3-DOF system is adopted of reduced dimensions in order to prove if this model gives the correct limit as its size is diminished. Therefore, the natural limit of this configuration must be a plate with 1-DOF system attached to it. The 1-DOF system is located at $(x_m, y_m) = (1.00 \text{ m}, 0.50 \text{ m})$ and has the same mass and equivalent stiffness as the 3-DOF system. Numerical tests are made setting $a_1 = a_2 = a_3 = a_4 = 0.01$ m, $(a_e = b_e = 0.02 \text{ m} \approx 0)$ and $k_1 = k_2 = k_3 = k_4 = k$, with $k = 250$ N/m for the 3-DOF system. Other constants of the attached mass are: thickness $h_e = 0.04$ m, density $\rho = 7.850 \times 10^3$ kg/m^3, mass $m_e = 0.1256$ kg and mass moments of inertia $I_{ex} = m_e(h_e^2 + a_e^2)/12 = I_{ey} = m_e(h_e^2 + b_e^2)/12 = 2.0933 \times 10^{-5}$ kg m^2.

The rows labelled as FEM corresponds to the Finite Element Method's results and LMM corresponds to the Lagrange Multiplier Method's results. The subsystem is located at $x_1 = x_m - a_1$, $y_1 = y_m - a_3$; $x_2 = x_m + a_2$, $y_2 = y_1$; $x_3 = x_2$, $y_3 = y_m + a_4$; $x_4 = x_1$, $y_4 = y_3$; with $x_m = 1.00$ m; $y_m = 0.50$ m (center of the plate) and its own natural frequencies are listed in a row labelled 3-DOF. Obviously, the first two frequencies can not be avoided in spite of the fact that the dimensions of the subsystem have been reduced. It can be observed that the LMM approach has an excellent agreement when compared with FEM 3-DOF (differences below 0.7%). On the other hand, the frequencies of the plate with 1-DOF system attached to it (FEM 1-DOF) coincide with the values presented in rows named as FEM 3-DOF and LMM.

It can be seen that the displacement amplitude with $N = 10$ terms in the expansion gives excellent agreement with accurate finite element results.

3.2 Eigenvalue perturbation analysis; nearly uncoupled system

Through observation of Table 1 it can be said that the first three natural frequencies of the compound system are nearly identical to those of the 3-DOF oscillator and the subsequent natural frequencies are merely tiny corrections to those of the bare plate.

For that reason one may think that a first order eigenvalue perturbation analysis can give an approximate result to the perturbed eigenfrequencies. The authors numerically study this situation and give a kind of heuristic criterion (without proving it formally) which may be applied to similar cases.

Table 2 First eight natural frequencies of a SSSS rectangular plate with a spring-mass system

Model		ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
Plate + Subsystem	FEM 1-DOF	-	-	87.612	97.100	152.670	248.211	324.417	381.684
	FEM 3-DOF	69.116	69.116	87.614	97.098	152.670	248.210	324.417	381.684
	LMM	9.116	69.116	88.214	96.323	152.916	248.137	324.411	381.660
	First-order-EP	69.116	69.116	88.721	96.472	152.916	248.137	324.411	381.660
3-DOF Exact solution		69.116	69.116	89.228	-	-	-	-	-
Bare plate		-	-	-	95.415	152.663	248.078	324.410	381.659

As it is not an intention of this paper to expose the theory of eigenvalue perturbation analysis, the reader may refer to Inman (2006) for a detailed presentation of it.

Taking the theory's main results, the increments of the perturbed eigenfrequencies are calculated as

$$\delta\omega_i^2 = \frac{\mathbf{x}_{0i}^{\mathbf{T}}(\delta\mathbf{K} - \omega_{0i}^2\delta\mathbf{M})\mathbf{x}_{0i}}{\mathbf{x}_{0i}^{\mathbf{T}}\mathbf{M}_0\mathbf{x}_{0i}} \tag{10}$$

where $\mathbf{M}_0$ is the unperturbed mass matrix; ω_{0i} and $\mathbf{x}_{0i}$ are respectively the $i-th$ eigenfrequency and eigenvector of the unperturbed problem. Finally, the perturbed eigenfrequencies are calculated with $\omega_i^2 = \omega_{0i}^2 + \delta\omega_i^2$.

