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Rayleigh-Ritz optimal design of orthotropic plates for buckling

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Abstract. This paper is concerned with the structural optimization problem of maximizing the compressive buckling load of orthotropic rectangular plates for a given volume of material. The optimality condition is first derived via variational calculus. It states that the thickness distribution is proportional to the strain energy density contrary to popular claims of constant strain energy density at the optimum. An engineers physical meaning of the optimality condition would be to make the average strain energy density with respect to the depth a constant. A double cosine thickness varying plate and a double sine thickness varying plate are then fine tuned in a one parameter optimization using the Rayleigh-Ritz method of analysis. Results for simply supported square plates indicate an increase of 89% in capacity for an orthotropic plate having 100% of its fibers in 0° direction.

Key words: thin orthotropic plates; optimal shapes; prebuckling optimization; variable thickness plates.

1. Introduction

Distributed parameter structural optimization problems involving shape as a design variable and maximal buckling loads as the objective function for such elements as thin rectangular plates having a behavioral partial differential equation of the fourth order and containing mixed terms will often exhibit uncomfortable complexities yet produce high strength to weight ratios of the structural elements under consideration and thus attract industries with weight sensitive products. A comprehensive review on such problems was done by Haug (1981) and Banichuk (1983). Of particular interest in this paper is the thickness variation of thin orthotropic plates under uniform compressive buckling loads.

Keller (1960) and Tadjbakhsh and Keller (1962) considered optimal shapes of beams under compressive loads for various constraints. Spillers and Levy (1990) extended Keller's technique to axially compressed plates. Axisymmetric cylindrical shells (Levy and Spillers 1989) and orthotropic rectangular plates (Levy 1990) for axial compression were later considered. Their work derived optimal plates of double curvature (a constant and the first term of a cosine Fourier series in each direction) having a parametric characterization for buckling represented by a symmetric double sine displacement mode with a 112% increase in the buckling load. A symmetric Rayleigh-Ritz analysis with a multiple of half sine waves in the direction of loading and only one half sine wave in the other direction predicted a 41% increase in the buckling load for the same plates. A general Rayleigh-Ritz analysis with half sine waves in both plate directions produced a "degenerate" locally buckled mode with no strength advantages.

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This paper reshapes the Spillers and Levy plate to avoid these local instabilities. It shows that the behavior is indeed symmetric with only slight variations in shape and critical buckling loads. The fine tuning is performed in a one parameter search using the Rayleigh-Ritz method repeatedly.

The same search is also imposed on a hybrid double sine plate having the first term (a constant) of a Fourier cosine series and the first term of a Fourier sine series.

2. Problem formulation

The partial differential equations (Timoshenko and Krieger 1959) which describe the buckling behavior of rectangular orthotropic plates which are acted upon by compressive forces parallel to the x-axis consist of the equilibrium equations

$$M_{x,xx} - 2M_{xy,xy} + M_{y,yy} = Pw_{,xx} \tag{1}$$

and the constitutive equation

$$\{\mathbf{M}\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ 2w_{,xy} \end{Bmatrix} = [\mathbf{D}]\,\{\mathbf{w}''\} \tag{2}$$

Eqs. (1) and (2) yield a governing partial differential equation in terms of lateral displacements, w, of the form

$$(D_x w_{,xx})_{,xx} + (D_1 w_{,yy})_{,xx} + 4(D_{xy} w_{,xy})_{,xy} + (D_1 w_{,xx})_{,yy} + (D_y w_{,yy})_{,yy} + Pw_{,xx} = 0 \tag{3}$$

The comma is used to indicate differentiation; M_x and M_y are the bending moments per unit length; M_{xy} is the twisting moment per unit length; P is the applied compressive force per unit length; w is the lateral displacement; D_x, D_y, D_1 and M_{xy} are the elastic rigidities all of which are proportional to the cube of the plate thickness; $\{\mathbf{M}\}$ is a vector containing the bending moments M_x, M_y and M_{xy}; $[\mathbf{D}]$ is a matrix of elastic rigidities containing D_x, D_y, D_1 and D_{xy}.

