Structural Engineering and Mechanics, Vol. 40, No. 3 (2011) 335-346
DOI: http://dx.doi.org/10.12989/sem.2011.40.3.335

Bifurcations of non-semi-simple eigenvalues and the zero-order approximations of responses at critical points of Hopf bifurcation in nonlinear systems

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(Received August 13, 2010, Revised July 22, 2011, Accepted August 17, 2011)

Abstract. This paper deals with the bifurcations of non-semi-simple eigenvalues at critical point of Hopf bifurcation to understand the dynamic behavior of the system. By using the Puiseux expansion, the expression of the bifurcation of non-semi-simple eigenvalues and the corresponding topological structure in the parameter space are obtained. The zero-order approximate solutions in the vicinity of the critical points at which the multiple Hopf bifurcation may occur are developed. A numerical example, the flutter problem of an airfoil in simplified model, is given to illustrate the application of the proposed method.

Keywords: eigenvalue bifurcations; non-semi-simple eigenvalues; Hopf bifurcation; zero-order approximation solution; nonlinear systems

1. Introduction

An emerging research area that has become very stimulating is the bifurcation control which aims at modifying the dynamical behavior of a system around critical points, delaying the onset of an inherent bifurcation, or stabilizing a bifurcation solution (Chen *et al.* 2000). For example, the control of the Hopf bifurcation in a class of nonlinear systems whose linear approximation has two distinct eigenvalues on the imaginary axis, was discussed without assuming the system is controllable (Verduzco and Alvarez 2006). An analytical method for the analysis and control of oscillations in non-linear control systems, whose linearization around the origin has k zero eigenvalues was presented (Verduzco 2007). The bifurcation and instability behavior are studied for the case, k = 3 (Yu 2003, 2004), and the normal forms are calculated for the case, k = 2 and k = 3 (Bi and Yu 1998). An efficient method for computing the normal forms for general semi-simple eigenvalues was presented (Yu and Leung 2003). The order of retarded nonlinear systems was studies (Nayfeh 2008). Modal interactions in contract-mode atomic force microscopes were discussed (Arafat *et al.* 2008). The steady-state dynamics of a linear structure weakly coupled to an

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essentially nonlinear oscillator was presented (Malatkar and Nayfeh 2007). Multi-stage design procedure for modal controller of multi-input defective systems was given (Chen 2007).

It should be noted that the previous studies mainly involve the control problems of the nonlinear system, whose linearization in the origin has non-semi-simple zero eigenvalues and the corresponding static bifurcation occurs. However, in actual engineering problems, the linearization of flutter analysis of aeroelasticity, the dynamic analysis of mobility and graspability of general manipulation systems may have non-semi-simple purely imaginary eigenvalues at a critical point, giving rise to multiple Hopf bifurcations (Chen *et al.* 2001). The case that the linear system has two pairs of purely imaginary eigenvalues at critical point giving rise to double Hopf bifurcations, and concerned with only the effect of time delayed feedbacks in a nonlinear systems with external forcing was developed (Yu *et al.* 2002).

One of the objectives of the bifurcation control is to modify the dynamical behavior of a system around critical point. Thus it is important to study the dynamical behavior of the nonlinear system for the bifurcation control (Chen *et al.* 2000). However, the responses at the Hopf bifurcation points in most published papers are obtained numerically which gives little physical insight of the dynamic behavior of the system. To this end, this paper is dedicated to find approximate analytical solutions for the bifurcations of non-semi-simple eigenvalues at critical point of Hopf bifurcation to understand the dynamic behavior of the system around the critical points. By using the Puiseux expansion, the expression for the bifurcations of non-semi-simple eigenvalues and the corresponding topological structure in the parameter space are obtained. The zero-order approximation solutions in the vicinity of the critical points at which the multiple Hopf bifurcation may occur are developed to show the instability behavior of the nonlinear system.

The paper is organized as follows. In the section 2, the basic equation for the nonlinear systems whose linear approximation has non-semi-simple eigenvalues are presented. By using the Puiseux expansion, the section 3 presents the bifurcations of the non-semi-simple eigenvalues and corresponding topological structure in the parameter space. The section 4 develops the zero-order approximation solutions at critical point to study the stability of the nonlinear system. In the section 5, a numerical example, the flutter analysis of an airfoil, is given to illustrate the application and validity of the presented method.

