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Large deflection analysis of laminated composite plates using layerwise displacement model

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Abstract. In this paper the geometrically nonlinear continuum plate finite element model, hitherto not reported in the literature, is developed using the total Lagrange formulation. With the layerwise displacement field of Reddy, nonlinear Green-Lagrange small strain large displacements relations (in the von Karman sense) and linear elastic orthotropic material properties for each lamina, the 3D elasticity equations are reduced to 2D problem and the nonlinear equilibrium integral form is obtained. By performing the linearization on nonlinear integral form and then the discretization on linearized integral form, tangent stiffness matrix is obtained with less manipulation and in more consistent form, compared to the one obtained using laminated element approach. Symmetric tangent stiffness matrixes, together with internal force vector are then utilized in Newton Raphson's method for the numerical solution of nonlinear incremental finite element equilibrium equations. Despite of its complex layer dependent numerical nature, the present model has no shear locking problems, compared to ESL (Equivalent Single Layer) models, or aspect ratio problems, as the 3D finite element may have when analyzing thin plate behavior. The originally coded MATLAB computer program for the finite element solution is used to verify the accuracy of the numerical model, by calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different load direction (unloading/loading). The obtained results are compared with available results from the literature and the linear solutions from the author's previous papers.

Keywords: geometrically nonlinear analysis; composite plates; continuum finite element

1. Introduction

The low mass density (ρ) and the high tensile strength (σ_u), usually expressed through the specific modulus of elasticity (E/ρ) and the specific strength (σ_u/ρ) (Altenbach *et al.* 2004), have made composite materials lighter and stronger compared with most traditional materials (such as steel, concrete, wood, etc.) and have increased their application not only for secondary, but during the last two decades also for primarily structural members in aerospace and automotive industry, ship building industry and bridge design. The advanced mechanical properties of composite materials, which are resulted in large weight savings, have given designers more flexibility in finding efficient solution for specific problem, but have also required formulation of mathematical model able to

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present their complex anisotropic nature. Although weight saving has eliminated constrain of slenderness and thickness and has made possible use of very thin plate elements, they have become susceptible to large deflections (Polat 2007, Zhang and Kim 2006). In such cases, the geometry of structures is continually changing during the deformation and nonlinear analysis should be adopted. The geometrically nonlinear analysis seems also to be necessary for obtaining the structural response of unsymmetrical laminated composite materials (Zhang *et al.* 2003). Namely, the nonlinear response of these laminates is present even for small displacements, due to complex coupling between in-plane and out-of plane deformation.

A considerable amount of research work has been carried out so far on the nonlinear analysis of laminated plates. Among the published works, the von Karman plate theory of plates undergoing large deflections has attracted outstanding attention and a number of papers have been published. The first authors investigating the nonlinear response using the von Karman nonlinear theory (Tanriover and Senocak 2004, Reddy and Chao 1983) were: Leissa, Bennett, Bert, Chandra and Raju, Zaghloul and Kennedy, Chia and Prabhakara, Noor and Hartley, and in the last decades Han, Tabiei and Park, Singh, Lal and Kumar, Reddy and Chao, Zhang Kim and others.

Mechanical response of laminated composite material is generally 3D problem of nonlinear mechanics. However, due to its mathematical complexity, analytical solutions using 3D theory of elasticity are usually difficult and some times even impossible to achieve, while numerical solutions are computationally inefficient and constrained to very specific domains. Thus, whenever possible, refined simplified mathematical models, with acceptable accuracy in a field of applications, should be used. It is shown that the Equivalent Single Layer theories (ESL) may give acceptable results when analyzing global response, such as gross deflections and gross stresses, critical buckling loads and fundamental frequencies of thin to moderate thick laminated composite plates (Vuksanovic 2000, Naserian-Nik and Tahani 2010). However, a continuous displacement function in ESL is not able to accurately present the discontinuous zigzag variation of displacements in highly anisotropic plates and give adequate stress distribution at local or ply level (Ćetković and Vuksanović 2009). A compromise between 3D theory of elasticity and ESL theories is then achieved with the use of Layer Wise theories (LW). In LW theories the in-plane displacement field, assumed for each layer, is interpolated through the thickness by appropriate layerwise Lagrange interpolation function or Heaviside step function (Reddy 2004), thus replacing 3D laminated element with N+1 2D plate elements (N is number of layers), which fulfills the continuity of displacement functions at the interfaces between adjacent layers.

From the continuum mechanics it is known that two different level of geometrical nonlinearity may be modeled, which are: geometrically nonlinear models with small strain and large displacements (von Karman theory) and geometrically nonlinear models with large strains. In the first case, the geometry of the structure before deformation remains unchanged after the deformation. However, the structure is subjected to large displacements and the equilibrium is achieved on the configuration displaced from the undeformed one. In the second case the geometry of the structure is changing during the deformation and the equilibrium is achieved on the deformed configuration. In both cases equilibrium equations are nonlinear.

In order to formulate nonlinear finite element model of laminated structures, which will be able to represent two above mentioned levels of geometrical nonlinearity, two distinct approaches have been reported in the literature (Reddy 2004). The first approach is based on laminate theory, in which 3D elasticity equations are reduced to 2D equations through certain kinematical assumptions and homogenization through the thickness. In this approach only first type of nonlinearity or small strain, large displacement assumption may be included. The finite elements based on such an assumptions are named the laminated elements. The second approach is based on 3D continuum formulation (total and updated Lagrange formulation) and both types on nonlinearity may be included. Finite elements based on this approach are called the continuum elements.

The aim of the author's research on composite materials so far was to implement Layerwise theory of Reddy or Generalized Layerwise Plate Theory-GLPT (Reddy *et al.* 1989) on different levels of analysis of laminated composite plates. The previous work has been concerned with the linear analysis (Ćetković and Vuksanović 2009), and the linear laminated plate element of GLPT has been formulated, while in the present paper the GLPT nonlinear continuum plate element with von Karman geometrical nonlinearity is presented. It may be seen from the literature that Reddy's or other researches have not reported any papers on nonlinear analysis using the continuum element approach based on GLPT. Only the laminated plate element based on GLPT (Barbero and Reddy 1990) continuum shell element based on FSDT and HSDT (Arciniega and Reddy 2007, Lee *et al.* 2006), or continuum plate element based on other laminated theories (CLPT, FSDT, HSDT) (Kuppusamy *et al.* 1984, Praveen and Reddy 1998, Laulusa and Reddy 2004, Reddy and Haung 1981, Reddy and Chao 1981, Kuppusamy and Reddy 1984, Reddy 1984, Putcha and Reddy 1986) have been reported so far. Two main reasons have driven the present authors to formulate continuum element based on GLPT. The first reason was to formulate general numerical model capable to include different levels nonlinearity, like large strain geometrical nonlinearity or material nonlinearity, in the future. The second reason was to represent the more sophisticated way to derive the tangent stiffness matrix (with less computational cost), from the one using laminate element approach.