These results are shown in Table 2 in row named "First-order-EP". In order to obtain accurate results for the general case, the norm of the matrices $|\delta\mathbf{K}|$ and $|\delta\mathbf{M}|$ have to be calculated first. If they are within a 10% of $|\mathbf{K}|$ or $|\mathbf{M}|$, or if $\mathbf{x}^{\mathbf{T}}_{0i}(\delta\mathbf{K} - \omega_{0i}^2\delta\mathbf{M})\mathbf{x}_{0i} \leq 0.1\omega_{0i}^2$, both systems are nearly uncoupled to each other and a first order eigenvalue perturbation analysis could be used to efficiently determine the natural frequencies of the compound system.

3.3 Influence of stiffness, mass, moment of inertia and location of 3-DOF system

The purpose of this subsection is to study the influence of the parameters of the 3-DOF system in the dynamic behaviour of a SFSF rectangular plate.

Table 3 shows the influence of the spring stiffness, k_1, on the first eight natural frequencies of the compound system. The mass is elastically attached at the center of the plate, i.e., $(x_m, y_m) = (1.0$ m, 0.5 m$)$ and its physical constants are $m_e = 78.5$ kg, $h_e = 0.75$ m, $I_{ex} = I_{ey} = 7.3594$ kg m^2. $a_1 = a_2 = a_3 = a_4 = 0.375$ m.

It can be observed that the results of FEM analysis and the analytical model LMM are in excellent agreement between them. For $k_1 = k_2 = k_3 = k_4 = k = 10^4$ N/m, compared with previous results Wu (2006), the present analytical results are proved to be more accurate -as the comparison with FEM shows- and provide a new available data in the literature.

Moreover, they are qualitatively correct viewed from the following argument due to Lord Rayleigh (1945).

In general, the frequencies of a bare plate change if a spring-mass system is added to it. The frequencies of the bare plate originally higher than the basic spring-mass frequencies (3-DOF) are increased. Those originally lower are decreased, and a new set of frequencies appears between the originally pair of frequencies nearest the spring-mass frequencies. In Table 3, when $k < 10^4$ N/m, the subsystem's frequencies are always below the frequencies of the bare plate. As a result, the frequencies of the whole system happen to be a bit higher than those of the bare plate (see row "Bare plate" in Table 3 and Fig. 3). On the contrary, for $k = 10^4$ N/m, the first frequency of the bare plate $\omega_1^{(bare)} = 18.392$ is below $\omega_1^{(3\text{-}DOF)} = 22.573$, which is the 3-DOF system first natural frequency (traslation). As a consequence, according to Rayleigh's statement, the first frequency of the whole system (ω_1 in Table 3) is lower than $\omega_1^{(bare)}$ and superior frequencies, ω_5, ω_6, ω_7, ω_8, are higher than $\omega_2^{(bare)}$, $\omega_3^{(bare)}$, $\omega_4^{(bare)}$, $\omega_5^{(bare)}$ respectively. Following the same argument, three "new" frequencies appear in between (in this case ω_2, ω_3, ω_4), due to the contribution of the degrees of freedom of the subsystem to the motion of the entire system.

This may be understood by the following intuitive approach based on the concept of "effective mass",

Table 3 Influence of the spring stiffness on the first eight natural frequencies of a SFSF rectangular plate with the 3-DOF system

k (N/m)	Model	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
10	FEM	0.7131	0.8741	0.8742	18.412	53.220	74.494	124.785	168.763
	LMM	0.7131	0.8741	0.8742	18.412	53.225	74.495	124.792	168.766
	3-DOF	0.7138	0.8743	0.8743	-	-	-	-	-
100	FEM	2.233	2.758	2.762	18.591	53.321	74.550	124.837	168.764
	LMM	2.233	2.758	2.762	18.590	53.326	74.551	124.845	168.767
	3-DOF	2.257	2.764	2.764	-	-	-	-	-
1000	FEM	6.394	8.556	8.662	20.508	54.341	75.111	125.361	168.775
	LMM	6.400	8.559	8.664	20.499	54.347	75.114	125.369	168.778
	3-DOF	7.138	8.742	8.742	-	-	-	-	-
10000	FEM	10.866	22.430	25.187	37.790	65.272	81.044	130.433	168.885
	LMM	10.891	22.486	25.228	37864	65.379	81.103	130.459	168.890
	Wu (2006)	-	21.169	25.210	33.996	59.760	-	124.996	168.890
	3-DOF	22.573	27.646	27.646	-	-	-	-	-
	Bare plate LMM	18.3924	53.2143	74.4896	124.7871	168.7663	203.9653	225.8961	284.0378