2. Technical background

Consider the following nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

where the variable $\mathbf{x} \in R^n$ is the state, $\mathbf{p}$ is the parameter. Assume that $\mathbf{f}(0, 0) = 0$, and the system can be linearized at $\mathbf{x} = 0$ with

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(0, \mathbf{p}) \in R^{n \times n} \tag{2}$$

As we know, the eigenvalues of $\mathbf{A}(\mathbf{p})$ are the functions of the parameter $\mathbf{p}$, and denoted as $\lambda_i(\mathbf{p})$, $i = 1, 2, \dots, n$.

This study discusses the case where the linearization system, $\mathbf{A}(\mathbf{p}_c)$, has multiple eigenvalues, that is $\lambda_i (i = 1, 2, \ldots, r)$ are m_i multiple eigenvalues, and $\mathrm{Re}(\lambda_i) = 0$, $\mathrm{Im}(\lambda_i) = \omega_{ic} \neq 0$ $(i = 1, 2, \ldots, r)$, respectively. Assume that A_m is used to denote the algebra multiplicity of the eigenvalue λ_i, and G_m is the number of the linearly independent eigenvectors corresponding to λ_i. If $A_m > G_m$, λ_i is a non-semi-simple eigenvalue and the system is unstable which describes the critical point $(\mathbf{p}_c, \omega_{ic})$ of Hopf bifurcation.

In the following, we give the basic equations of the linearization system which has multiple non-semi-simple eigenvalues at the critical points. From the algebra theory, there exist non-singular matrix **U**, such that

$$\mathbf{AU} = \mathbf{UJ} \tag{3}$$

where **U** is the generalized modal matrix of **A**, **J** is the Jordan block of **A**, given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_r \end{bmatrix}, \quad r \leq n \tag{4}$$

where

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{m_i \times m_i}, \quad \sum_{i=1}^{r} m_i = n \tag{5}$$

Eq. (3) can be written in the following manner

$$\begin{cases} (\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_1 = 0, & i = 1, \ldots, r \\ (\mathbf{A} - \lambda_i \mathbf{I})\mathbf{u}_j = \mathbf{u}_{j-1}, & j = 2, \ldots, m_i \end{cases} \tag{6}$$

The conjugate transpose of **A** is called adjoined system, i.e., for $\mathbf{A}^H$ the generalized modes satisfy the following equation

$$\mathbf{A}^H \mathbf{V} = \mathbf{V}\mathbf{J}^H \tag{7}$$

where $\mathbf{A}^H$, $\mathbf{J}^H$ are the conjugate transpose of **A** and **J**, respectively, **V** is the generalized modal matrix of the adjoined system.

Eq. (7) can be also written as the following form

$$\begin{cases} (\mathbf{A}^H - \bar{\lambda}_i \mathbf{I})\mathbf{v}_j = \mathbf{v}_{j+1}, & j = 1, 2, \ldots, m_{i-1} \\ (\mathbf{A}^H - \bar{\lambda}_i \mathbf{I})\mathbf{v}_{m_i} = 0 \end{cases} \tag{8}$$

where $\bar{\lambda}_i$ are the conjugate of λ_i. In general, $\mathbf{u}_i (i = 1, 2, \ldots, r)$ are known as the right eigenvectors, $\mathbf{v}_i (i = 1, 2, \ldots, r)$ the left eigenvectors, $\mathbf{u}_{i+1}, \ldots, \mathbf{u}_{i+m_i-1}$ and $\mathbf{v}_{i+1}, \ldots, \mathbf{v}_{i+m_i-1}$ are the right and left generalized eigenvectors of λ_i, respectively.

The right and left generalized modal matrices **U** and **V** satisfy the following orthogonal condition

$$\mathbf{V}^H\mathbf{U} = \mathbf{I} \tag{9}$$

Using the modal transformation

$$\mathbf{x} = \mathbf{U}\xi \tag{10}$$

we obtain the linearization system of Eq. (1) in the Jordan form

$$\dot{\xi} = \mathbf{J}\xi \tag{11}$$

3. Bifurcations of eigenvalues at crucial points

In this section, we use the Puiseux expansion to discuss the bifurcations of non-semi-simple eigenvalues at the critical points.