The aim of this paper is to present the mathematical and numerical model for geometrically nonlinear, small strain, large displacements problem of laminated composite plates. The 3D elasticity equations are reduced to 2D problem using kinematical assumptions based on layerwise displacement field of Reddy (GLPT). With the assumed displacement field, nonlinear Green-Lagrange small strain large displacements relations and linear orthotropic material properties for each lamina, the total Lagrangian formulation is used to derive the weak form of the problem. The weak form or nonlinear integral equilibrium equations are discretized using isoparametric finite element approximation. The obtained nonlinear incremental algebric equilibrium equations are solved using the Newton Raphson's method. The originally coded MATLAB computer program for the finite element solution is used to investigate the effects of geometrical nonlinearity on displacement and stress field of thin and thick, isotropic, orthotropic and anisotropic laminated composite plates with various boundary conditions and loading direction (loading/unloading). The accuracy of the numerical model is verified by being compared with available results from the literature and the linear solutions from the previous paper (Ćetković and Vuksanović 2009). The appropriate conclusions are derived.

2. Theoretical formulation

2.1 Displacement field

In the LW theory of Reddy (Reddy *et al.* 1989), or Generalized Layerwise Plate Theory (GLPT), in-plane displacements components (u, v) are interpolated through the thickness using 1D linear Lagrangian interpolation function $\Phi^I(z)$, while transverse displacement component ω is assumed to be constant through the plate thickness.

$$
\begin{aligned}
u_1(x,y,z) &= u(x,y) + \sum_{I=1}^{N+1} U^I(x,y)\cdot\Phi^I(z) \\
u_2(x,y,z) &= v(x,y) + \sum_{I=1}^{N+1} V^I(x,y)\cdot\Phi^I(z) \\
u_3(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

2.2 Strain-displacement relations

The Green Lagrange strain tensor associated with the displacement field (1) can be computed using von Karman strain-displacement relation to include geometric nonlinearities as follows

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \sum_{I=1}^{N+1}\frac{\partial U^I}{\partial x}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_{yy} &= \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \sum_{I=1}^{N+1}\frac{\partial V^I}{\partial y}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\gamma_{xy} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N+1}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\
\gamma_{xz} &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{I=1}^{N+1} U^I\frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x} \\
\gamma_{yz} &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \sum_{I=1}^{N+1} V^I\frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y}
\end{aligned}
\tag{2}
$$

2.3 Constitutive equations

For Hook's elastic material, the stress-strain relations for k-th orthotropic lamina have the following form

$$
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^{(k)} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)}
\tag{3}
$$

where $\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx}\ \sigma_{yy}\ \tau_{xy}\ \tau_{xz}\ \tau_{yz}\}^{(k)T}$ and $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx}\ \varepsilon_{yy}\ \gamma_{xy}\ \gamma_{xz}\ \gamma_{yz}\}^{(k)T}$ are stress and strain components respectively, and $\mathbf{Q}_{ij}^{(k)}$ are transformed elastic coefficients, of k-th lamina in global coordinates.

2.4 Incremental equilibrium equations

The geometrically nonlinear problem of laminated composite plates subjected to external loading may be obtained using incremental continuum formulation. In incremental continuum formulation geometry of the structure is continually changing and all quantities are referred to unknown deformed configuration C_2 (Reddy 2008). Equilibrium equations may be obtained from the Principle of Virtual Displacements (PVD), in which sum of external virtual work done on the body and internal virtual work stored in the body should be equal zero (Malvern 1969)

$$\int_{^2V} \delta({}_2e) \cdot {}^2\sigma \cdot d^2V = \int_{^2V} \delta u \cdot {}^2f \cdot d^2V + \int_{^2S} \delta u \cdot {}^2t \cdot d^2S \tag{4}$$

The previous equation can not be solved directly, while all quantities are given with respect to unknown configuration C_2. Therefore, all quantities should be expressed with respect to previously known configuration introducing appropriate stress and strain measures. If initial configuration C_0 is used as reference configuration, with respect to which all quantities are measured, it is called the total Lagrangian description, and if previous configuration C_1 is adopted as reference configuration, it is called the updated Lagrangian formulation. In this paper the total Lagrangian formulation is adopted and PVD, Eq. (4), has the following form

$$\int_{^0V} \delta({}_0^2E) \cdot {}_0^2S \cdot d^0V = \int_{^0V} \delta u \cdot {}_0^2f \cdot d^0V + \int_{^0S} \delta u \cdot {}_0^2t \cdot d^0S \tag{5}$$

The virtual Green Lagrange strain tensor δ_0^2E is

$$\delta_0^2E_{ij} = \delta({}_0^1E_{ij} + {}_0\varepsilon_{ij}) = \delta({}_0\varepsilon_{ij}) = \delta({}_0e_{ij}) + \delta({}_0\eta_{ij}) \tag{6}$$

where $\delta({}_0^1E) = 0$ because it is not function of unknown displacements, while linear $\delta({}_0e_{ij})$ and nonlinear $\delta({}_0\eta_{ij})$ parts of Green Lagrange strain incremental tensor $\delta({}_0\varepsilon_{ij})$ are

$$\delta({}_0e_{ij}) = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^0x_j} + \frac{\partial \delta u_j}{\partial {}^0x_i} + \frac{\partial {}_0^1u_k}{\partial {}^0x_i}\frac{\partial \delta u_k}{\partial {}^0x_j} + \frac{\partial \delta u_k}{\partial {}^0x_i}\frac{\partial {}_0^1u_k}{\partial {}^0x_j}\right) \tag{7}$$

$$\delta({}_0\eta_{ij}) = \frac{1}{2}\left(\frac{\partial \delta u_k}{\partial {}^0x_i}\frac{\partial u_k}{\partial {}^0x_j} + \frac{\partial u_k}{\partial {}^0x_i}\frac{\partial \delta u_k}{\partial {}^0x_j}\right) \tag{8}$$

The Second Piola-Kirchhoff stress tensor ${}_2^0S$ in configuration C_2 is given in terms of Second Piola-Kirchhoff stress tensor ${}_0^1S$ in configuration C_1 and Kirchhoff stress incremental tensor ${}_0S_{ij}$

$${}_2^0S_{ij} = {}_0^1S_{ij} + {}_0S_{ij} \tag{9}$$

The body and boundary stress (or traction) vectors are ${}_2^0f$ and ${}_2^0t$, δu is virtual displacement vector, while d^0V and d^0S are volume and area, respectively, which body occupies in the configuration C_0.

Substituting Eq. (6) for δ_0^2E and Eq. (9) for ${}_0^2S$ into Eq. (5), one arrives at

$$\int_{^0V} \delta({}_0\varepsilon_{ij}) \cdot {}_0S_{ij} \cdot d^0V + \int_{^0V} \delta({}_0\eta_{ij}) \cdot {}_0^1S_{ij} \cdot d^0V = \delta({}_0^2R) - \delta({}_0^1R) \tag{10}$$

where the virtual strain energy stored in the body at configuration C_1 is

$$\delta({}_0^1R) = \int_{{}^0V} \delta({}_0e_{ij}) \cdot {}_0^1S_{ij} \cdot d^0V \tag{11}$$

and virtual work done by applied forces is

$$\delta({}_0^2R) = \int_{{}^0V} \delta u \cdot {}_0^2f \cdot d^0V + \int_{{}^0S} \delta u \cdot {}_0^2t \cdot d^0S \tag{12}$$

The first term of Eq. (10) represents the change in virtual strain energy due to virtual incremental displacements u_i between configurations C_1 and C_2. The second term represents virtual work done due to initial stresses ${}_0^1S_{ij}$. The last two terms together denote the change in virtual work done by applied body forces and surface tractions in moving from C_1 to C_2. In mathematical sense, first term makes Eq. (10) nonlinear in displacement increments. To make it computationally tractable, one can assume that displacements u_i are small (which is provided with small load steps) in moving from C_1 to C_2, so that the following approximation holds

$$ {}_0S_{ij} \approx {}_0C_{ijkl} \cdot {}_0e_{kl} \quad \text{and} \quad \delta({}_0\varepsilon_{ij}) \approx \delta({}_0e_{ij}) \tag{13}$$