which in this case represents the mass that must be added to the attached subsystem to correct predict the behaviour of the complete system and depends on frequency ω and the parameters of the attached subsystem. With the help of this concept -which yields very accurate results if the frequencies of the bare plate are far apart-, it is possible to explain Rayleigh's statement. In the case of a 1-DOF system added to a plate, the effective mass is $m_{ef} = k/(\omega^2_a - \omega^2)$, where k and ω_a are the elastic constant and the natural frequency of the 1-DOF. Accordingly, the expression for the "new" frequencies of the compound system is

$$\omega^2 \approx \frac{\omega_p^2}{1 + \phi_p^2(\mathbf{x})\dfrac{m_{ef}}{m_p}} \tag{11}$$

where ω_p is the natural frequency of the plate whose value is near to ω_a, $\phi_p^2(\mathbf{x})$ is the displacement amplitude of mode p where the subsystem is attached and m_p is mode's p modal mass of the bare plate.

Then, if it is made $\omega \approx \omega_p$ on the definition for m_{ef} (providing will be slightly modified), $m_{ef} > 0$ for $\omega_a > \omega_p$ and the frequency of the compound system decreases. For $\omega_a < \omega_p$, $m_{ef} < 0$ and the "new" frequency increases.

If multiple uncoupled DOF systems are added at ($\mathbf{x}_1$,....., $\mathbf{x}_r$,.....), expression Eq. (11) generalizes to

$$\omega^2 \approx \frac{\omega_p^2}{1 + \dfrac{1}{m_p}(\phi_p^2(\mathbf{x}_1)m_{ef1} + \ldots + \phi_p^2(\mathbf{x}_r)m_{efr} + \ldots)} \tag{12}$$

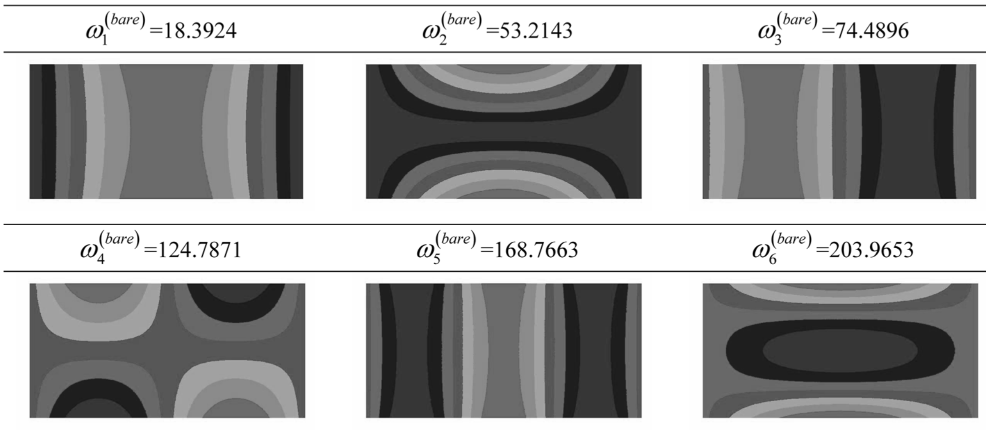


Fig. 3 First six mode shapes of a rectangular SFSF bare plate

which provides a simply approach to predict the “new” frequencies depending on the variation of the parameters of multiple DOF added systems. Nevertheless, caution must be taken if one wants to apply Eq. (12) to the 3-DOF (coupled) subsystem since it is no longer valid. In this case, a similar equation can be calculated resulting in a much more complicated expression. Obtaining and solving this equation means solving the problem. Since the present analysis is only qualitatively at this stage, it is only provided here its general form

$$\omega^2 \approx \frac{\omega_p^2}{1 + \frac{1}{m_p}\left(\sum_{i=1}^{4} \phi_p^2(\mathbf{x}_i) m_{efi} + \sum_{i<j}^{3} \phi_p(\mathbf{x}_i)\phi_p(\mathbf{x}_j) m_{efij}\right)} \tag{13}$$

Obviously, the m_{efi} (i = 1,....,4) -which depend on 3-DOF system’s parameters k_1, I_{ex}, I_{ey}, m_e, as well as on frequency ω-have not the same expression as in Eq. (12) and also the m_{efij}’s have been added (which represent the cross effective mass between $\mathbf{x}_i$ and $\mathbf{x}_j$).

A more complete explanation exceeds the purpose of the present paper. Nonetheless, Eq.13 - although approximate - shows how a variation of 3-DOF system’s parameters (stiffness, inertia and location) affects the natural frequencies of the whole system in a different way compared with the simpler case of multiple 1-DOF system attached to a plate.