Assume that at critical value $\mathbf{p} = \mathbf{p}_c$, λ is a non-semi-simple eigenvalue of $\mathbf{A}(\mathbf{p}_c)$, and seek the change of the eigenvalue λ depending on a change of the parameter vector $\mathbf{p}$. To this end, suppose the parameter vector is given a change, i.e., $\mathbf{p} = \mathbf{p}_c + \varepsilon\mathbf{p}_1 = \mathbf{p}_c + \Delta\mathbf{p}$, where ε is a small number, $\mathbf{p}_1$ is a real vector, then the eigenvalue problem is perturbed into

$$(\mathbf{A} + \varepsilon\mathbf{A}_1)\tilde{\mathbf{u}} = \tilde{\lambda}\tilde{\mathbf{u}} \tag{12}$$

and $\varepsilon\mathbf{A}_1$ is given by

$$\varepsilon\mathbf{A}_1 = \sum_{j=1}^{L}\frac{\partial\mathbf{A}}{\partial p_j}\Delta p_j \tag{13}$$

where L is the number of the parameters.

Due to the λ is an m-multiple non-semi-simple eigenvalue, the small parameter expansion for semi-simple eigenvalue can not be used and we have to use the Puiseux expansion[11]. The perturbed eigenvalue and eigenvector, $\tilde{\lambda}$ and $\tilde{\mathbf{u}}$, are as follows

$$\tilde{\lambda}_k = \lambda + \lambda_k^{(1)}\delta + \lambda_k^{(2)}\delta^2 + \ldots + \lambda_k^{(m)}\delta^m + \lambda_k^{(m+1)}\delta^{m+1} + \ldots \tag{14}$$

$$\tilde{\mathbf{u}}_k = \mathbf{u}_1 + \mathbf{u}_k^{(1)}\delta + \mathbf{u}_k^{(2)}\delta^2 + \ldots + \mathbf{u}_k^{(m)}\delta^m + \mathbf{u}_k^{(m+1)}\delta^{m+1} + \ldots \tag{15}$$

where $\delta = \varepsilon^{1/m}$. Substituting Eqs. (14) and (15) into Eq. (12), yields

$$(\mathbf{A} + \delta\mathbf{A}_1)(\mathbf{u}_1 + \delta\mathbf{u}_k^{(1)} + \ldots) = (\lambda + \delta\lambda_k^{(1)} + \ldots)(\mathbf{u}_1 + \delta\mathbf{u}_k^{(1)} + \ldots) \tag{16}$$

By grouping the same power of δ, we get

$$\left.\begin{aligned}
&\delta^0\colon \mathbf{A}\mathbf{u}_1 = \lambda\mathbf{u}_1\\
&\delta^1\colon \mathbf{A}\mathbf{u}_k^{(1)} = \lambda\mathbf{u}_k^{(1)} + \lambda_k^{(1)}\mathbf{u}_1\\
&\delta^2\colon \mathbf{A}\mathbf{u}_k^{(2)} = \lambda\mathbf{u}_k^{(2)} + \lambda_k^{(2)}\mathbf{u}_k^{(1)} + \lambda_k^{(2)}\mathbf{u}_1\\
&\ldots\ldots.\\
&\delta^m\colon \mathbf{A}\mathbf{u}_k^{(m)} + \mathbf{A}_1\mathbf{u}_1 = \lambda\mathbf{u}_k^{(m)} + \lambda_k^{(1)}\mathbf{u}_k^{(m-1)} + \ldots + \lambda_k^{(m)}\mathbf{u}_1
\end{aligned}\right\} \tag{17}$$

From the above equations, it can be seen that in order to calculate $\mathbf{u}_k^{(1)}$, we first need to calculate $\lambda_k^{(1)}$, to calculate $\mathbf{u}_k^{(2)}$, we need to know $\lambda_k^{(1)}$, $\mathbf{u}_k^{(1)}$ and $\lambda_k^{(2)}$ and so on.