Then Eq. (10) can be simplified to

$$\int_{{}^0V} \delta({}_0\mathbf{e}_{ij}) \cdot {}_0\mathbf{C}_{ijkl} \cdot {}_0\mathbf{e}_{kl} \cdot d^0V + \int_{{}^0V} \delta({}_0\boldsymbol{\eta}_{ij}) \cdot {}_0^1\mathbf{S}_{ij} \cdot d^0V = \delta({}_0^2\mathbf{R}) - \delta({}_0^1\mathbf{R}) \tag{14}$$

Eq. (14) is the weak form used for the development of nonlinear finite element model based on total Lagrangian formulation. The total stress components ${}_0^1S_{ij}$ from Eq. (14) may be evaluated using the constitutive relation

$${}_0^1S = {}_0C_{ijkl} \cdot {}_0^1E_{kl} \tag{15}$$

where ${}_0^1E_{ij}$ is Green Lagrange strain tensor defined in Eq. (2) and ${}_0C_{ijkl}$ is material elasticity tensor.

3. Finite element model

3.1 Displacement field

The GLPT finite element consists of middle surface plane and $I = 1,\ N+1$ planes through the

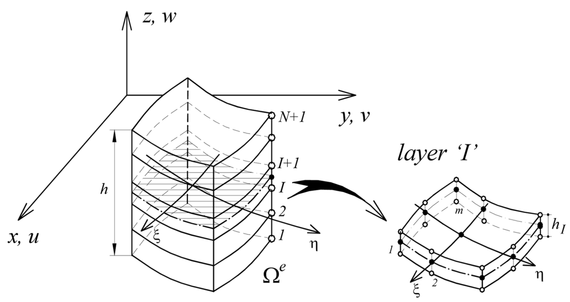

Fig. 1 Plate finite element with n layers and m nodes

plate thickness Fig. 1. The element requires only the C^0 continuity of major unknowns, thus in each node only displacement components are adopted, that are (u, v, w) in the middle surface element nodes and (U^I, V^I) in the I-th plane element nodes. The generalized displacements over element Ω^e can be expressed as

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} u_j \Psi_j \\ \sum_{j=1}^{m} v_j \Psi_j \\ \sum_{j=1}^{m} w_j \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\boldsymbol{\Psi}_j]^e \{\mathbf{d}_j\}^e, \quad \begin{Bmatrix} U^I \\ V^I \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} U_j^I \Psi_j \\ \sum_{j=1}^{m} V_j^I \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\bar{\boldsymbol{\Psi}}_j]^e \{\mathbf{d}_j^I\}^e \tag{16}$$

where $\{\mathbf{d}_j\}^e = \{u_j^e \ v_j^e \ w_j^e\}^T$, $\{\mathbf{d}_j^I\}^e = \{U_j^I \ V_j^I\}^T$ are displacement vectors, in the middle plane and I-th plane, respectively, Ψ_j^e are interpolation functions, while $[\Psi_j]^e, [\bar{\Psi}_j]^e$ are interpolation function matrix for the j-th node of the element Ω^e, given in (Ćetković and Vuksanović 2009).

3.2 Strain field

With the known element displacement field, strain field over the element may be expressed in terms of element nodal displacements using the strain displacement relations Eq. (2). Then, the linear and non linear parts of strain-displacement relations in middle and I-the plane of the plate have the following form

$$\{\boldsymbol{\varepsilon}^0\} = \sum_{j=1}^{m} [\mathbf{H}_j]\{\mathbf{d}_j\}^e, \ \{\boldsymbol{\varepsilon}^m\} = \frac{1}{2}\sum_{j=1}^{m} [\mathbf{H}_{NLj}]\{\mathbf{d}_j\}^e, \ \{\boldsymbol{\varepsilon}^I\} = \sum_{j=1}^{m} [\bar{\mathbf{H}}_j]\{\mathbf{d}_j^I\}^e \tag{17}$$

where

$$[\mathbf{H}] = \begin{bmatrix} \frac{\partial \Psi_j^e}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \Psi_j^e}{\partial y} & 0 \\ \frac{\partial \Psi_j^e}{\partial y} & \frac{\partial \Psi_j^e}{\partial x} & 0 \\ 0 & 0 & \frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & \frac{\partial \Psi_j^e}{\partial y} \end{bmatrix}, \quad [\mathbf{H}_{NL}] = \begin{bmatrix} 0 & 0 & \frac{\partial w_0}{\partial x}\frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & \frac{\partial w_0}{\partial y}\frac{\partial \Psi_j^e}{\partial y} \\ 0 & 0 & \frac{\partial w_0}{\partial x}\frac{\partial \Psi_j^e}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\bar{\mathbf{H}}] = \begin{bmatrix} \frac{\partial \Psi_j^e}{\partial x} & 0 \\ 0 & \frac{\partial \Psi_j^e}{\partial y} \\ \frac{\partial \Psi_j^e}{\partial y} & \frac{\partial \Psi_j^e}{\partial x} \\ \Psi_j^e & 0 \\ 0 & \Psi_j^e \end{bmatrix} \tag{18}$$

3.3 Equilibrium equations

Discretization of the weak form Eq. (14) with the assumed displacement field Eq. (16), and the use of fundamental Lemma of calculus of variations (i.e., variations of displacement vector increments in middle surface and I-th plane element nodes $\{\delta\bar{d}\}, \{\delta\bar{d}^I\}$ are arbitrary), give the following nonlinear incremental finite element GLPT model associated with the total Lagrangian formulation

$$([\mathbf{K}_{NL}]^e + [\mathbf{K}_G]^e)\cdot\{\bar{\mathbf{d}}\}^e = \{{}_0^2\mathbf{R}\}^e - \{{}_0^1\mathbf{R}\}^e \tag{19}$$

The stiffness matrix for the element Ω^e is

$$[\mathbf{K}_{NL}]^e = \begin{bmatrix} [\mathbf{K}^{11}]^e & [\mathbf{K}^{12}]^e \\ [\mathbf{K}^{21}]^e & [\mathbf{K}^{22}]^e \end{bmatrix} \tag{20}$$

where

$$[\mathbf{K}^{11}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e}([\mathbf{H}_i^e]^{\mathrm{T}}\cdot[\mathbf{A}]\cdot[\mathbf{H}_j^e] + [\mathbf{H}_i^e]^T\cdot[\mathbf{A}]\cdot[\mathbf{H}_{j\mathrm{NL}}^e] + [\mathbf{H}_{i\mathrm{NL}}^e]^T\cdot[\mathbf{A}]\cdot\left[\mathbf{H}_j^e\right] + [\mathbf{H}_{i\mathrm{NL}}^e]^T\cdot[\mathbf{A}]\cdot[\mathbf{H}_{j\mathrm{NL}}^e])\ d\Omega^e$$

$$[\mathbf{K}^{12}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e}([\mathbf{H}_i^e]^{\mathrm{T}}\cdot[\mathbf{B}^{\mathbf{I}}]\cdot[\bar{\mathbf{H}}_j^e] + [\mathbf{H}_{i\,\mathrm{NL}}^e]^T\cdot[\mathbf{B}^{\mathbf{I}}]\cdot[\bar{\mathbf{H}}_j^e])\ d\Omega^e$$

$$[\mathbf{K}^{21}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e}([\bar{\mathbf{H}}_i^e]^{\mathrm{T}}\cdot[\mathbf{B}^I]\cdot[\mathbf{H}_j^e] + [\bar{\mathbf{H}}_i^e]^T\cdot[\mathbf{B}^I]\cdot[\mathbf{H}_{jNL}^e])\ d\Omega^e$$

$$[\mathbf{K}^{22}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\Omega^e}[\bar{\mathbf{H}}_i^e]^T\cdot[\mathbf{D}^{IJ}]\cdot[\bar{\mathbf{H}}_j^e]\ d\Omega^e \tag{21}_{1,2,3,4}$$

