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Static and dynamic stability of a single-degree-of-freedom autonomous system with distinct critical points

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Abstract. The dynamic buckling mechanism of a single-degree-of-freedom dissipative/nondissipative gradient system is thoroughly studied, employing energy criteria. The model is chosen in such a manner, that its corresponding static response is associated with all types of distinct critical points. Under a suddenly applied load of infinite duration, it is found that dynamic buckling, occurring always through a saddle, leads to an escaped motion, which is finally attracted by remote stable equilibrium positions, belonging sometimes also to complementary paths. Moreover, although the existence of initial imperfection changes the static behaviour of the system from limit point instability to bifurcation, it is established that the proposed model is dynamically stable in the large, regardless of the values of all other parameters involved.

Key words: critical points; dynamic buckling; saddle; stable in the large; snapping.

1. Introduction

The use of properly selected and developed simple models of a few degrees of freedom (DOF), and especially of single DOF ones, enables the investigators to examine the static and dynamic response of real (continuous) structures. Provided of course that for a correct modelling the silent features of the structure are known and well represented. Since the equations involved are rather simple, there are no severe difficulties when dealing with the problem, and thus a variety of phenomena, some of immense importance, are rather easily revealed and thoroughly studied.

Numerous researchers have contributed in this area, by developing several simple models, with or without initial imperfections, each one of them exhibiting a distinct static critical point (limit point, symmetric stable and unstable branching point, asymmetric branching point). Among these the names of Koiter, Chilver, Augusti, Thompson, Huseyin, Croll, Walker etc. must be quoted, while a very good bibliography on the subject can be found in the book by Gioncu and Ivan (1984).

Furthermore, dynamic buckling of such simple autonomous gradient systems and pertinent energy criteria have been recently reported by Kounadis (1991, 1993, 1994) and Kounadis, Sophianopoulos (1995). According to the aforementioned analyses, dynamic buckling of autonomous single DOF systems takes place always via saddle points of the unstable postbuckling equilibrium

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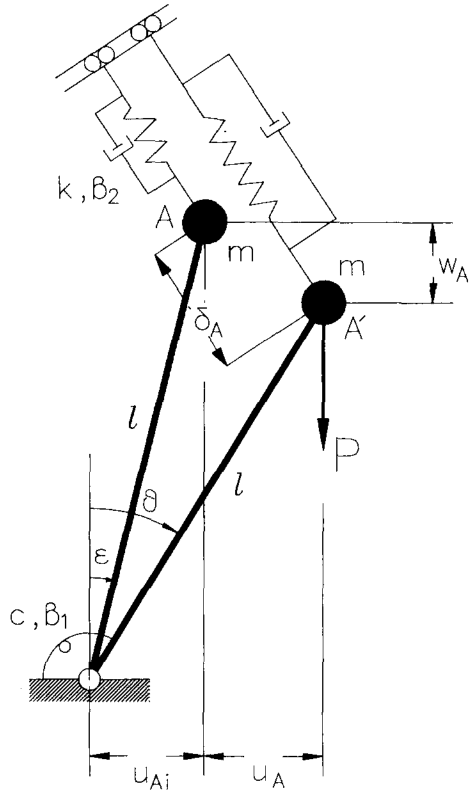

Fig. 1 Geometry, sign convention and properties of the proposed single DOF model.

path. These saddles are precisely determined in the absence of damping, while for dissipative systems one must resort to approximate evaluations (lower or upper bound buckling estimates).

In this paper, a combination of three simple one DOF models, first studied by Zanaboni (1962) and Augusti (1964a,b), is introduced, and the proposed system exhibits all kinds of distinct critical points (depending on the values of initial imperfection and other parameters). It is found that regardless of the variation of these values, the system is dynamically stable in the large, since the existence of remote stable equilibrium paths leads to a large amplitude bounded motion (point attractor).

Moreover from several cases considered, it was observed that remote point attractors sometimes belong to complementary equilibrium paths, a phenomenon reported for multi DOF systems.