The perturbed eigenvalue $\tilde{\lambda}_k$ is given (Deif 1991, Chen 2007)

$$\tilde{\lambda}_i = \lambda + [(\mathbf{v}_m^H\mathbf{A}_1\mathbf{u}_1)^{1/m} e^{(2i\pi/m)j}]\varepsilon^{1/m} \tag{18}$$

and

$$\lambda_i^{(1)} = [(\mathbf{v}_m^H\mathbf{A}_1\mathbf{u}_1)^{1/m} e^{(2i\pi/m)j}]\varepsilon^{1/m}, \quad i = 1, 2, \ldots, m \tag{19}$$

where $\mathbf{u}_1$ and $\mathbf{u}_m$ can be obtained from Eqs. (6) and (8).

Now we turn to calculate the perturbation of the generalized modes, $\mathbf{u}_k^{(1)}$, corresponding to non-semi-simple eigenvalue λ. The second equation of Eq. (17) can be rewritten as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_k^{(1)} = \lambda_k^{(1)}\mathbf{u}_1, \quad k = 1, 2, \ldots, m \tag{20}$$

Using the definitions of the generalized modes, i.e., Eq. (3), we have

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_2 = \mathbf{u}_1$$

that is

$$(\mathbf{A} - \lambda\mathbf{I})\lambda_k^{(1)}\mathbf{u}_2 = \lambda_k^{(1)}\mathbf{u}_1 \tag{21}$$

Hence, we get

$$\mathbf{u}_k^{(1)} = \lambda_k^{(1)}\mathbf{u}_2 \tag{22}$$

where $\mathbf{u}_2$ is the generalized mode.

Therefore, the perturbed eigenvectors $\tilde{\mathbf{u}}_i$ corresponding to eigenvalues $\tilde{\lambda}_i$ can be approximated by

$$\tilde{\mathbf{u}}_i = \mathbf{u}_i + \lambda_i^{(1)}\mathbf{u}_2\varepsilon^{1/m}, \quad i = 1, 2, \ldots, m \tag{23}$$

From the above discussion, it is clear that if the non-semi-simple matrix has a small perturbation $\varepsilon\mathbf{A}_1$, the m-multiple eigenvalue can be separated into m distinct eigenvalues, which is known as the bifurcations of the multiple eigenvalues.

In the following, we give the geometric interpretation of the bifurcation of the m-multiple eigenvalue. To this end, we rewrite the Eq. (18) in the following form

$$\tilde{\lambda}_i = \lambda + (Z)^{1/m} e^{(2i\pi/m)j} \varepsilon^{1/m}, \quad i = 1,2,\ldots,m \tag{24}$$

where

$$Z = \mathbf{v}_m^H \mathbf{A}_1 \mathbf{u}_1 = R \cdot e^{\phi j}, \quad j = \sqrt{-1} \tag{25}$$

Thus, we have

$$\lambda_i^{(1)} = R^{1/m} e^{\frac{\phi + 2i\pi}{m} j} \varepsilon^{1/m}, \quad i = 1,2,\ldots,m \tag{26}$$

If $m = 2$, we have

$$\lambda_i^{(1)} = R^{1/2} e^{(\phi/2 + i\pi)j} \varepsilon^{1/2}, \quad i = 1,2 \tag{27}$$

By considering one parameter case and taking $\varepsilon = p - p_c$, from Eq. (27), we obtain

$$\begin{cases} \lambda_i^{(1)} = re^{(\phi/2 + i\pi)j} (p - p_c)^{1/2}, & p > p_c,\ i = 1,2 \\ \lambda_i^{(1)} = re^{(\phi/2 + i\pi)j} (p_c - p)^{1/2} j, & p < p_c,\ i = 1,2 \end{cases} \tag{28}$$

where $r = R^{1/2}$. Using Eq. (28), we obtain that $\lambda_i^{(1)}$ are at the points 1, 2, 3 and 4 placed on the circle of radius $r\sqrt{|p - p_c|}$, which is shown in Fig. 1.

From Eq. (28) and Fig. 1, it also can be seen that when the parameter p is increased for $p > p_0$, the increment of eigenvalues, $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$, diverge along the line 1-2; and for $p < p_0$, the increment of eigenvalues, $\lambda_3^{(1)}$ and $\lambda_4^{(1)}$, approach to zero along the line 3-4, which is perpendicular to the line 1-2. The arrows in the Fig. 1 show the movement of the eigenvalue λ, when the parameter p is changed.