The Geometric stiffness matrix for the element Ω^e is

$$[\mathbf{K}_G]^e = \begin{bmatrix} [\mathbf{K}_G^{11}]^e & [0] \\ [0] & [0] \end{bmatrix} \tag{22}$$

where

$$[\mathbf{K}_G^{11}]^e = \sum_{i=1}^{m}\int_{\Omega^e}[\mathbf{G}_i^e]^T\cdot[\mathbf{N}^0]\cdot[\mathbf{G}_i^e]\ d\Omega^e$$

$$[\mathbf{G}_i^e] = \begin{bmatrix} 0 & 0 & \partial\Psi_i^e/\partial x \\ 0 & 0 & \partial\Psi_i^e/\partial y \end{bmatrix}, \qquad [\mathbf{N}^0] = \begin{bmatrix} N_{xx}^0 & N_{xy}^0 \\ N_{xy}^0 & N_{yy}^0 \end{bmatrix} \tag{23}_{1,2,3}$$

The internal and external (body force and traction) force vectors are

$$\{{}_0^1\mathbf{R}\} = \sum_{i=1}^{m}\int_{{}^0\Omega^e}[\tilde{\mathbf{H}}_i^e]^T\cdot\{\tilde{\mathbf{N}}\}\ d\,{}^0\Omega^e \tag{24}$$

$$\{{}_0^2\mathbf{R}\} = \sum_{i=1}^{m}\left(\int_{\Omega^e} [\Psi_i^e]^T \cdot \{{}_0^1\mathbf{f}\}\, d\,{}^0\Omega^e + \int_{\Gamma^e} [\Psi_i^e]^T \cdot \{{}_0^1\mathbf{t}\}\, d\,{}^0\Gamma^e \right) \tag{25}$$

The Kirchhoff stress resultant vector is

$$\{\tilde{\mathbf{N}}\} = \begin{Bmatrix} \{\mathbf{N}^{11}\} \\ \{\tilde{\mathbf{N}}^{12}\} \end{Bmatrix} \tag{26}$$

and sub vectors

$$\{\mathbf{N}^{11}\} = \{ N_{xx} \quad N_{yy} \quad N_{xy} \quad N_{xz} \quad N_{yz} \}^T = [\mathbf{A}] \cdot (\{\varepsilon^0\} + \{\varepsilon^m\}) + \sum_{I=1}^{N+1} [\mathbf{B}^I] \cdot \{\varepsilon^I\} \qquad (27)_1$$

$$\{\mathbf{N}^{12}\} = \{ N_{xx}^I \quad N_{yy}^I \quad N_{xy}^I \quad N_{xz}^I \quad N_{yz}^I \}^T = [\mathbf{B}^I] \cdot (\{\varepsilon^0\} + \{\varepsilon^m\}) + \sum_{J=1}^{N+1} [\mathbf{D}^{JI}] \cdot \{\varepsilon^J\} \qquad (27)_2$$

The total strain components $\{\varepsilon^0\}, \{\varepsilon^m\}$ and $\{\varepsilon^I\}$ from Eqs. (27), (28) are given in Eq. (18), constitutive matrix $[\mathbf{A}], [\mathbf{B}^I], [\mathbf{D}^{JI}]$ are given as

$$[\mathbf{A}] = [A_{ij}] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [Q_{ij}^{(k)}] dz, \quad i,j = 1,2,3,4,5 \qquad (28)_1$$

$$[\mathbf{B}^I] = [B_{ij}^I] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [Q_{ij}^{(k)}]\, \Phi^I dz, \quad i,j = 1,2,3 \qquad (28)_2$$

$$[\mathbf{B}^I] = [\bar{B}_{ij}^I] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [Q_{ij}^{(k)}] \frac{d\Phi^I}{dz} dz, \quad i,j = 4,5 \qquad (28)_3$$

$$[\mathbf{D}^{JI}] = [D_{ij}^{JI}] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [Q_{ij}^{(k)}]\, \Phi^J \Phi^I dz, \quad i,j = 1,2,3 \qquad (28)_4$$

$$[\mathbf{D}^{JI}] = [\bar{D}_{ij}^{JI}] = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} [Q_{ij}^{(k)}] \frac{d\Phi^J}{dz} \frac{d\Phi^I}{dz} dz, \quad i,j = 4,5 \qquad (28)_5$$

while

$$[\tilde{\mathbf{H}}] = \begin{bmatrix} [\mathbf{H}] + [\mathbf{H}_{NL}] & 0 \\ 0 & [\bar{\mathbf{H}}] \end{bmatrix} \tag{29}$$

The stiffness matrix Eq. (21) and the geometric stiffness matrix Eq. (23) together denote the tangential stiffness

matrix, $\{\bar{\mathbf{d}}\}^e$ is the incremental displacement vector, while the difference between two vectors Eqs. (24), (25) on the right side of Eq. (19) together denotes the incremental change in load vector during the deformation process. The obtained finite element incremental equations Eq. (19) may then be assembled into system incremental equations and after imposing appropriate boundary conditions may be solved using the Newton Rapson°Øs method for unknown incremental displacements (Bathe 1996, Hughes 1987).

3.4 Stress field

With the known displacement field, the stress field over the element may be obtained as a part of a postprocessor, using strain displacement and constitutive relations, Eqs. (2), (3) as

$$\{\boldsymbol{\sigma}_b\}_U^{(k)e} = [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} ([\mathbf{H}_{bj}] + [\mathbf{H}_{bj}^{NL}])\{\mathbf{d}_j\}^e + [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} [\bar{\mathbf{H}}_{b_j}]\{\mathbf{d}_j^I\}^e$$

$$\{\boldsymbol{\sigma}_b\}_O^{(k)e} = [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} ([\mathbf{H}_{bj}] + [\mathbf{H}_{bj}^{NL}])\{\mathbf{d}_j\}^e + [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} [\bar{\mathbf{H}}_{b_j}]\{\mathbf{d}_j^{I+1}\}^e$$

$$\{\boldsymbol{\sigma}_s\}_{const}^{(k)e} = [\mathbf{Q}_s]^{(k)} \sum_{j=1}^{m} [\mathbf{H}_{sj}]\{\mathbf{d}_j\}^e + [\mathbf{Q}_s]^{(k)} \sum_{j=1}^{m} [\bar{\mathbf{H}}_{sj}](\{\mathbf{d}_j^{I+1}\}^e - \{\mathbf{d}_j^I\}^e)/h_k \qquad (30)_{1,2,3}$$

where $\{\boldsymbol{\sigma}_b\}_U^{(k)e}$ and $\{\boldsymbol{\sigma}_b\}_O^{(k)e}$ are in-plane normal stresses (σ_{xx}, σ_{yy}, τ_{xy}) at bottom and upper plane in k-th layer of plate element 'e', while $\{\boldsymbol{\sigma}_s\}_{const}^{(k)e}$ are average transverse shear stresses (τ_{xz}, τ_{yz}) in k-the layer of plate element 'e'.