The optimization problem is now stated as find $t(x, y)$ that minimizes the volume V for a fixed buckling load P given the plate dimensions. It is formally written as minimize the functional

$$J = \int_S t(x, y)\, ds \equiv V \tag{4}$$

subject to

$$L(t)w = 0 \tag{5}$$

where

$$L(t) = \bar{D}_x[t^3(\,)_{,xx}]_{,xx} + \bar{D}_1[t^3(\,)_{,yy}]_{,xx} + 4\bar{D}_{xy}[t^3(\,)_{,xy}]_{,xy} + \bar{D}_1[t^3(\,)_{,xx}]_{,yy} + \bar{D}_x[t^3(\,)_{,yy}]_{,yy} + P(\,)_{,xx} \tag{6}$$

In Eq. (6), the following new variables have been introduced so as to highlight the dependence of the problem on t: $\bar{D}_x = D_x t^{-3}$; $\bar{D}_y = \bar{D}_y t^{-3}$; $\bar{D}_1 = D_1 t^{-3}$ and $\bar{D}_{xy} = D_{xy} t^{-3}$. In matrix form these new variables are given as

$$[\mathbf{D}] = t^3[\bar{\mathbf{D}}] \tag{7}$$

Where $[\bar{\mathbf{D}}]$ is a matrix of elastic rigidities as in Eq. (2) but containing $\bar{D}_x$, $\bar{D}_y$, $\bar{D}_1$ and $\bar{D}_{xy}$ instead.

2.1. The optimality condition

The optimality condition is obtained by applying variational calculus in a straightforward manner. Taking the first variation of the governing equation, Eq. (5), multiplied by an adjoint variable and integrated over the area added to the first variation of the objective functional, Eq. (4), form a new Lagrangian functional which is equal to zero. Using the self adjoint property of the operator $L(t)$ and the fact that the original objective functional varies with the design variable t and, therefore δt only, and is independent of w, the optimality condition is obtained as

$$\frac{1}{2}\frac{\{\mathbf{w}''\}^T t^3[\bar{\mathbf{D}}]\{\mathbf{w}''\}}{t}=\alpha \tag{8}$$

where is α constant; $\{\mathbf{w}''\}$ is a vector whose elements are $-w_{,xx}$, $-w_{,yy}$ and $2w_{,xy}$.

The optimality condition may be obtained, in a somewhat simpler manner by applying the stationary condition of Rayleigh's quotient with respect to the variation of thickness. It states that the plate thickness is proportional to the strain energy density. Alternatively it states that the depth averaged strain energy density is a constant at the optimum.

Prior claims (Masur 1970) argue that constant strain energy density at the optimum for prebuckling statically determinate stress fields.

2.2. The approximate "optimal" plate

When solved simultaneously Eqs. (3) and (8) will yield an optimal thickness distribution function and a lateral displacement function. A truncated Fourier series that satisfies the simply supported boundary conditions is proposed (Spillers and Levy 1990). For a plate defined over $0\leq x\leq a$, $0\leq y\leq b$, let

$$w=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b} \tag{9}$$

$$t=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\beta_{pq}\cos\frac{p\pi x}{a}\cos\frac{q\pi y}{b} \tag{10}$$

$$t^2=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\gamma_{pq}\cos\frac{p\pi x}{a}\cos\frac{q\pi y}{b} \tag{11}$$

For a symmetric approximation use

$$w=\alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{p\pi y}{b} \tag{12}$$

$$t=\left[B_1+B_2\cos\frac{2p\pi x}{a}\right]\left[B_3+B_4\cos\frac{2q\pi y}{b}\right] \tag{13}$$

$$t^2=\left[C_1+C_2\cos\frac{2p\pi x}{a}\right]\left[C_3+C_4\cos\frac{2q\pi y}{b}\right] \tag{14}$$