2. Mathematical formulation

Let us consider the single degree of freedom system shown in Fig. 1, having in general an initial angular imperfection ε, which consists of a weightless bar of length l, partially hinged at one end (via a linear rotational spring with corresponding stiffness equal to c). Its other end, where a concentrated mass m is located, is connected to an also linear translational spring of stiffness k, which can freely slide along inclined tracks. Both springs are able to absorb an amount of energy, and the values of their viscous damping coefficients are equal to β_1 and β_2 respectively. If the system under a step loading P, vertically applied at the centre of the mass, deforms at a new equilibrium position defined by point A', the horizontal and vertical displacement components related to this deformation are equal to

$$u_A = l(\sin\theta - \sin\varepsilon) \tag{1a}$$

$$w_A = l(\cos\varepsilon - \cos\theta) \tag{1b}$$

while the change of length of the translational spring is given by

$$\delta_A = l\{\sin(\alpha-\varepsilon) - \sin(\alpha-\theta)\} \tag{2a}$$

and the moment developing on the rotational spring is

$$M = c(\theta - \varepsilon) \tag{2b}$$

Assuming that the system is initially at rest, the differential equation of Lagrange governing its motion is expressed by the following relation (valid for conservative systems):

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot\theta}\right) - \frac{\partial K}{\partial\theta} + \frac{\partial V}{\partial\theta} + \frac{\partial F}{\partial\dot\theta} = 0 \tag{3}$$

In this equation angle θ, measured from the vertical configuration, is used as generalised coordinate, and the corresponding expressions of the energy functions involved are:
Kinetic energy $K = \frac{1}{2}m\left(\dot u_A^2 + \dot w_A^2\right)$ which yields

$$K = \frac{1}{2}ml^2\dot\theta^2 \tag{4}$$

Total potential $V_T = U + \Omega$ with

$U = \frac{1}{2}k\delta_A^2 + \frac{1}{2}M(\theta-\varepsilon)$ and $\Omega = -Pw_A$ which give

$$V_T = \frac{1}{2}kl^2[\sin(\alpha-\varepsilon) - \sin(\alpha-\theta)]^2 + \frac{1}{2}c(\theta-\varepsilon)^2 - Pl(\cos\varepsilon - \cos\theta) \tag{5}$$

Rayleigh dissipation function $F = \frac{1}{2}\beta_1\left[\frac{d}{dt}(\theta-\varepsilon)\right]^2 + \frac{1}{2}\beta_2\dot\delta_A^2 \Rightarrow$

$$F = \frac{1}{2}\beta_1\dot\theta^2 + \frac{1}{2}\beta_2 l^2\cos^2(\alpha-\theta)\,\dot\theta^2 \tag{6}$$

Introducing the derivatives of the above functions in Eq. (3) we get

$$ml^2\ddot\theta + [\beta_1 + \beta_2 l^2\cos^2(\alpha-\theta)]\,\dot\theta + kl^2\{\sin(\alpha-\varepsilon) - \sin(\alpha-\theta)\}\cos(\alpha-\theta)\} + c(\theta-\varepsilon) - Pl\sin\theta = 0 \tag{7}$$

subjected to initial conditions

$$\theta(0) = \varepsilon \text{ and } \dot\theta(0) = 0 \tag{8}$$

Nondimensionalizing, using the following parameters

$$\bar c = \frac{c}{kl^2} \qquad \bar\beta_1 = \beta_1/\sqrt{mcl^2} \qquad \bar\beta_2 = \beta_2/\sqrt{km} \qquad \lambda = \frac{P}{kl} \tag{9}$$

dimensionless time $\tau=\sqrt{\frac{k}{m}}t$ $\quad \vartheta(\tau)=\theta(t)$

one can write the DE of motion in dimensionless form as follows

$$\ddot{\vartheta}+[\bar{\beta}_1\sqrt{\bar{c}}+\bar{\beta}_2\cos^2(\alpha-\vartheta)]\,\dot{\vartheta}+[\sin(\alpha-\varepsilon)-\sin(\alpha-\vartheta)]\cos(\alpha-\vartheta)+\bar{c}(\vartheta-\varepsilon)-\lambda\sin\vartheta=0 \tag{10a}$$

$$\vartheta(0)=\varepsilon, \qquad \dot{\vartheta}(0)=0 \tag{10b}$$

Eliminating the inertia factors from Eq. (10a) we reach to the static buckling equation, given in dimensionless form by