4. Numerical results and discussion

Based on the previously derived continuum finite element model for the geometrically nonlinear analysis of laminated composite plates, the original computer program is coded using MATLAB programming language. The finite element stiffness matrix and the geometric stiffness matrix are evaluated using Gauss-Legendre quadrature rule, which are 3 × 3 Gauss integration schemes or 2D quadratic Lagrange rectangular element for in-plane interpolation and 1D linear Lagrange element for through the thickness interpolation (Hinton *et al.* 1988). The Newton Rapson's numerical method is used to solve nonlinear incremental equilibrium equations. The effects of plate thickness, lamination scheme, boundary conditions and load direction on nonlinear response of isotropic, orthotropic and anisotropic plates are analyzed. The accuracy of the present formulation is demonstrated through a number of examples and by comparison with results available from the literature.

The following boundary conditions at the plate edges are analyzed (Thankam *et al.* 2003).

Simply supported (SS)

$$\textbf{SS:} \quad \begin{cases} x = 0,\ a: & v_0 = w_0 = V^I = N_{xx} = N_{xx}^I = 0 \\ y = 0,\ b: & u_0 = w_0 = U^I = N_{yy} = N_{yy}^I = 0 \end{cases} \qquad I = 1,\ldots,N+1 \qquad (31)$$

Simply supported-hinged (HH)

$$\textbf{HH:} \quad \begin{cases} x = 0,\ a: & u_0 = v_0 = w_0 = V^I = N_{xx}^I = 0 \\ y = 0,\ b: & u_0 = v_0 = w_0 = U^I = N_{yy}^I = 0 \end{cases} \qquad I = 1,\ldots,N+1 \qquad (32)$$

Clamped (CC)

$$\textbf{CC:}\quad \begin{cases} x = 0,\ a: & u_0 = v_0 = w_0 = U^I = V^I = 0 \\ y = 0,\ b: & u_0 = v_0 = w_0 = U^I = V^I = 0 \end{cases} \quad I = 1, \ldots, N+1 \tag{33}$$

When analyzing a quarter of a plate, boundary conditions in the plane of symmetry become:

For cross ply laminates

$$\textbf{SS1:}\quad \begin{cases} x = a/2: & u_0 = U^I = N_{yy} = N^I_{yy} = 0 \\ y = b/2: & v_0 = V^I = N_{xx} = N^I_{xx} = 0 \end{cases} \quad I = 1, \ldots, N+1 \tag{34}$$

For angle ply laminates

$$\textbf{SS2:}\quad \begin{cases} x = a/2: & v_0 = U^I = N_{xx} = N^I_{yy} = 0 \\ y = b/2: & u_0 = V^I = N_{yy} = N^I_{xx} = 0 \end{cases} \quad I = 1, \ldots, N+1 \tag{35}$$

Example 4.1. A nonlinear bending of square, simply supported (SS1), isotropic plate, with $a = b = 10$ in and $h = 1$ in made of material

$$E = 7.8 \cdot 10^6 \text{ psi}, \quad \nu = 0.3 \tag{36}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\bar{q} = q(x, y) \cdot (a/h)^4 \cdot 1/E_2$, the incremental load vector is chosen to be

$$\{\Delta q\} = \{6.25, 6.25, 12.5, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0\} \cdot \bar{q} \tag{37}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0,8$. The displacements and stresses are given in following nondimensional form

$$\bar{w} = w_0 \cdot E_2 h^3/(q \cdot a^4), \quad \bar{\sigma}_{xx} = \sigma_{xx} \cdot (a/h)^2 \cdot 1/E \tag{38}$$

A 3×3 quarter plate GLPT continuum model is compared with 4×4 quadratic FSDT model (Reddy 2004), The results for linear and nonlinear deflections and nonlinear stresses are presented in Table 1 and Fig. 2. It is shown that proposed GLPT model closely agree with FSDT model. The Fig. 2 also demonstrates the physical nature of geometrically nonlinear response. The study has proved that depending of applied load level, the plate goes from the state of pure bending, at small displacement ($w \le 0.30$ h) to the phase of bending-stretching coupling, at large displacements. Namely, when the lateral displacement reaches approximately one half of plate thickness ($w \approx 0.5$ h), they take part in stretching, together with bending of the plate middle surface (nonlinear terms in Eqs. (2)). This activates the tensile forces, thus enlarging the stiffness of the plates, and reducing displacements and stresses from the values predicted by linear theory. This may be the reason why this phenomena is also known as “plate stiffening” or “stress relaxation”. Moreover, the activation of tensile forces in laminated composite plates is of utmost importance, due to their high available specific tensile strength.

Table 1 Central displacement and stresses $\bar{\sigma}_{xx}$ versus load parameter of square simply-supported (SS1) isotropic plate with a/h = 10

$\bar{q}$	$\bar{w}$					$\bar{\sigma}_{xx}$		
	Linear solution		Nonlinear solution			Nonlinear solution		
	Reddy (2004)	Present	Reddy (2004)	Reddy (2004)	Present	Reddy (2004)	Reddy (2004)	Present
6.25	0.2917	0.2889	0.2813	0.2813	0.2788	1.779	1.780	1.8460
12.50	0.5834	0.5778	0.5186	0.5186	0.5155	3.396	3.398	3.5486
25.0	1.1668	1.1556	0.8673	0.8673	0.8629	5.882	5.885	6.0960
50.0	2.3336	2.3112	1.3149	1.3149	1.2884	9.162	9.165	9.1316
75.0	3.5004	3.4668	1.6239	1.6241	1.6055	11.462	11.465	11.9282
100.0	4.6672	4.6224	1.8683	1.8687	1.8473	13.307	13.308	13.3641
125.0	5.8340	5.7780	2.0751	2.0758	2.0555	14.890	14.889	14.8062
150.0	7.0008	6.9336	2.2556	2.2567	2.2335	16.293	16.290	16.2348
175.0	8.1676	8.0892	2.4177	2.4194	2.4337	17.572	17.567	17.4796
200.0	9.3344	9.2448	2.5657	2.5681	2.5720	18.755	18.748	18.6326

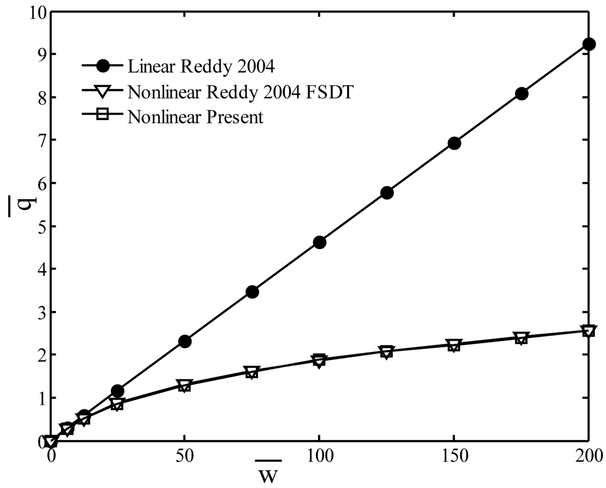

Fig. 2 Nonlinear bending of square simply supported (SS1) isotropic plate with a/h = 10; central displacement versus load parameter