Eqs. (8) and (12) are now substituted into Rayleigh's representation of the governing equations, i.e.

$$P_{cr}=\frac{\int_s t^3\{\mathbf{w}''\}^T[\bar{\mathbf{D}}]\{\mathbf{w}''\}\,ds}{\int_s (w_{,x})^2 ds} \tag{15}$$

to yield an expression for the volume in terms of P_{cr} and α. Since α may be directly expressed in terms of C_1, C_2, C_3 and C_4 from the optimality condition, their values that minimize V are obtained. First order relations between B_1, B_2, B_3, B_4 and C_1, C_2, C_3, C_4 are extracted by equating the square of Eq. (13) and Eq. (14) as identities. Thus "optimal" values are obtained as

$$t=\frac{V}{ab}\left[1+(-2+\sqrt{2})\cos\frac{2p\pi x}{a}\right]\left[1+(-2+\sqrt{2})\cos\frac{2q\pi y}{b}\right] \tag{16}$$

and

$$P_{cr}=2(3-2\sqrt{2})\frac{V^3}{a^3b^3}(\psi+9\phi)\left(\frac{a}{p\pi}\right)^2 \tag{17}$$

where

$$\psi=4\bar{D}_{xy}\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2 \tag{18}$$

and

$$\phi=\bar{D}_x\left(\frac{p\pi}{a}\right)^4+2\bar{D}_1\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{b}\right)^2+\bar{D}_y\left(\frac{q\pi}{b}\right)^2 \tag{19}$$

For $q=1$, p may be found from

$$p=\frac{a}{b}\left(\frac{D_{y0}}{D_{x0}}\right)^{1/4} \tag{20}$$

where D_{x0} and D_{y0} are averaged uniform elastic constants.

3. One parameter optimization

Two plates are considered in this section as candidates for optimization. The first is the double cosine symmetric plate of the form

$$t(x,\ y)=c_1\left(1+c_2\cos\frac{2\pi x}{a}\right)\left(1+c_2\cos\frac{2\pi y}{b}\right) \tag{21}$$

where $c_1=t_{average}$ and $-1\leq c_2\leq 1$. The second is the hybrid double sine symmetric plate of the form

$$t(x,\ y)=c_1\left(1+c_2\sin\frac{\pi x}{a}\right)\left(1+c_2\sin\frac{\pi y}{b}\right) \tag{22}$$

where $0\leq c_1\leq t_{average}/(1-2/\pi)^2$ and $c_2=\pi/2(-1+\sqrt{t_{average}/c_1})$.

The total potential $\Pi = U - W$, is first written in terms of the coefficients α_{pq} of a double sine Fourier infinite series for the displacement, w. The variation of Π is equated to zero as a requirement for equilibrium to yield

$$\delta\Pi = 0 \quad \text{or} \quad \left(\frac{\partial \Pi}{\partial \alpha_{pq}}\right)\delta\alpha_{pq} \tag{23}$$

Due to the arbitrariness of $\delta\alpha_{pq}$ obtain

$$\frac{\partial \Pi}{\partial \alpha_{pq}} = 0 \qquad \text{for all } p,\ q \tag{24}$$

The total potential of non uniform orthotropic plates is given by

$$\begin{aligned}\Pi = \frac{1}{2}\Bigg[\int_0^b\int_0^a &\{\bar{D}_x t^3 (w,_{xx})^2 + \bar{D}_y t^3 (w,_{yy})^2 + 2\bar{D}_1 t^3 (w,_{xx} w,_{yy}) \\ &+ 4\bar{D}_{xy} t^3 (w,_{xy})^2\}\, dxdy - P\int_0^b\int_0^a (w,_x)^2\, dxdy\Bigg]\end{aligned} \tag{25}$$