$$\frac{d\bar{V}_T}{d\vartheta}=[\sin(\alpha-\varepsilon)-\sin(\alpha-\vartheta)]\cos(\alpha-\vartheta)+\bar{c}(\vartheta-\varepsilon)-\lambda\sin\vartheta=0 \tag{11}$$

where

$$\bar{V}_T=\frac{V_T}{kl^2}=\frac{1}{2}[\sin(\alpha-\varepsilon)-\sin(\alpha-\vartheta)]^2+\frac{1}{2}\bar{c}(\vartheta-\varepsilon)^2-\lambda(\cos\varepsilon-\cos\vartheta) \tag{12}$$

For the stability of equilibrium states, one must also determine the value of $d^2\bar{V}_T/d\vartheta^2$ being equal to

$$V_{\vartheta\vartheta}=\left[\frac{d^2\bar{V}_T}{d\vartheta^2}\right]_E=\cos2(\alpha-\vartheta)+\sin(\alpha-\varepsilon)\sin(\alpha-\vartheta)+\bar{c}$$
$$-\frac{[\sin(\alpha-\varepsilon)-\sin(\alpha-\vartheta)]\cos(\alpha-\vartheta)+\bar{c}(\vartheta-\varepsilon)}{\tan\vartheta} \tag{13}$$

If $V_{\vartheta\vartheta}>0$ there is a local minimum of V_T and the equilibrium state is stable, while if $V_{\vartheta\vartheta}<0$ there is a local maximum of V_T and the state is unstable. If, in any case the second derivative of V_T is also equal to zero (which usually happens for points of bifurcation), one must seek the sign of higher order derivatives, in order to determine whether the corresponding state is stable or unstable.

Furthermore, integrating the equation of motion Eq. (10) and employing the well known *inflection point criterion* the following system is valid

$$\frac{d\bar{V}_T}{d\vartheta}=0$$

$$\int_0^\tau[\bar{\beta}_1\sqrt{\bar{c}}+\bar{\beta}_2\cos^2(\alpha-\vartheta)]\,\dot{\vartheta}^2+\bar{V}_T=0 \tag{14}$$

For a nondissipative system, the solution of Eq. (14) provides us with the exact values of the dynamic buckling load λ_D and the critical dynamic displacement ϑ_D. These are lower and upper bound estimates respectively of the corresponding load λ_{DD} and displacement ϑ_{DD} when damping is present.

3. Numerical results and discussion

3.1. Nonlinear static analysis

At first, aiming to represent all kinds of static buckling, which could be exhibited by the foregoing model, all possible combinations of the foregoing parameter $\bar{c}(=10, 0.75, 1, 0)-\varepsilon$ $(=0, 0.05)$ and a $(=0, \pi/4, \pi/3)$ and the corresponding equilibrium paths (primary and complementary) are drawn. These can be clearly seen throughout Figs. 2, 3, 4 and 5, where the above 24 characteristic system cases are considered.