Example 4.2. A nonlinear bending of square simply supported (SS1), orthotropic plate made of high modulus glass-epoxy fiber reinforced material

$$E_1/E_2 = 25,\ G_{12}/E_2 = 0.5,\ G_{13}/E_2 = 0.5,\ G_{23}/E_2 = 0.2,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{39}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\bar{q} = q(x,y) \cdot (a/h)^4 \cdot 1/E_2$, the incremental load vector is chosen to be

$$\{\Delta q\} = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140\} \cdot \bar{q} \tag{40}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0,3$. The displacements and stresses are given in following nondimensional form

$$\bar{w} = w_0 \cdot E_2 h^3/(q \cdot a^4), \quad (\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\tau}_{xy}) = (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \cdot \left(\frac{h}{a}\right)^2 \cdot \frac{1}{E_2}, \quad \bar{\tau}_{xx} = \tau_{xz} \cdot \frac{h}{a} \cdot \frac{1}{E_2} \tag{41}$$

Table 2 Central displacement and stresses $\bar{\sigma}_{xx}$ versus load parameter of square simply-supported (SS1) orthotropic plate with *a/h* = 10

$\bar{q}$	Nonlinear solution							
	$\bar{\sigma}_{xx}$		$\bar{\sigma}_{yy}$		$\bar{\sigma}_{xy}$		$\bar{\tau}_{xz}$	
	Reddy (2004)	Present	Reddy (2004)	Present	Reddy (2004)	Present	Reddy (2004)	Present
10	7.453	7.7843	0.3771	0.3581	0.4800	0.4755	0.0540	0.05152
20	14.852	15.4336	0.7827	0.7466	0.9845	1.0902	0.1077	0.1028
30	22.146	22.8972	1.2117	1.1605	1.5113	1.4801	0.1608	0.1534
40	29.291	30.1395	1.6590	1.5948	2.0583	2.0859	0.2130	0.2033
50	36.253	37.1112	2.1198	2.0427	2.6229	2.6925	0.2641	0.2523
60	43.010	43.8498	2.5900	2.5031	3.2026	3.0450	0.3139	0.3001
70	49.546	50.3274	3.0660	2.9715	3.7952	3.9478	0.3624	0.3467
80	55.856	56.5466	3.5450	3.4449	4.3985	4.4896	0.4096	0.3920
90	61.940	62.5144	4.0248	3.9213	5.0106	5.0883	0.4554	0.4361
100	67.802	68.2402	4.5037	4.3989	5.6300	5.7141	0.4998	0.4789
110	73.450	73.7364	4.9804	4.8764	6.2551	6.3526	0.5430	0.5204
120	78.893	79.0148	5.4540	5.3529	6.8849	6.9965	0.5849	0.5608
130	84.141	84.0879	5.9239	5.8278	7.5184	7.6427	0.6256	0.6000
140	89.205	88.9686	6.3894	6.3003	8.1548	8.2913	0.6653	0.63803

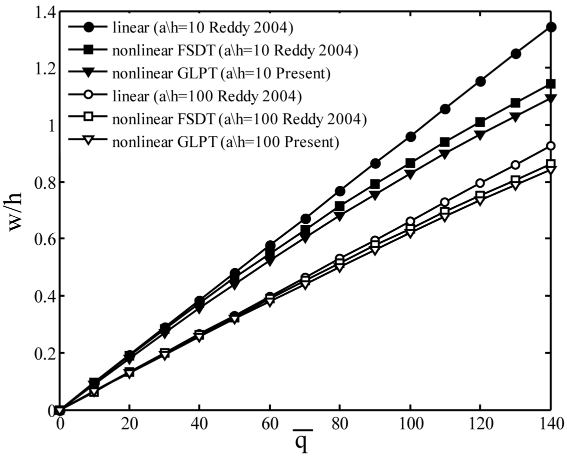

Fig. 3 Nonlinear bending of square simply supported (SS1) orthotropic plate; central displacement versus load parameter

A 2×2 quarter plate GLPT continuum model is compared with 8×8 CPT nonconforming and 4×4 quadratic FSDT models (Reddy 2004). The results for thick and thin plates ($a/h = 10$ and $a/h = 100$) of linear and nonlinear deflections and nonlinear stresses are presented in Table 2 and Fig. 3. It is shown that proposed GLPT model closely agree with CLPT and FSDT models (Reddy 2004). The more significant difference between linear and nonlinear solutions is observed for thick plates, while in thick plates larger lateral deflections have greater influence on nonlinear response, as it can be seen from the underlined nonlinear terms in Eq. (2).

Example 4.3. A nonlinear bending of square cross ply 0/90 and angle ply 45/–45 plates, with $a = b = 1$ and $h = 0.1$, with three different boundary conditions (SS, HH and CC, Eqs. (31), (32), (33)), made of material

$$E_1/E_2 = 40,\ G_{12}/E_2 = 0.6,\ G_{13}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{42}$$

subjected to uniform transverse pressure $\bar{q} = q(x,y) \cdot (a/h)^4 \cdot 1/E_2$ are analyzed. The incremental load vector is

$$\{\Delta\bar{q}\} = \{-100, -20, -20, -20, -20, 40, 20, 20, 20, 20\} \cdot \bar{q} \tag{43}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0,5$. The displacements and stresses are given in following nondimensional form

$$\bar{w}_{LIN} = w \times \frac{h^3 E_2}{a^4 q} \cdot 100, \quad (\bar{\sigma}_{xx}, \bar{\sigma}_{yy}) = (\sigma_{xx}, \sigma_{yy}) \times \left(\frac{a}{h}\right)^2 \cdot \frac{1}{E_2} \tag{44}$$

A 2×2 quarter plate and 4×4 full plate continuum GLPT models are analyzed and compared with full 8×8 plate FSDT models (Thankam *et al.* 2003). The results for linear and nonlinear deflections are presented in Tables 3, 4, 5, 6, 7, 8. It is shown that proposed GLPT model closely agree with FSDT model form literature, with the faster convergence. Also, the discrepancy between

Table 3 Stresses versus load parameter of square simply-supported (SS1) orthotropic plate a/h = 10

$\bar{q}$	$\bar{w}_{NL} \cdot 10$			$\bar{\sigma}_{XX}$			$\bar{\sigma}_{YY}$		
	Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present	
		Full plate	Quarter plate		Full plate	Quarter plate		Full plate	Quarter plate
-100	-9.723	-10.4584	-10.3068	-85.81	-87.7687	-91.4865	6.389	-6.2803	-6.1190
-80	-8.421	-8.8924	-8.8388	-74.23	-80.2677	-83.2767	5.546	-5.4995	-5.3706
-60	-6.888	-7.1942	-7.1501	-60.66	-64.9796	-66.6366	4.551	-4.5429	-4.4573
-40	-5.026	-5.1870	-5.1778	-44.24	-46.1771	-47.1425	3.334	-3.3316	-3.3083
-20	-2.722	-2.7559	-2.7537	-23.93	-24.2330	-24.4841	1.814	-1.8280	-1.8198
20	2.722	2.7559	2.7537	23.76	22.8399	22.5550	1.827	1.8946	1.8201
40	5.026	5.1870	5.1778	43.66	41.5389	40.5038	3.379	3.5618	3.3021
60	6.888	7.1942	7.1501	59.57	56.2325	54.1600	4.635	4.9345	4.4838
80	8.421	8.8924	8.8388	72.59	68.1087	65.2326	5.674	6.0725	5.4087
100	9.723	10.4584	10.3068	83.61	77.9324	74.6326	6.559	6.5274	6.1561
$\bar{w}_{LIN}$	1.410	1.4130	1.4130	1.2370	1.2115	1.2114	0.0940	0.0960	0.0960