After substituting Eq. (9) into Eq. (25) obtain

$$\begin{aligned}\Pi = {} & \frac{\bar{D}_x}{2}\int_0^b\int_0^a t^3\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{\pi^2 p^2}{a^2}\ \alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right)^2 dxdy \\ & + \frac{\bar{D}_y}{2}\int_0^b\int_0^a t^3\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{\pi^2 p^2}{b^2}\ \alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right)^2 dxdy \\ & + \bar{D}_1\int_0^b\int_0^a t^3\Bigg[\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{\pi^2 p^2}{a^2}\ \alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right) \\ & \qquad \times\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{\pi^2 q^2}{b^2}\ \alpha_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right)\Bigg] dxdy \\ & + 2\bar{D}_{xy}\int_0^b\int_0^a t^3\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{\pi^2 pq}{ab}\ \alpha_{pq}\cos\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right)^2 dxdy \\ & - \frac{P}{2}\int_0^b\int_0^a\left(\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\frac{p\pi}{a}\ \alpha_{pq}\cos\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\right)^2 dxdy\end{aligned} \tag{26}$$

Note that four of the double integrals contain t^3 which in itself consists of 16 terms to be multiplied by the sum resulting from a particular choice of p and q. The choice is such that an increase in the number of terms will yield insignificant additional accuracy. A truncated, sufficiently accurate series is thus used instead of an infinite series. Moreover, mixed double summations or double summations squared result in quadruple summations that render themselves more suitable for programming.

3.1. The "cosine" plate

Eq. (26) is further reduced to tailor to the "cosine" plate of Eq. (21) to yield

$$\Pi = \sum_p\sum_q\sum_m\sum_n\left\{\sum_{i=1}^{16} B_i\left[\left(\frac{1}{2}\bar{D}_x K_1(\beta) + \bar{D}_1 K_2(\beta) + \frac{1}{2}\bar{D}_y K_3(\beta)\right) I_i(\beta)\right.\right.$$

$$+2\bar{D}_{xy}K_4(\beta)J_i(\beta)\Big]\Big\}\alpha_{pq}\,\alpha_{mn}-\frac{P}{8}\sum_p\sum_q K_5(\beta)\alpha_{pq}^2 \tag{27}$$

where B_i, $i=1\rightarrow16$ comes from the thickness cubed, $I_i(\beta)$, $i=1\rightarrow16$ and $J_i(\beta)$, $i=1\rightarrow16$ are two types of integrals, β is a given combination, p, q, m, n and $K_i(\beta)$, $i=1\rightarrow5$ contains constants stemming from internal differentiation of sines and cosines. For example, and without going into unnecessary details

$$B_{16}=C_1^3 \tag{28}$$

$$I_{16}(\beta)=\int_0^b\int_0^a \gamma dxdy \tag{29}$$

where $\gamma=\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ and

$$J_{16}(\beta)=\int_0^b\int_0^a \eta dxdy \tag{30}$$

where $\eta=\cos\frac{p\pi x}{a}\cos\frac{q\pi y}{b}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$ and

$$K_5(\beta)=\frac{bp^2\pi^2}{a} \tag{31}$$

3.2. The hybrid "sine" plate

The hybrid "sine" plate of Eq. (22) is substituted in Eq. (26) to yield an expression for the total potential of the form:

$$\Pi=\sum_p\sum_q\sum_m\sum_n\Big\{\sum_{i=1}^{16}B_i\Big[\Big(\frac{1}{2}\bar{D}_xK_1(\beta)+\bar{D}_1K_2(\beta)+\frac{1}{2}\bar{D}_yK_3(\beta)\Big)I_i^*(\beta)$$

$$+2\bar{D}_{xy}K_4(\beta)J_i^*(\beta)\Big]\Big\}\alpha_{pq}\,\alpha_{mn}-\frac{P}{8}\sum_p\sum_q K_5(\beta)\alpha_{pq}^2 \tag{32}$$

where integrals $I_i^*(\beta)$ and $J_i^*(\beta)$ are obtained by replacing the cosine terms with identical sine terms in Eq. (27).