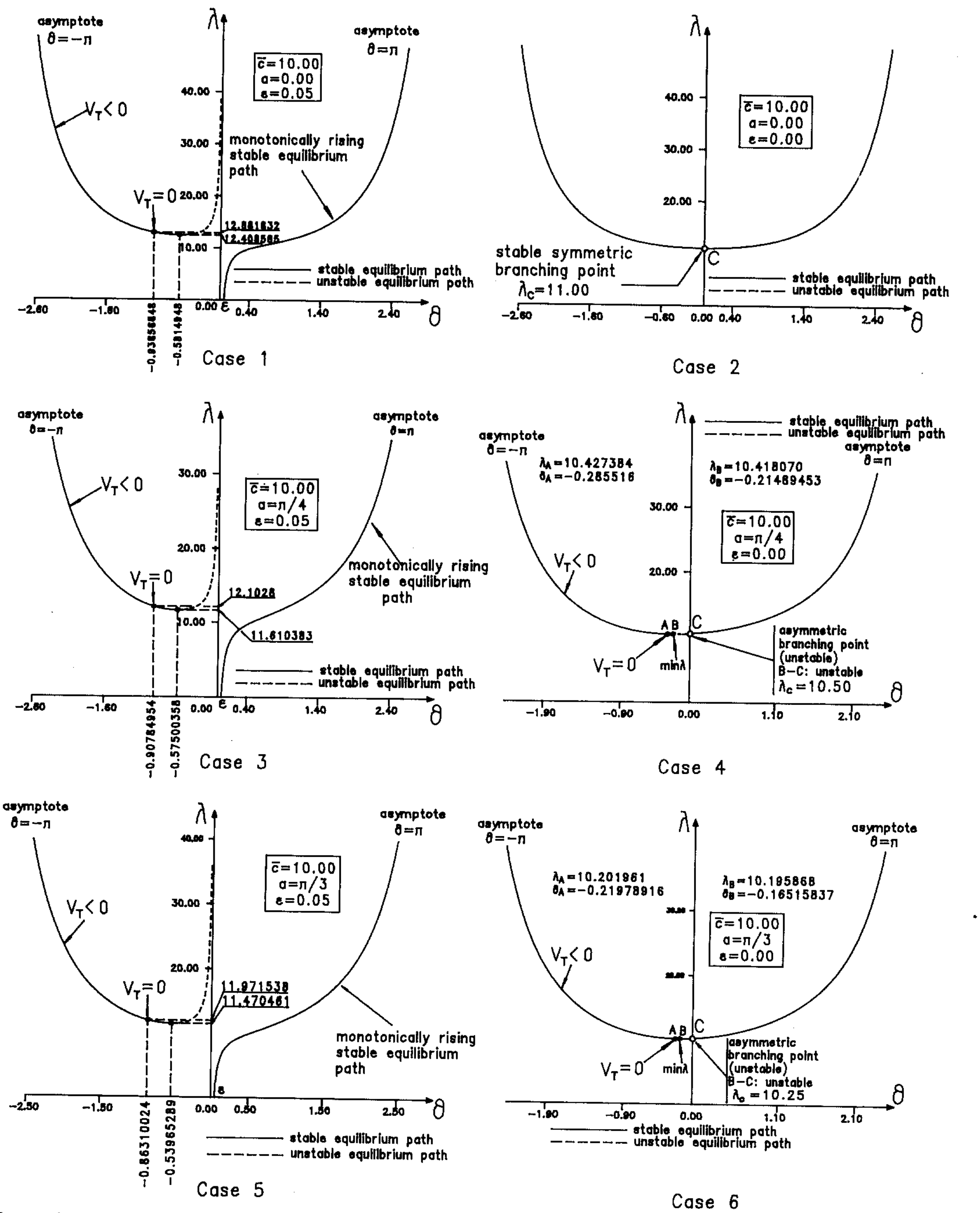

Fig. 2 Natural and complementary equilibrium paths of a model with $\bar{c}=10$ and various values of angle a $(=0, \pi/4, \pi/3)$ and initial imperfection ε $(=0, 0.05)$; cases 1-6.