Table 4 Central displacement and stresses versus load parameter of square hinged (HH) cross ply 0/90 plate with a/h = 10

$\bar{q}$	$\bar{w}_{NL} \cdot 10$			$\bar{\sigma}_{XX}$			$\bar{\sigma}_{YY}$		
	Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present	
		Full plate	Quarter plate		Full plate	Quarter plate		Full plate	Quarter plate
-100	-4.7650	-4.5939	-4.9490	-39.0700	-37.8815	-37.1852	-2.5720	-2.7585	-2.7143
-80	-4.2080	-4.0508	-4.2688	-37.0900	-35.6542	-35.2788	-2.3440	-2.4965	-2.4756
-60	-3.5360	-3.4747	-3.7107	-33.7000	-32.8219	-32.0224	-2.0350	-2.1942	-2.1512
-40	-2.6720	-2.5864	-2.7981	-27.7900	-26.8872	-26.3827	-1.5940	-1.7097	-1.6858
-20	-1.5090	-1.4334	-1.5765	-17.2700	-16.6923	-16.5961	-0.9330	-1.0001	-0.9954
20	1.5090	1.4334	1.5765	20.4900	20.1643	20.2555	0.9570	1.0238	1.0249
40	2.6720	2.5864	2.7981	38.0800	37.6015	38.0757	1.6730	1.7686	1.7877
60	3.5360	3.4747	3.7107	52.1200	51.5368	52.4040	2.1860	2.2869	2.3170
80	4.2080	4.0508	4.2688	63.6200	63.6309	64.2721	2.5760	2.7131	2.7150
100	4.7650	4.5939	4.9490	73.3600	71.8442	74.4556	2.8940	2.9282	3.0357
$\bar{w}_{LIN}$	0.8020	0.7544	0.8385	1.0100	0.9847	0.9848	0.0510	0.0545	0.0545

Table 5 Central displacement and stresses versus load parameter of square clamped (CC) cross ply 0/90 plate with a/h = 10

$\bar{q}$	$\bar{w}_{NL} \cdot 10$			$\bar{\sigma}_{XX}$			$\bar{\sigma}_{YY}$		
	Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present		Thankam *et al.* (2003)	Present	
		Full plate	Quarter plate		Full plate	Quarter plate		Full plate	Quarter plate
-100	-3.5120	-3.3819	-3.4784	-21.4000	-20.5632	-19.2411	-2.1120	-2.1710	-2.2723
-80	-3.0180	-2.8846	-2.9700	-20.2300	-19.1365	-18.3254	-1.8910	-1.9245	-2.0278
-60	-2.4380	-2.3055	-2.3810	-18.0200	-16.8767	-16.4203	-1.5970	-1.6135	-1.7035
-40	-1.7460	-1.6342	-1.6902	-14.2500	-13.1940	-12.9932	-1.1970	-1.1990	-1.2737
-20	-0.9240	-0.8567	-0.8889	-8.3160	-8.4576	-8.4757	-0.6620	-0.6576	-0.7026
20	0.9240	0.8567	0.8889	9.7460	9.2713	9.3123	0.7064	0.7002	0.7525
40	1.746	1.6342	1.6902	19.4220	18.8206	18.9384	1.3556	1.3562	1.4554
60	2.4380	2.3055	2.3810	28.2170	27.7584	28.0222	1.9089	1.9293	2.0654
80	3.0180	2.8846	2.9700	36.0310	35.8938	36.2816	2.3748	2.4206	2.5803
100	3.5120	3.3819	3.4784	43.0000	43.2176	43.8145	2.7725	2.8376	3.02543
$\bar{w}_{LIN}$	0.4730	0.4535	0.4535	0.4650	0.4353	0.4352	0.0350	0.0373	0.0373

linear and nonlinear solutions are larger for flexible plates, which are the plates with simply supported boundary conditions (SS), compared to hinged (HH) and clamped (CC) boundary conditions. The study has verified that the change in load direction gives unsymmetrical stress field and symmetrical displacement field, due to non-coincidence of the neutral plane and the mid-plane in laminated composite plates.

Table 6 Central displacement and stresses versus load parameter of square simply-supported (SS) angle ply 45/−45 plate with $a/h = 10$

$\bar{q}$	$\bar{w}_{NL}$		$\bar{\sigma}_{XX}$		$\bar{\sigma}_{YY}$	
	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present
-100	-4.811	-4.8803	-14.2100	-16.2177	-14.21	-15.1525
-80	-4.266	-4.4157	-13.3280	-14.4082	-13.28	-14.4082
-60	-3.607	-3.5423	-11.9700	-13.1740	-11.97	-13.1740
-40	-2.765	-2.5328	-9.8840	-10.1196	-9.884	-10.1196
-20	-1.605	-1.3778	-6.3090	-5.7065	-6.309	-5.7065
20	1.605	1.3778	7.7410	6.7321	7.741	6.7321
40	2.765	2.5328	14.1160	13.1960	14.116	13.1960
60	3.607	3.5423	19.1260	19.0340	19.126	19.0340
80	4.266	4.4157	23.2730	24.1754	23.273	24.1754
100	4.811	4.8803	26.8130	28.7658	26.813	28.7658
$\bar{w}_{LIN}$	0.8740	0.8385	0.3840	0.4136	0.3840	0.4136

Table 7 Central displacement and stresses versus load parameter of square hinged (HH) angle ply 45/−45 plate with $a/h = 10$

$\bar{q}$	$\bar{w}_{NL}$		$\bar{\sigma}_{XX}$		$\bar{\sigma}_{YY}$	
	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present
-100	-4.279	-4.4926	-16.9800	-16.6682	-16.9800	-16.6037
-80	-3.713	-3.9842	-15.3600	-15.0492	-15.3600	-14.9743
-60	-3.039	-3.2500	-13.1700	-12.8871	-13.1700	-12.8074
-40	-2.216	-2.3568	-10.1100	-9.8387	-10.1100	-9.7654
-20	-1.195	-1.2646	-5.7630	-5.5954	-5.7630	-5.5457
20	1.195	1.2646	6.3330	6.1515	6.3330	6.0767
40	2.216	2.3568	12.0960	11.8130	12.0960	11.6517
60	3.039	3.2500	16.9700	16.6419	16.9700	16.3936
80	3.713	3.9842	21.1100	20.7937	21.1100	20.4630
100	4.279	4.4926	24.7100	24.0000	24.7100	24.0042
$\bar{w}_{LIN}$	0.618	0.6514	0.3140	0.3041	0.3140	0.3009

Example 4.4. A nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0/45/-45/90)_s$, laminated plate with $a = b = 1$ and $h = 0.1$, made of material

$$E_1/E_2 = 40,\ G_{12}/E_2 = 0.6,\ G_{13}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \quad (45)$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\bar{q} = q(x,y) \cdot (a/h)^4 \cdot 1/E_2$, the incremental load vector is chosen to be

Table 8 Central displacement and stresses versus load parameter of square clamped (CC) angle ply 45/−45 plate with *a/h* = 10