Within Eq. (24) lies a generalized eigenvalue problem of the form

$$[[\mathbf{A}]-P[\mathbf{B}]]\,\{\alpha\}=\mathbf{0} \tag{33}$$

where $\{\alpha\}$ is a vector of coefficients α_{pq}. An automated procedure evaluates integrals and assembles $[\mathbf{A}]$ and $[\mathbf{B}]$. Actual solution of Eq. (33) is performed using IMSL subroutines. Repeated analyses are performed using a suitable large number of terms that ensures convergence for each choice of c_1 and c_2 as a one parameter (since c_1 and c_2 are mutually dependent) optimization problem.

4. Examples

An isotropic aluminum square plate of dimension 25.4 cm (10.0 in)×25.4 cm (10.0 in) and

Table 1 Mechanical constants of graphite epoxy plates

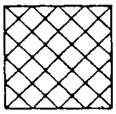
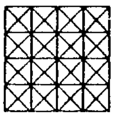
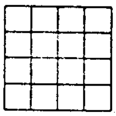
Plate type				Mechanical constants $\times 10^6$, *psi*			
				E_x'	E_y'	E''	G
I		100%	0°	17.076	1.8084	0.3798	0.65004
II		100%	±45°	5.916	5.916	4.5	4.5
III		25% 25% 50%	0° 90° ±45°	7.4724	7.4724	2.2416	2.6004
IV		50% 50%	0° 90°	9.42	9.42	0.38004	0.65004

Table 2 Optimized "COSINE" plates

Plate type	Shape coefficients		$(P)_{cr}^{var}/(P)_{cr}^{uniform}$
	c_1	c_2	
Aluminium		−0.295	1.234
I		−0.370	1.745
II	0.05	−0.113	1.0107
III		−0.295	1.228
IV		−0.365	1.745

Table 3 Optimized "SINE" plates

Plate type	Shape coefficients		$(P)_{cr}^{var}/(P)_{cr}^{uniform}$
	c_1	c_2	
Aluminium	0.0135	1.452	1.323
I	0.00980	1.977	1.887
II	0.0244	0.678	1.0329
III	0.0130	1.509	1.315
IV	0.010	1.942	1.873

having a total volume of 81.9 cm^3 (5.0 in^3) with $E=70.6$ GPa$=10.2\times10^6$ *psi* and $\nu=0.32$ and four different orthotropic configurations whose mechanical constants are given in Table 1 and of the same edge dimensions and volume as the isotropic plate were optimized for both the

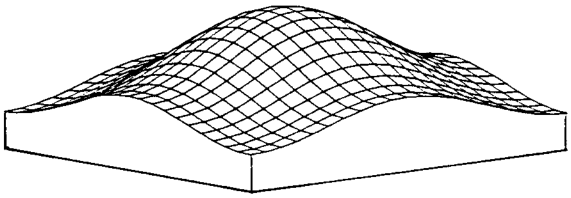


Fig. 1 Typical optimal "cosine" plate.

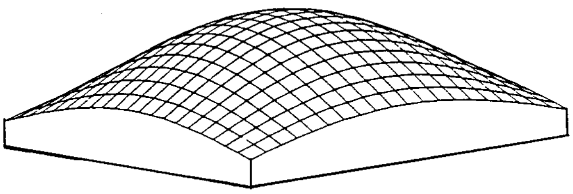


Fig. 2 Typical optimal "sine" plate.

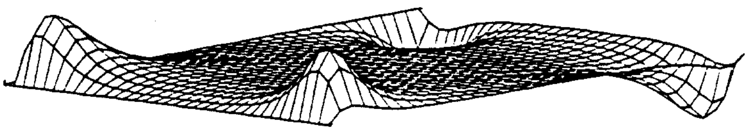


Fig. 3 Degenerate buckling mode of approximate optimal plate.