Table 4 illustrates the influence of mass moments of inertia of the subsystem, on the first eight natural frequencies of the compound system. The subsystem is located at the center of the plate. Three different cases are analyzed:

Case	h_e (m)	m_e (kg)	$I_{ex} \equiv I_{ey}$	$k_1 = k_2 = k_3 = k_4 =$ (N/m)
1	0.750	78.50	7.35940	10000
2	0.375	39.25	2.29980	10000
3	0.075	7.85	0.37165	10000

Again, the values obtained by the analytical model LMM and FEM-based computations show excellent agreement between them. For case 1, $\omega_1^{(3-DOF)}$, $\omega_2^{(3-DOF)}$, $\omega_3^{(3-DOF)}$ lie between $\omega_1^{(bare)}$ and $\omega_2^{(bare)}$. After applying Rayleigh's statement (or effective mass concept), it can be conclude that $\omega_1 < \omega_1^{(bare)}$ $\omega_5 < \omega_2^{(bare)}$ and there must be three frequencies ω_2, ω_3, ω_4, appearing between them. A similar reasoning to explain the "new" shifted frequencies could be applied for case 2 and 3.

Fig. 4 shows the first eight modes and mode shapes of a rectangular SFSF plate with a 3-DOF system at its center. The Figure presents both, a perspective view of modal shapes as well as their projection on plane $x - z$.

Referring to the analysis of the relative ratios of the normal coordinates **q**, that have been obtained from solving Eq. (8), it is possible to analyze the relative contribution of bare plate's and subsystem's modes to the motion of the compound system.

In particular, looking at the first six modes (case 1) in Table 3 and transcribing only the ratios distinct from zero. For example analyzing the first mode $\omega_1 = 10.891$, Fig. 4(a), there is, $q_1^{(1)}/q_{N+1}^{(1)} = 0.837$, $q_{N+1}^{(1)}/q_{N+2}^{(1)} = q_{N+1}^{(1)}/q_{N+3}^{(1)} = 1$. This clearly means that the first mode's motion is mainly due to the contribution of the first mode of the bare plate plus a translation of the center of mass of the subsystem (first mode of the 3-DOF system). The second mode, Fig. 4(b), has $\omega_2 = 22.486$ and $q_{N+3}^{(2)}/q_{N+1}^{(2)} = q_{N+3}^{(2)}/q_{N+2}^{(2)} = -1$, $q_2^{(2)}/q_{N+3}^{(2)} = -0.491$, showing a non-vanishing contribution of the second mode of the bare plate plus a rotation around y axis of the subsystem (second mode of the 3-DOF system). A similar analysis can be carried out for the third mode, Fig. 4(c), $\omega_3 = 25.288$, $q_{N+1}^{(3)}/q_{N+2}^{(3)} = q_{N+1}^{(3)}/q_{N+3}^{(3)} = -1$, $q_3^{(3)}/q_{N+1}^{(3)} = 0.161$, which is mainly due mainly due to a rotation around y axis of the subsystem (third mode of the subsystem) and a small contribution of the third mode of the bare plate. Following the same arguments modes 4, 5 and 6 can be studied in a similar fashion: $\omega_4 = 37.864$, $q_{N+1}^{(4)}/q_{N+2}^{(4)} = q_{N+1}^{(4)}/q_{N+3}^{(4)} = 1$, $q_{N+1}^{(4)}/q_1^{(4)} = -0.491$, represents a contribution of the first mode of the bare plate and the first mode of the subsystem, Fig. 4(d); $\omega_5 = 65.379$, $q_{N+3}^{(5)}/q_{N+1}^{(5)} = q_{N+3}^{(5)}/q_{N+2}^{(5)} = -1$, $q_{N+1}^{(5)}/q_2^{(5)} = -0.149$ means a contribution of the second mode of the plate and the first mode of the subsystem, Fig 4(e), and $\omega_6 = 81.103$, $q_{N+1}^{(6)}/q_{N+2}^{(6)} = q_{N+1}^{(6)}/q_{N+3}^{(6)} = -1$, $q_{N+1}^{(6)}/q_3^{(6)} = -0.125$ means that its movement is mainly due to the third mode of the plate plus a rotation around y axis of the subsystem, Fig. 4(f). Finally, higher modes (7-8) seem to remain undisturbed by the presence of the

Table 4 Influence of different moments of inertia of the 3-DOF system on the frequency values of a SFSF rectangular plate