Now we turn to discuss the bifurcations of eigenvalues at the critical point. Assume that the eigenvalue λ is a 2-multiple non-semi-simple eigenvalue, and $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) = \omega_c$. By using Eq. (28), we have

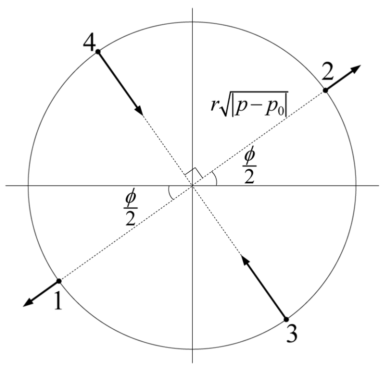


Fig. 1 Bifurcations of eigenvalues

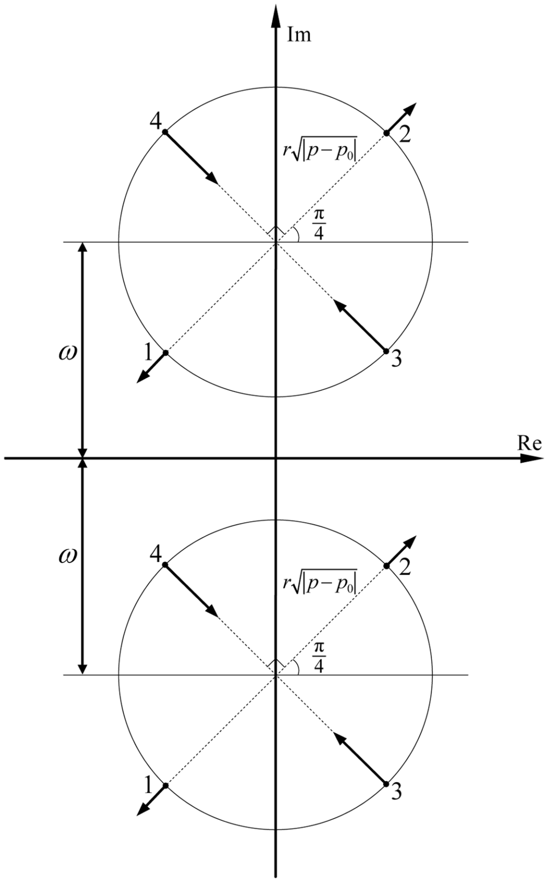


Fig. 2 Eigenvalue bifurcation at the critical point for the example

$$\begin{cases} \tilde{\lambda}_i = \omega_c j + re^{(\phi/2+i\pi)j}(p-p_c)^{1/2}, & p>p_c,\ i=1,2 \\ \tilde{\lambda}_i = \omega_c j + re^{(\phi/2+i\pi)j}(p_c-p)^{1/2}j, & p<p_c,\ i=1,2 \end{cases} \tag{29}$$

From Eq. (29) and Fig. 2, it can be seen that when the parameter p is increased for $p > p_0$, the second eigenvalue with positive real part diverges from the point 2; the first eigenvalue with negative real part diverges from the point 1; the eigenvalue 3 and 4 approach to original point (the critical point). Thus, we conclude that the system is unstable which is characterized by the second eigenvalue at the point 2 (Fig. 2). In this case, $\mathrm{Re}(\tilde{\lambda}_2)>0$, $\mathrm{Im}(\tilde{\lambda}_2) = \tilde{\omega}_{c2} \neq 0$, which describe the flutter occurs and $\tilde{\omega}_{c2}$ is the corresponding flutter frequency. This is the mechanism of the instability of Hopf bifurcation with non-semi-simple eigenvalues.

In the following section, we will discuss the zero-order approximate solution at critical point to illustrate the instability behavior of the nonlinear system.