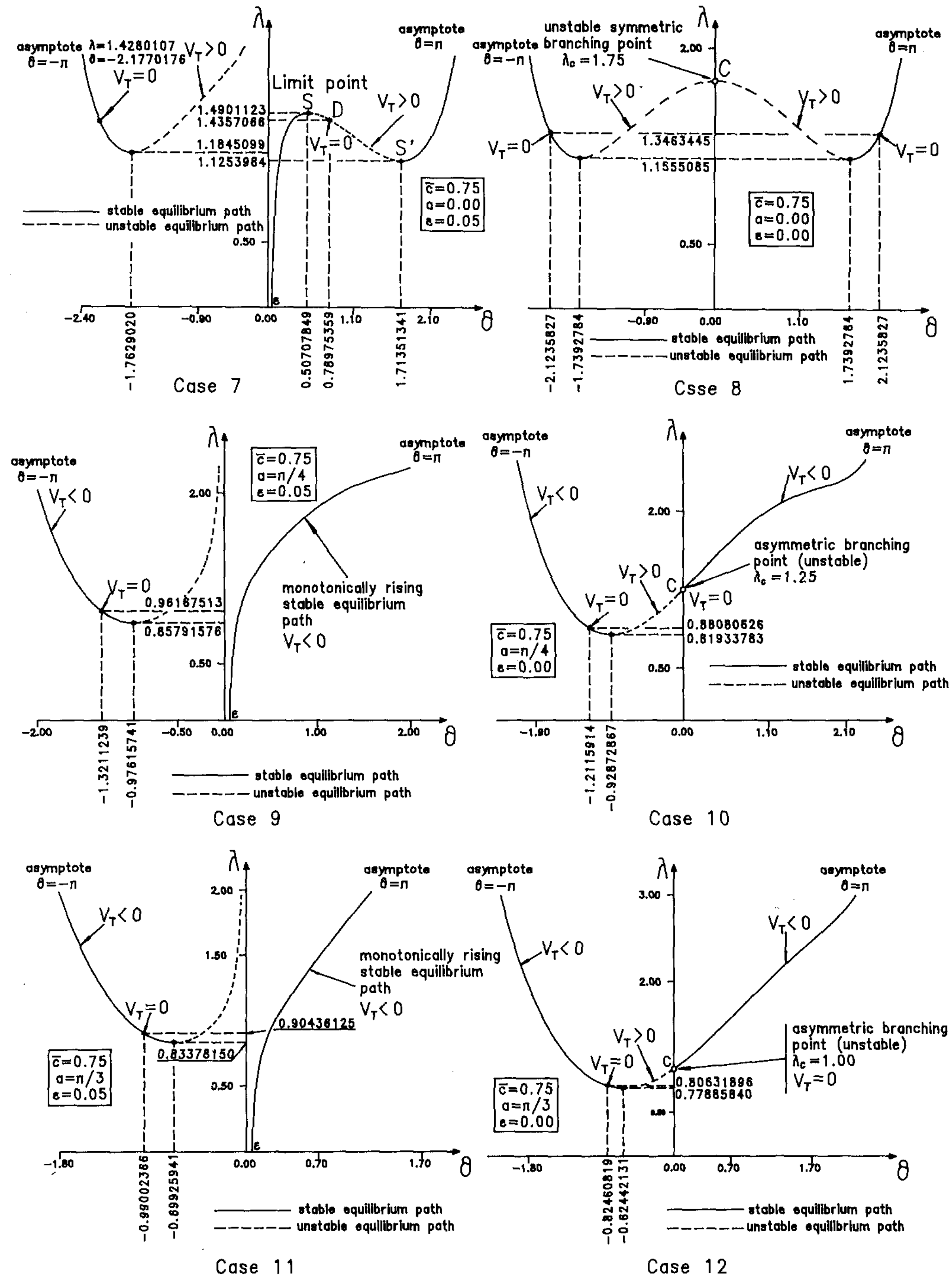

Fig. 3 Natural and complementary equilibrium paths of a model with $\bar{c}=0.75$ and various values of angle a ($=0$, $\pi/4$, $\pi/3$) and intial imperfection ε ($=0$, 0.05); cased 7-12.

From all these drawings, one can observe at first, that for great values of $\bar{c}$ ($=10$), meaning that the effect of the translational spring is minimal, and in the presence of an initial imperfection, the primary equilibrium path is monotonically rising (always stable) while for the perfect system