Case	Model	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
1	FEM	10.866	22.429	25.186	37.789	65.271	81.044	130.433	168.885
	LMM	10.891	22.486	25.228	37.864	65.379	81.103	130.459	168.890
	3-DOF	22.573	27.646	27.646	-	-	-	-	-
2	FEM	13.453	36.751	43.167	43.857	71.262	83.249	130.433	168.887
	LMM	13.473	36.810	43.280	43.915	71.443	83.334	130.459	168.892
	3-DOF	31.923	49.455	49.455	-	-	-	-	-
3	FEM	17.142	50.855	71.271	75.682	127.273	128.053	130.433	168.908
	LMM	17.148	50.862	71.267	75.977	127.588	128.586	130.459	168.914
	3-DOF	71.383	123.025	123.025	-	-	-	-	-
	Bare plate LMM	18.3924	53.2143	74.4896	124.7871	168.7663	203.9653	225.8961	284.0378

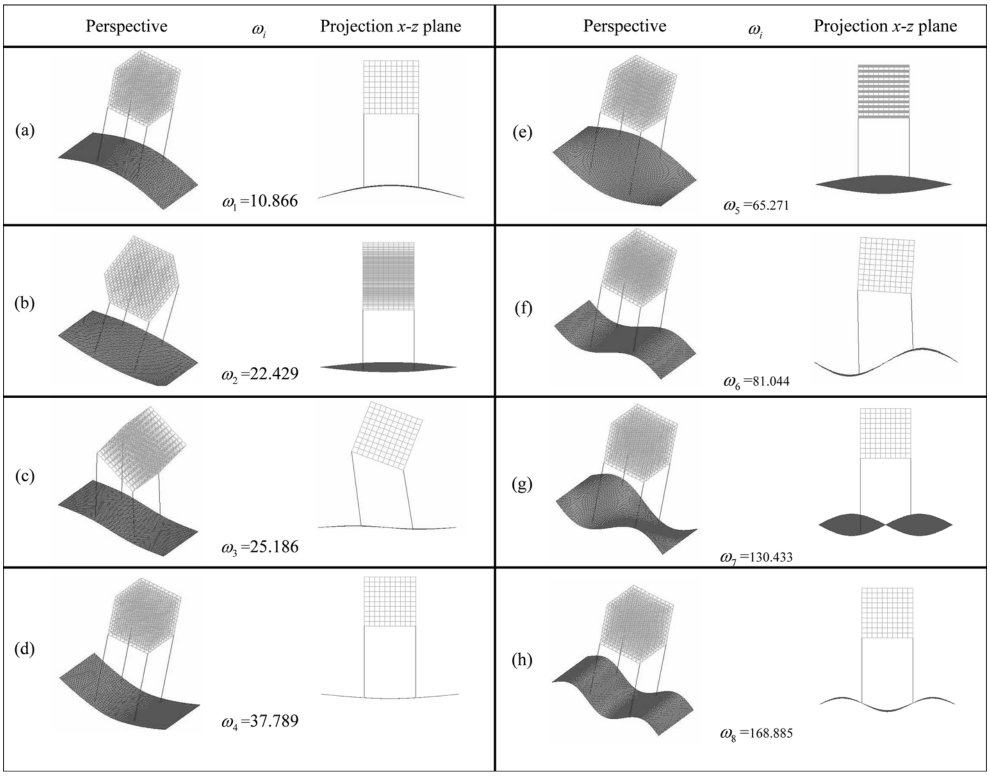


Fig. 4 First 8 modes of a rectangular SFSF plate with 3-DOF system at its center

subsystem and they can be represented with a high degree of accuracy by the corresponding modes of the bare plate (plus a tiny displacement of the subsystem's center of mass), Fig. 4 (g) and (h).

From this analysis, it can be observed that the "new" modes are due to the contribution of bare plate's modes and subsystem's modes that present the same symmetry. Another example, mode 1 of the SFSF bare plate presents negative curvature along x axis (mainly a sine function) on its whole domain, see Fig. 3; on the other side, the first mode of the subsystem is a pure translation of its center of mass. Then, the first and fourth "new" modes are constituted by simply the combination of these two modes. Moreover, "new" mode 2 is due to a combination of mode 2 of the bare plate (alternative negative and positive curvature along y axis) and a rotation around x axis of the 3-DOF system. Accordingly, the rest of the modes can be constructed following similar analyses.