4. The zero-order approximate solutions at crucial point of the nonlinear system with non-semi-simple eigenvalues

Assume that the linearization system of Eq. (1) in the Jordan form is

$$\dot{\boldsymbol{\xi}} = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{m \times m} \boldsymbol{\xi} \tag{30}$$

in which we assume that λ is an m-multiple non-semi-simple eigenvalue. Eq. (30) can be written as the following form

$$\begin{cases} \dot{\xi}_1 = \lambda\xi_1 + \xi_2 \\ \dot{\xi}_2 = \lambda\xi_2 + \xi_3 \\ \quad\ldots\ldots \\ \dot{\xi}_m = \lambda\xi_m \end{cases} \tag{31}$$

Solving the Eq. (31), and using the modal transformation (10), the solutions $\boldsymbol{\xi}$ and $\mathbf{x}(t)$ can be obtained in the following forms

$$\begin{cases} \xi_m = c_m e^{\lambda t} \\ \xi_{m-1} = (c_{m-1} + c_m t)e^{\lambda t} \\ \quad\ldots\ldots \\ \xi_1 = \left[c_1 + c_2 t + \ldots + c_{m-1}\dfrac{t^{m-2}}{(m-2)!} + c_m \dfrac{t^{m-1}}{(m-1)!}\right]e^{\lambda t} \end{cases} \tag{32}$$

and

$$\mathbf{x}(t) = \sum_{i=1}^{m} \mathbf{u}_i \xi_i = c_1\mathbf{u}_1 e^{\lambda t} + c_2(\mathbf{u}_1 t + \mathbf{u}_2)e^{\lambda t} + \ldots + c_m\left[\frac{\mathbf{u}_1 t^{m-1}}{(m-1)!} + \frac{\mathbf{u}_2 t^{m-2}}{(m-2)!} + \ldots + \mathbf{u}_m\right]e^{\lambda t} \tag{33}$$

where c_1, c_2, ..., c_m are arbitrary constants depending on the initial conditions. From Eq. (33), we can see that although $\lambda = (= \omega j)$ have no real part, the solution depends on t, t^{m-1}, ..., which means that solution $\mathbf{x}(t)$ is unbound, i.e., $\mathbf{x}(t) \to \infty$, when $t \to \infty$, and the system is unstable.

5. Numerical example

In order to illustrate the application of the present procedure, we consider the flutter problem of an airfoil in simplified formulation. The airfoil is replaced by a rigid rectangular panel with two degrees of freedom, the vertical displacement h and the rotation α. It is assumed that aerodynamic lift force is proportional to the angle of attack α and to the square of the velocity v of flight. The nonlinear differential equations of motion are (Chen 2007)

$$\begin{cases} m\ddot{h} + s\ddot{\alpha} + K_h h + \rho v^2 ab\alpha = \varepsilon Q_1(h_1, \alpha) \\ s\ddot{h} + J_\alpha\ddot{\alpha} + K_\alpha\alpha - \rho v^2 abe\alpha = \varepsilon Q_2(h_1, \alpha) \end{cases} \tag{34}$$

where m is the mass of the panel, s the static moment of the cross section area of the penal, J_α the moment of inertia, K_h the bending stiffness, K_α the torsional stiffness, ε is a small parameter, respectively, and εQ_1, εQ_2 are the nonlinear force.

If the parameters are given as $m/(\rho ab^2) = 5$, $s/(mb) = 0.25$, $J_\alpha/(mb^2) = 0.5$, $e/b = 0.4$, $K_h/m = 0.25$, $K_\alpha/J_\alpha = 1$ and $p = v(J_\alpha b/K_\alpha)^{1/2}$, then the linearized equations become

$$\mathbf{M}\ddot{\mathbf{q}} + [\mathbf{K} + \mathbf{H}(p^2)]\mathbf{q} = 0 \tag{35}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}, \quad \mathbf{K} + \mathbf{H}(p^2) = \begin{bmatrix} 0.25 & 0.2p^2 \\ 0 & 0.5 - 0.08p^2 \end{bmatrix} \tag{36}$$

Assuming the parameter $p_c = 1.32567735$, the state matrix has the following form

$$\mathbf{A} = \begin{bmatrix} 0 & -\mathbf{M}^{-1}(\mathbf{K}+\mathbf{H}) \\ \mathbf{I} & 0 \end{bmatrix} = \begin{bmatrix} 0.0 & 0.0 & -0.28571429 & -0.19642103 \\ 0.0 & 0.0 & 0.14285714 & -0.62065221 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \tag{37}$$

where **M** is the mass matrix, **K** the stiffness matrix, **H** the asymmetric aerodynamic matrix.