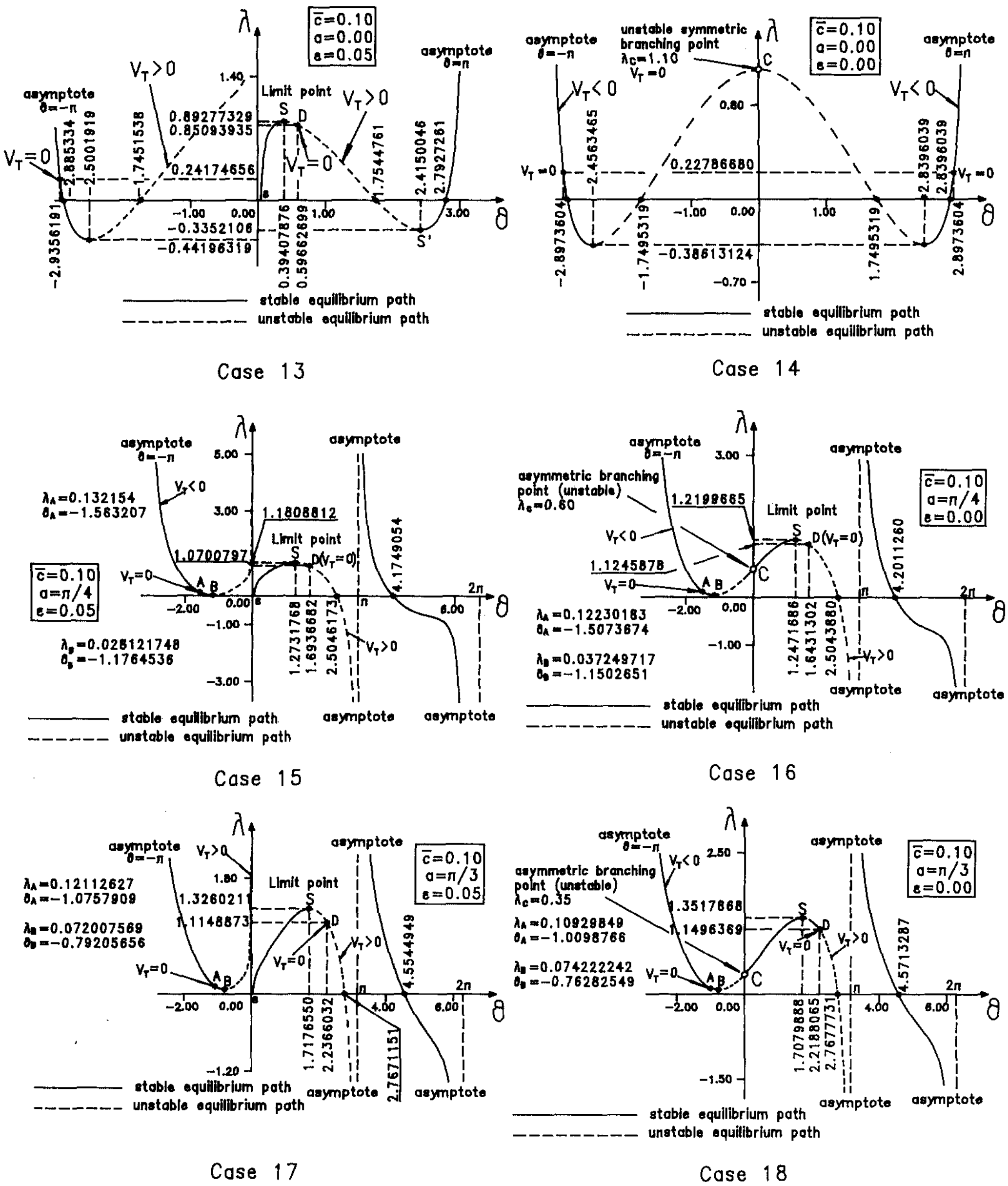

Fig. 4 Natural and complementary equilibrium paths of a model with $\bar{c}=0.10$ and various values of angle a $(=0, \pi/4, \pi/3)$ and intial imperfection ε $(=0, 0.05)$; cased 13-18.

there is a stable symmetric branching point for $\alpha=0$ and an asymmetric one for $\alpha\neq0$, which is itself unstable. Furthermore, for a rational value of $\bar{c}$ $(=0.75)$, implying that both springs are effective, the imperfect system with $\alpha=0$ exhibits a typical limit point instability (*snapping*), while for $\alpha\neq0$ a monotonically rising path. If again $\varepsilon=0.00$ (perfect system) the same as for $\bar{c}=10$ static response is estabilished. Moreover, for the case of small $\bar{c}$ $(=0.10)$, in which only a slight effect of the rotational spring is accounted for, there is always a limit point, regardless