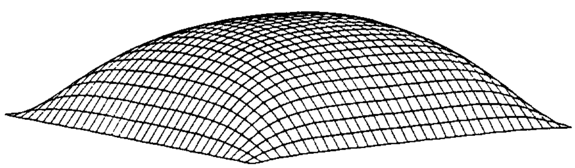


Fig. 4 Optimal buckling mode isotropic "cosine" plate.

"cosine" shape and "hybrid "sine" shape. The plates are of symmetric layout with off diagonal terms D_{16} and D_{26} taken as zero to yield a specially orthotropic matrix of constants. Optimal results in the form of magnification ratios are presented in Tables 2 and 3. Figs. 1 and 2 show three dimensional plots of thickness results of typical optimal plate configurations for the "cosine" and "sine" plates respectively for all cases under consideration. Only one half of each of a plate is shown. The other half is a mirror image on the other side of their base planes. The thickness is highly exaggerated in these plots. Figs. 4-8 and Figs. 9-13 show of the buckling modes for all cases considered. It should be noted that all modes come out symmetric.

A point of interest is the approximate plate of Eq. (16) for $p=q=1$. When a modified version of Eq. (27) is used to handle the special case of taking $q=1$ and $p=1, 3, 5, 7, \cdots$ (symmetric case) a magnification ratio of 1.41 results with a degenerate mode of anti symmetric nature (Fig. 3) for a full analysis. Slight "tuning" brings back the symmetric stable mode at the optimum.

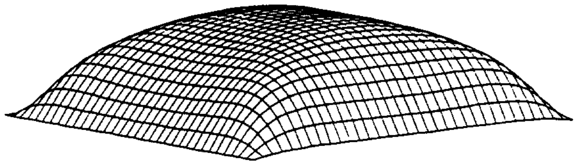


Fig. 5 Optimal buckling mode case I "cosine" plate.

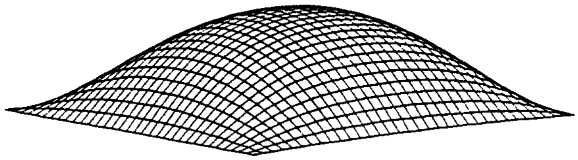


Fig. 6 Optimal buckling mode case II "cosine" plate.

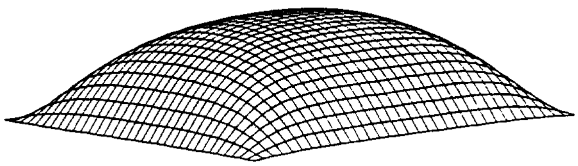


Fig. 7 Optimal buckling mode case III "cosine" plat.

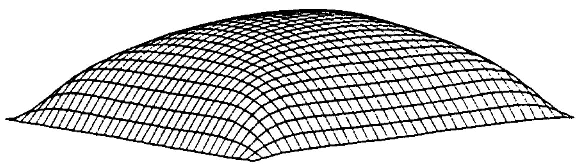


Fig. 8 Optimal buckling mode case IV "cosine" plate.

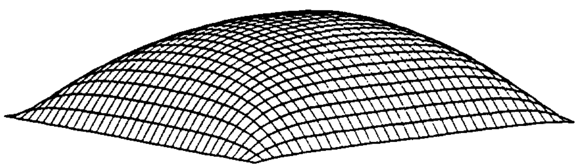


Fig. 9 Optimal buckling mode isotropic "sine" plate.

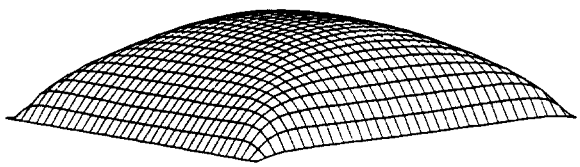


Fig. 10 Optimal buckling mode case I "sine" plate.

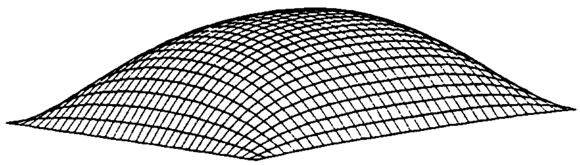


Fig. 11 Optimal buckling mode case II "sine" plate.