The flutter of the airfoil is characterized by the conditions that if $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) \neq 0$, which describe the critical state of the flutter, and if $\mathrm{Re}(\lambda) > 0$, $\mathrm{Im}(\lambda) \neq 0$, which describe the flutter occurs, and eigenvalue is also the corresponding flutter frequency.

The eigenvalues of **A** are

$$\begin{gathered} \lambda_1 = 0.67318887j, \quad \lambda_2 = 0.67318887j \\ \lambda_3 = -0.67318887j, \quad \lambda_4 = -0.67318887j \end{gathered} \tag{38}$$

where $j = \sqrt{-1}$, λ_1, λ_2 and λ_3, λ_4 are two pairs of 2-multiple non-semi-simple eigenvalues. Because $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) \neq 0$, the system is in the critical state of the flutter, and the Hopf bifurcation occurs at the critical points $(p_c, \omega_{ci})(i = 1, 2)$.

The Jordan matrix of the system is

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \tag{39}$$

The right and left modal matrices **U** and **V** can be computed as follows

$$\begin{cases} \mathbf{U} = \begin{bmatrix} -0.46540319 & -0.27308419 & -0.46540319 & -0.27308419 \\ -0.39700585 & 0.55959648 & -0.39700585 & 0.55929648 \\ 0.69134117j & 0.66056754j & -0.69134117j & -0.66056754j \\ 0.58973919j & -0.61336914j & -0.58973919j & 0.61336914j \end{bmatrix} \\ \mathbf{V} = \begin{bmatrix} -0.68374619 & -0.53836501 & -0.68374619 & -0.53836501 \\ -0.45788329 & 0.63111613 & -0.45788329 & 0.63111613 \\ 0.32665905j & 0.36242134j & -0.32665905j & -0.36242134j \\ 0.46489559j & -0.42486035j & -0.46489559j & 0.42486035j \end{bmatrix} \end{cases} \tag{40}$$

In the following, we discuss the bifurcations of eigenvalues λ_1 and λ_3. For the first Jordan block, if $\varepsilon = p - p_c = 1.32567735 - 1.32467735 = 0.001$, by using Eq. (29), we obtain the eigenvalues of the perturbed system

$$\begin{cases} \tilde{\lambda}_{11} = 0.67318887j + 0.02289364e^{-\frac{3\pi}{4}j} \\ \tilde{\lambda}_{12} = 0.67318887j + 0.02289364e^{\frac{\pi}{4}j} \end{cases} \tag{41}$$

and if $\varepsilon = p - p_c = -0.001$, we obtain

$$\begin{cases} \tilde{\lambda}_{13} = 0.67318887j + 0.02289364e^{-\frac{\pi}{4}j} \\ \tilde{\lambda}_{14} = 0.67318887j + 0.02289364e^{\frac{3\pi}{4}j} \end{cases} \tag{42}$$

For the second Jordan block, we have the eigenvalues of the perturbed system by the same way