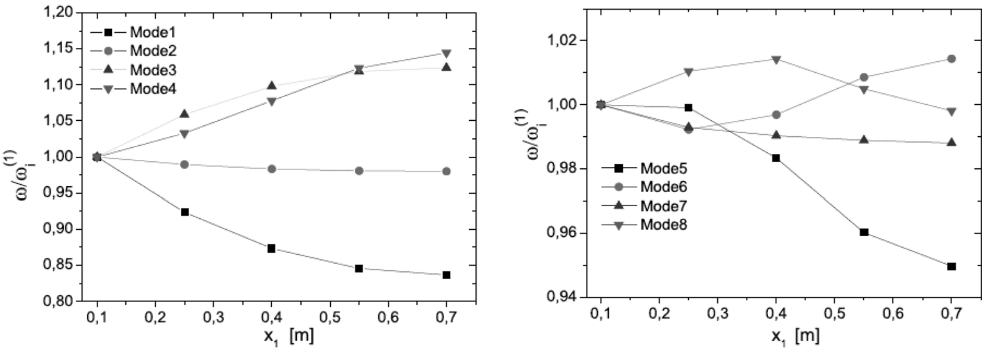
In order to study the influence of different subsystem's locations on the frequencies and modes of the compound system, Fig. 5 plots the first eight frequencies as a function of the location of the 3-DOF system. The subsystem is defined by $a_1 = a_2 = 0.30$ m; $a_3 = a_4 = 0.15$ m, $h_e = 0.40$ m, $I_{ex} = 1.63546666$ kg m^2, $I_{ey} = 3.40166666$ kg m^2, $k = k_l = 10^4$ N/m. Five different 3-DOF system' locations are analyzed, which is defined by the coordinates of points 1 and 4

Each frequency is normalized to the frequency of the system, with the 3-DOF system at location 1; which are: $\omega_1 = 12.7347$, $\omega_2 = 22.8678$, $\omega_3 = 27.0151$, $\omega_4 = 33.7895$, $\omega_5 = 58.7265$, $\omega_6 = 78.1048$, $\omega_7 = 127.0677$, $\omega_8 = 169.1550$, respectively.

Location	(x_1, y_1)	(x_4, y_4)
1:	(0.10 m, 0.650 m);	(0.10 m, 0.950 m)
2:	(0.25 m, 0.575 m);	(0.25 m, 0.875 m)
3:	(0.40 m, 0.500 m);	(0.40 m, 0.800 m)
4:	(0.55 m; 0.425 m);	(0.55 m, 0.725 m)
5:	(0.70 m, 0.350 m);	(0.70 m, 0.650 m)

First of all, it is important to remark that the results show no general tendency and the behaviour depends on the mode under consideration. Whereas for mode 1 and 2 the frequency drops, as the 3-DOF system's location moves to the center of the plate, for mode 3 and 4 its value is increased. From that fact, it can be concluded that the effect of adding the subsystem is to increase the total system's stiffness as it is moved away from the center of the plate for the first two modes and to decrease the total system's stiffness for modes 3 and 4, see Fig. 5(a). Of course, this can be understood since different plate's modes enable distinct motions of the 3-DOF system and contributes in different ways to the overall mass and stiffness. Another consequence of the same results can be put in this way: apparently, the more the subsystem moves to the center of the plate, the more symmetry of the overall motion is required inducing the total system's modes to be a combination of the modes of bare plate and 3-DOF system that possesses the same symmetry.

For modes 5 to 8, Fig. 5(b), the situation is similar in the sense that there exists no general tendency to increase/decrease the frequency values for all modes as the location of the 3-DOF system is modified. Briefly, it can be seen that mode 5 and 7 drop their frequency values as the subsystem is moved away from the corner side $x_1 = 0.10$ m to the center $x_1 = 0.70$ m. On the contrary mode 6 and 8 behave in a different manner. Whereas mode 6, decreases its frequency value up to $x_1 = 0.25$ m and then it raises as it moves to the center, mode 8 makes the opposite, i.e., it increases the frequency until $x_1 = 0.40$ m and then drops its value as the subsystem moves to the center of the plate.



(a) (b)

Fig. 5 First 8 natural frequencies (modes 1-8) of a SFSF plate with a 3-DOF system as a function of the position the subsystem x_1

4. Conclusions

In this paper the free vibration characteristics of a rectangular plate carrying a 3-DOF spring-mass system were calculated and analyzed by means of Lagrange multiplier method (analytical approach). The obtained results showed that this method exhibits excellent agreement when compared with a finite element formulation's results. This study not only extends previous results, (Wu 2006), but also presents a solution to the title problem, that requires solving a linear eigenvalue problem.

In order to perform a systematic study of problem, the influence of the parameters of the 3-DOF system (stiffness, mass, moment of inertia, location) in the dynamic behaviour of the system. Generally speaking it can be concluded that, apart from providing three more modes to the whole system, the 3-DOF system mainly modifies the modes of the bare plate whose frequencies lie near its natural frequencies. From the analysis of the variation of the frequencies, the influence of the spring constants, mass and moment of inertia can be understood under the effective mass concept or Rayleigh's statement. Different locations of the 3-DOF system were presented in Figs. 5(a) and (b).