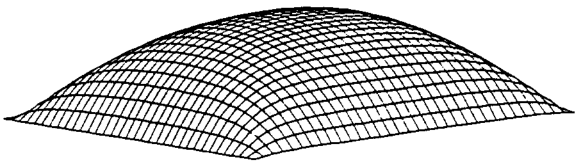


Fig. 12 Optimal buckling mode case III "sine" plate.

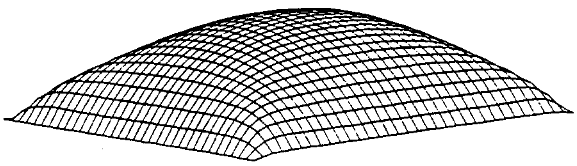


Fig. 13 Optimal buckling mode case IV "sine" plate.

5. Conclusions

Plates of variable thickness exhibit obvious strength/weight advantages in compressive buckling. The hybrid "sine" plate turned out to perform better than the "cosine" plate. Isotropic variable thickness plates outperform uniform plates by 32.3% whereas the specially orthotropic plate having 100% of its fibers at 0° direction maintained an 88.7% advantage.

In problems where Fourier series "works" such as the problem at hand a clear picture of the behavior in terms of the original problem parameters may be obtained using a one-term truncated series.

With regard to the constant state of compressive stress that was assumed to exist one should exercise further consideration and modifications when thickness variations are of the same order of magnitude as the plate dimensions (in this paper the order is 1/100 !) or when the plates are of high aspect ratios.

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Notation

The following symbols are used in this paper:

a	plate dimension in x-direction
$\{\alpha\}$	vector of coefficients α_{pq}
$[\mathbf{B}]$	generalized eigenvalue matrices
b	plate dimension in y-direction
B_1, B_2, B_3, B_4	coefficients of trigonometric terms
c_1, c_2	cosine and hybrid sine plate coefficients
C_1, C_2, C_3, C_4	coefficients of trigonometric terms
$[\mathbf{D}]$, $[\bar{\mathbf{D}}]$	matrix of elastic constants for plate.
D_1, D_x, D_y, D_{xy}	elastic constants for plate
D_{x0}, D_{y0}	averaged elastic constants for plate
$\bar{D}_1$, $\bar{D}_x$, $\bar{D}_y$, $\bar{D}_{xy}$	elastic constants for plate per cubic thickness
ds	differential area
E	Young's modulus
$I_i^*(\beta)$, $I_i(\beta)$	integral types
$J_i^*(\beta)$, $J_i(\beta)$	integral types
J	functional
$K_i(\beta)$, K_1, K_2, K_3, K_4	constants
$L(\,)$	differential operator
$\{\mathbf{M}\}$	vector of bending and twisting moments
$M_1(y)$	function of y
M_x, M_y, M_{xy}	bending and twisting moments per unit length
m, p	number of waves in x-direction
n, q	number of waves in y-direction
P	compressive loading
P_{cr}	critical buckling load
S	area
T	transpose of a matrix
$t_{average}$	uniform thickness
t, $t(x, y)$	thickness varying with x and y
U	strain energy
V	volume
W	work done
w	lateral displacement of plate
$[\mathbf{w}'']$	vector $(-w_{,xx};\ -w_{,yy};\ 2w_{,xy})$
x	coordinate direction

y	coordinate direction
α_{pq}, β_{pq}, γ_{pq}	Fourier double sine coefficients
β	p, q, m, n combination
γ	sine multiples
ν	Poisson's ratio
η	cosine multiples
Π	total potential
ρ	ratio of wavelengths squared
ϕ	function of $\bar{D}_{xy}$, p and q (Eq. (8)), and
ψ	function of $\bar{D}_1$, $\bar{D}_x$, $\bar{D}_y$, p and q (Eq. (19))