$\bar{q}$	$\bar{w}_{NL}$		$\bar{\sigma}_{XX}$		$\bar{\sigma}_{YY}$	
	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present	Thankam *et al.* (2003)	Present
-100	-3.6950	-3.7427	-13.5812	-14.6400	-13.4659	-14.6400
-80	-3.1470	-3.1993	-12.2698	-13.1800	-12.1617	-13.1800
-60	-2.5140	-2.5680	-10.4569	-11.1500	-10.3614	-11.1500
-40	-1.7770	-1.8251	-7.9629	-8.3790	-7.8876	-8.3790
-20	-0.9290	-0.9609	-4.4927	-4.6560	-4.4483	-4.6560
20	0.9290	0.9609	5.0711	5.1570	5.0145	5.1570
40	1.7770	1.8251	10.0477	10.2150	9.9274	10.2150
60	2.5140	2.5680	14.5717	14.8220	14.3851	14.8220
80	3.1470	3.1993	18.5663	18.9130	18.3136	18.9130
100	3.6950	3.7427	22.1323	22.5590	21.8148	22.5590
$\bar{w}_{LIN}$	0.4730	0.4906	0.2510	0.2423	0.2510	0.2423

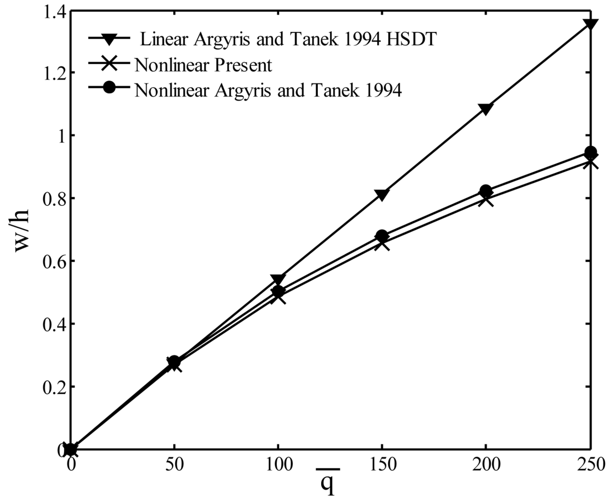

Fig. 4 Nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0/45/-45/90)_s$ laminated plate with *a/h* = 10; central displacement versus load parameter

$$\{\Delta q\} = \{50, 50, 50, 50, 50\} \cdot \bar{q} \tag{46}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0, 8$.

A 2 × 2 quarter plate continuum GLPT model is compared with 8 × 8 full plate HSDT model (Argyris and Tanek 1994). The results for linear and nonlinear deflections are presented in Fig. 4 and Table 9. It is shown that proposed GLPT model closely agree with HSDT model form literature, with the faster convergence.

Table 9 Central displacement versus load of square simply supported (SS1) general quasi-isotropic $(0/45/-45/90)_s$ laminated plate with a/h = 10

$\bar{q}$	Linear	Nonlinear	
	Argyris and Tanek (1994)	Argyris and Tanek (1994)	Present
50	0.2717	0.2691	0.2980
100	0.5435	0.4862	0.5582
150	0.8152	0.6573	0.7276
200	1.0870	0.7975	0.8631
250	1.3587	0.9179	0.9780

Table 10 Central deflection versus load of simply supported 8-layer unidirectional $(0^0)_8$ square laminate subjected to a uniformly distributed load

$\bar{q}$	Linear solution				Nonlinear solution			
	Argyris and Tanek (1994)	Zhang and Kim (2005)	Zhang and Kim (2005)	Present	Argyris and Tanek (1994)	Zhang and Kim (2005)	Zhang and Kim (2005)	Present
0.40	0.0912	0.0906	0.0900	0.0882	0.0840	0.0779	0.0774	0.0800
0.80	0.1824	0.1812	0.1799	0.1764	0.1408	0.1313	0.1306	0.1359
1.20	0.2720	0.2718	0.2698	0.2647	0.1768	0.1708	0.1699	0.1764
1.60	0.3608	0.3624	0.3598	0.3529	0.2152	0.2023	0.2013	0.2086
2.00	0.4536	0.4529	0.4498	0.4411	0.2364	0.2287	0.2275	0.2373

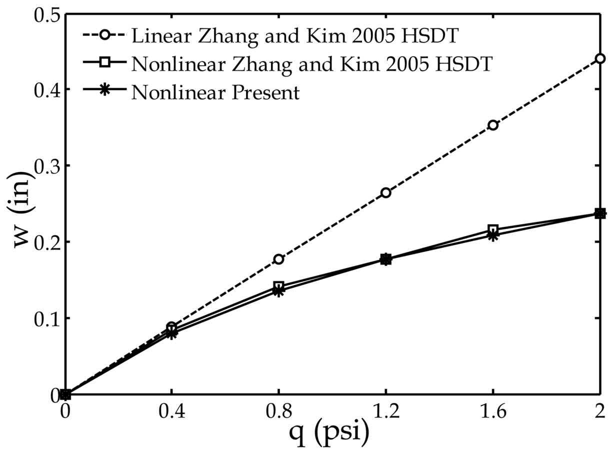

Fig. 5 Nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0)_8$ laminated plate; central displacement versus load

Example 4.5. A nonlinear bending of square simply supported (SS1) general unidirectional $(0/0/0/0)_s$, laminated plate with $a = b = 12$ in and $h = 0.138$ in, made of material

$$E_1 = 3\times10^6 \text{ psi}, E_2 = 1.28\times10^6 \text{ psi}, G_{12} = G_{13} = G_{23} = 0.37\times10^6 \text{ psi}, \ \nu_{12} = \nu_{13} = \nu_{23} = 0.32 \quad (47)$$

subjected to uniform transverse pressure applied in five load increments

$$\{\Delta q\} = \{0.40, 0.40, 0.40, 0.40, 0.40\} \quad \text{(psi)} \tag{48}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0, 5$ is analyzed.

A 2×2 quarter plate continuum GLPT model is compared with HSDT models (Zhang and Kim 2005). The results for linear and nonlinear deflections are presented Table 10 and on Fig. 5. It is shown that proposed GLPT model closely agree with HSDT model form literature.

5. Conclusions

In this paper a continuum layerwise finite element model for geometrically nonlinear small strain, large deflection analysis of laminated composite plates is derived using the total Lagrangian formulation. The total Lagrangian formulation is utilized to obtain the symmetric tangent stiffness matrix, which is computationally desirable and directly applicable in Newton Raphson's method for the numerical solution of system of nonlinear incremental finite element equilibrium equations. The accuracy of the model is verified calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different load direction (unloading/loading). In despite of its mathematical complexity, proposed model has shown better convergence characteristics than ESL models of CLPT, FSDT and HSDT, still with less computational cost than 3D elasticity model. Moreover, present model has no shear locking problems, compared to ESL models, or aspect ratio problems, as the 3D finite element may have when analyzing thin plate behavior. The analysis has also shown that the discrepancy of nonlinear from linear response is greater for flexible plates, such as thick compared to thin plates, or plates with SS compared to hinged (HH) and clamped (CC) boundary conditions. It is verified that the change of load direction (unloading/loading) has no influence on displacement field, while stress field is load direction dependent. Finally, the total Lagrangian formulation of GLPT model, derived from 3D continuum mechanics formulation, by introduction of appropriate kinematical assumptions, gives the general procedure to include different kinds of nonlinearity in the future.

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