As a conclusion, it can be sustained that the increase or decrease of the frequency values as the subsystem approaches to or moves away from the center of the plate depends on the mode and no general tendency is observed. From the analysis of the "new" modes, which is possible through the computation of the corresponding ratio of normal amplitudes, it can be affirmed that, when the 3-DOF system is located at the center of the plate, the "new" modes result in a combination of the subsystem's modes (rotation around x axis, y axis, translation) and the bare plate's modes that posses the same symmetry. This situation no longer exists as the subsystem is apart from the center of the plate. In these more general cases, different plate's modes enable distinct motions of the 3-DOF system contributing differently to the "new" modes as its location is modified.

As a complement, the authors also provide a way to efficiently determine the natural frequencies of the compound system when the plate and subsystem are nearly uncoupled. This was performed by means of a first order eigenvalue perturbation analysis.

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Appendix A. Normal mode shape functions

The normal mode shape functions $\phi_{ij}(x, y)$ used for the calculations are in all cases the exact solution of the classical differential equation of motion for the transverse displacement $w(x, y, t)$ of a plate (Leissa 1993). These are

• simply supported plate at all sides (SSSS): $\phi_{ij}(x, y) = \sin\left(i\pi\frac{x}{a}\right)\sin\left(j\pi\frac{y}{b}\right)$

• simply supported at $x = 0$ and $x = a$, and free at $y = 0$ and $y = b$, and (SFSF):

$\phi_{ij}(x, y) = \Phi_i(x)\Psi_j(y)$; where $\phi_i(x) = \sin\left(i\pi\frac{x}{a}\right)$ and $\Psi_j(y)$ is a linear combination of trigonometric and hyperbolic functions given by;

$$\Psi_j(y) = A_j\sin(\sqrt{\kappa^2-\alpha^2}y) + B_j\cos(\sqrt{\kappa^2-\alpha^2}y) + C_j\sin(\sqrt{\kappa^2-\alpha^2}y) + D_j\cos(\sqrt{\kappa^2-\alpha^2}y)$$

with $\alpha = j\pi/b$. The constants A_j, B_j, C_j, D_j as well as the eigenvalue $\kappa^2 = \sqrt{\rho h/D}$, are calculated substituting the equation for the displacement amplitude Eq. (4) into the boundary conditions for a SFSF plate, see Leissa (1993).

Appendix B. Mass and Stiffness matrices

The $(N + 3 \times N + 3)$ **M** and **K** matrices of Eq. (8) are

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0}^T & \mathbf{M}_{3DOF} \end{bmatrix}; \qquad \mathbf{K} = \begin{bmatrix} \mathbf{K}_p + \mathbf{K}_{sub} & \mathbf{K}_c \\ \mathbf{K}_c^T & \mathbf{K}_{3DOF} \end{bmatrix}$$, where $\mathbf{M}_p$ and $\mathbf{K}_p$

are $N \times N$ diagonal matrices whose elements are m_k and $m_k\omega_k^2$ respectively. The $\mathbf{K}_{sub}$ matrix represents the contribution of 3-DOF system, it is

$$\mathbf{K}_{sub} = \sum_{l=1}^{4} k_l \mathbf{\Phi}(x_l, y_l)\Phi^T(x_l, y_l)$$

where vector $\Phi(x_l, y_l)$ contains of the eigenfunctions of the plate evaluates at the points where the 3-DOF is attached.

$$\Phi(x_l, y_l) = [\phi_1(x_l, y_l), \ldots, \phi_N(x_l, y_l)]^T$$

$\mathbf{M}_{3DOF}$ and $\mathbf{K}_{3DOF}$ are 3 × 3 symmetric matrices that correspond to the mass and stiffness matrices of the attached 3-DOF system. The calculation yields