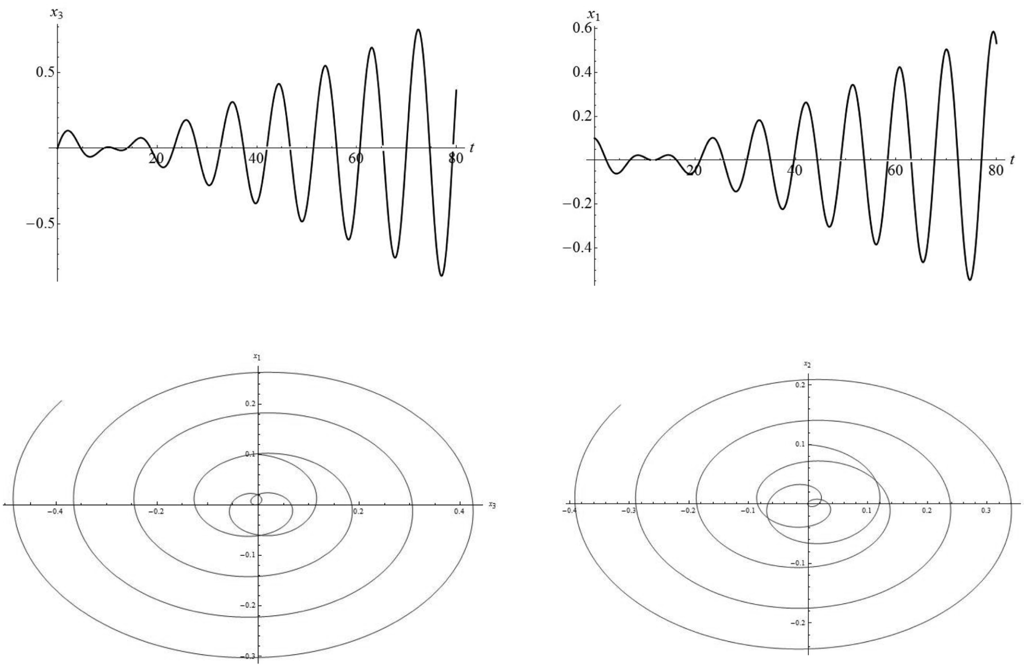
$$\begin{cases} \tilde{\lambda}_{31} = -0.67318887j + 0.02289364e^{-\frac{3\pi}{4}j} \\ \tilde{\lambda}_{32} = -0.67318887j + 0.02289364e^{\frac{\pi}{4}j} \\ \tilde{\lambda}_{33} = -0.67318887j + 0.02289364e^{-\frac{\pi}{4}j} \\ \tilde{\lambda}_{34} = -0.67318887j + 0.02289364e^{\frac{3\pi}{4}j} \end{cases} \tag{43}$$

The above results given by Eqs. (41), (42) and (43), are also shown in Fig. 2.

Now we turn to compute the responses of the zero-order approximations at the critical points (p_c, ω_{c1}) and (p_c, ω_{c3}). Using Eq. (33), we have

$$\mathbf{x}(t) = [c_1\mathbf{u}_1 + c_2(\mathbf{u}_1 t + \mathbf{u}_2)]e^{\lambda_1 t} + [c_3\mathbf{u}_3 + c_4(\mathbf{u}_3 t + \mathbf{u}_4)]e^{\lambda_3 t} \tag{44}$$

If the initial conditions are given by $x_1(0) = x_2(0) = 0.1$, $x_3(0) = x_4(0) = 0.0$, the solutions can be obtained



(c) (d)

Fig. 3 Zero-order solutions at the critical point

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} (0.1-0.00863006t)\cos 0.673189t \\ (0.1-0.00736176t)\cos 0.673189t \\ (0.14382-0.0128197t)\sin 0.673189t \\ (0.144506-0.0109357t)\sin 0.673189t \end{Bmatrix} \tag{45}$$

The solutions are plotted in Fig. 3 in which curves (a) and (b) are in amplitude-time plane, curves (c) and (d) in the state plane.

The results given by Eqs. (41) and (42) describe the bifurcation of the eigenvalue λ_1, and the results given by Eq. (43) the bifurcation of the eigenvalue λ_3, they are also shown in Fig. 2. From the above results and Fig. 2, it can be seen that if the parameter has a change, $\varepsilon = p - p_0$, the perturbed eigenvalues, $\tilde{\lambda}_{12}$ and $\tilde{\lambda}_{32}$, have positive real part, as the ε goes up, the $\tilde{\lambda}_{12}$ and $\tilde{\lambda}_{32}$ diverge, thus leading to the instability of the system. This conclusions are also illustrated by the curves in the amplitude-time plane (Fig. 3(a), (b)) and the state plane (Fig. 3(c), (d))

6. Conclusions

The expression for bifurcations of the non-semi-simple eigenvalues at the critical points for the nonlinear system is developed by using the Puiseux expansion. The geometric interpretation of the bifurcations of the eigenvalues has been given. The results indicate that if the parameter has a small change ε, the perturbed eigenvalues may have positive real part and as the ε goes up, they diverge, thus leading to instability of the nonlinear system. The explicit approximate solutions of responses of the system at the critical points with non-semi-simple eigenvalues are developed. Because the solution depends on $t, t^{m-1}, \ldots$, the solution is unbound which also indicates the system is unstable. These results provide good physical insight of the dynamic behavior. The proposed method has been applied to the flutter problem of the airfoil and the results support the conclusions.

Acknowledgements

Project supported by the Foundation of the Science and Technology of Jilin Province (20070541) and Innovation Team of Jilin University.

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