$$\mathbf{M}_{3DOF} = \begin{bmatrix} \dfrac{m_e a_2^2 + I_{ey}}{a_e} & \dfrac{m_e a_2 (a_1 a_4 - a_3 a_2) + I_{ey} b_e}{a_e^2 b_e} & \dfrac{m_e a_2 a_3}{a_e b_e} \\ \dfrac{m_e a_2 (a_1 a_4 - a_3 a_2) + I_{ey} b_e}{a_e^2 b_e} & \dfrac{m_e (a_1 a_4 - a_3 a_2)^2 + I_{ex} a_e^2 + I_{ey} b_e^2}{a_e^2 b_e} & \dfrac{m_e a_3 (a_1 a_4 - a_3 a_2) + I_{ex} a_e}{a_e b_e^2} \\ \dfrac{m_e a_2 a_3}{a_e b_e} & \dfrac{m_e a_3 (a_1 a_4 - a_3 a_2) + I_{ex} a_e}{a_e b_e^2} & \dfrac{m_e a_3^2 + I_{ex}}{b_e} \end{bmatrix}$$

$$\mathbf{M}_{3DOF} = \begin{bmatrix} k_1 + k_4 & -k_4 & k_4 \\ -k_4 & k_2 + k_4 & -k_4 \\ k_4 & -k_4 & k_3 + k_4 \end{bmatrix}$$

The $N \times 3$ rectangular $\mathbf{K}_c$ matrix rest to be defined. It results

$$\mathbf{K}_c = [-k_1\Phi(x_1, y_1) - k_4\Phi(x_4, y_4) - k_2\Phi(x_2, y_2) - k_4\Phi(x_4, y_4) - k_3\Phi(x_3, y_3) - k_4\Phi(x_4, y_4)]$$

Appendix C. Nomenclature

m_e	the lumped mass of the 3-DOF
I_{ex}, I_{ey}	mass moments of inertia about x and y axes
ρ_e	mass' density
k_1, k_2, k_3 and k_4	spring constants of the 3-DOF
a_e, b_e, h_e	mass' dimensions of the 3-DOF
a_1, a_2, a_3 and a_4	distances between the barycenter and the sides of the 3-DOF system
T	total kinetic energy
V	total strain energy
x, y, z	coordinates
t	time
$w(x, y, z)$	transverse displacement of the plate
$w_i = w_i(t) = w(x_i, y_i, t)$	plate's displacement at point i (x_i, y_i)
z_{mi}	mass's displacement at i spring contact
ω_{ij}	eigenfrequencies of the bare plate
m_{ij}	modal mass of the bare plate
δ_{ij}	Kronecker's delta

$a \times b$	plate dimensions
ρ	plate's mass density
ν	plate's Poisson coefficient
h	plate's thickness
E	plate's Young modulus
N	first modes considered
f_1	l restriction functions with $l = 1, \ldots, 4$
$c_{ij}(t)$	time dependent generalized coordinates
f_5	restriction function imposed because the mass is considered as rigid body
$\gamma = 5$	total number of restriction functions
x_m, y_m, z_m	barycenter of the attached mass
x_e, y_e	the mass barycentric coordinate system
L	Lagrangian
s_p	non-independent degrees of freedom
λ_l	Lagrange multiplier
N	the number of modes considered
$\mathbf{q} \equiv [q_1, \ldots q_N, q_{N+1}, q_{N+2}, q_{N+3}]$ $\equiv [c_{11}, \ldots, c_{n,n'}, z_{m1}, z_{m2}, z_{m3}]$	independent set of coordinates
$\mathbf{M}$, $\mathbf{K}$	mass and stiffness matrices
$q_p(t) = \bar{q}_p e^{i\omega t}$	harmonic decomposition of the generalized coordinates
$\bar{q}_p$	eigenvector
$W^{(i)}(x, y)$	(i) indicates the plate + subsystem natural mode shape
LMM	corresponds to Lagrange multiplier method
FEM	corresponds to finite element method
ω_{0i} and $\mathbf{x}_{0i}$	i – th eigenfrequency and eigenvector of the unperturbed problem
$\omega_i^2 = \omega_{0i}^2 + \delta\omega_i^2$	perturbed eigenfrequencies
$\lvert\delta\mathbf{K}\rvert$ and $\lvert\delta\mathbf{M}\rvert$	norm of the corresponding matrices
$\mathbf{M}_0$	unperturbed mass matrix
$\omega_i^{(bare)}$	i – th frequency of the bare plate
$\omega_i^{(3\text{-}DOF)}$	i – th frequency of the 3-dof spring-mass-system
ω_i	i – th frequency of the plate with 3-dof spring-mass-system attached
m_{ef}	effective mass

ω_a	frequency of the 1-dof spring-mass-system
ω_p	is plate's frequency whose value is near to ω_a
$\phi_p\ (\boldsymbol{x})$	displacement amplitude of mode p where the subsystem is attached
m_p	mode's p modal mass of the bare plate
SFSF	simple supported-Free-Simple supported-Free
SSSS	simple supported-Simple supported -Simple supported-Simple supported