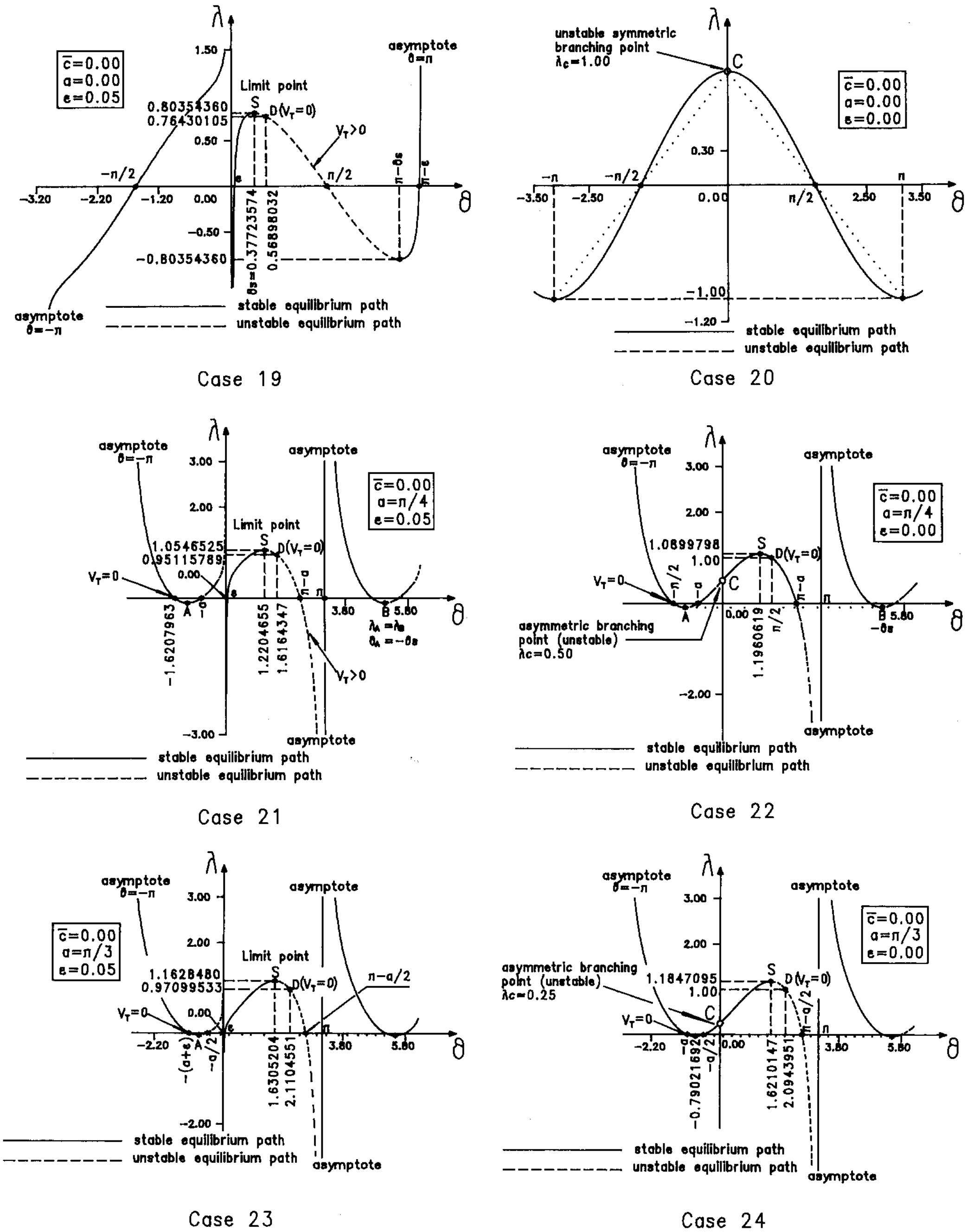

Fig. 5 Natural and complementary equilibrium paths of a model with $\bar{c}=0.00$ and various values of angle a ($=0$, $\pi/4$, $\pi/3$) and intial imperfection ε ($=0$, 0.05) cased 19-24.

of the value of angle α in the corresponding imperfect systems. Meanwhile if $\varepsilon=0.00$ the system's primary path is a continuous function of ϑ with both an asymmetric branching point (unstable) and a limit point (for $\alpha\neq0$), and only an unstable symmetric point of bifurcation for $\alpha=0$.

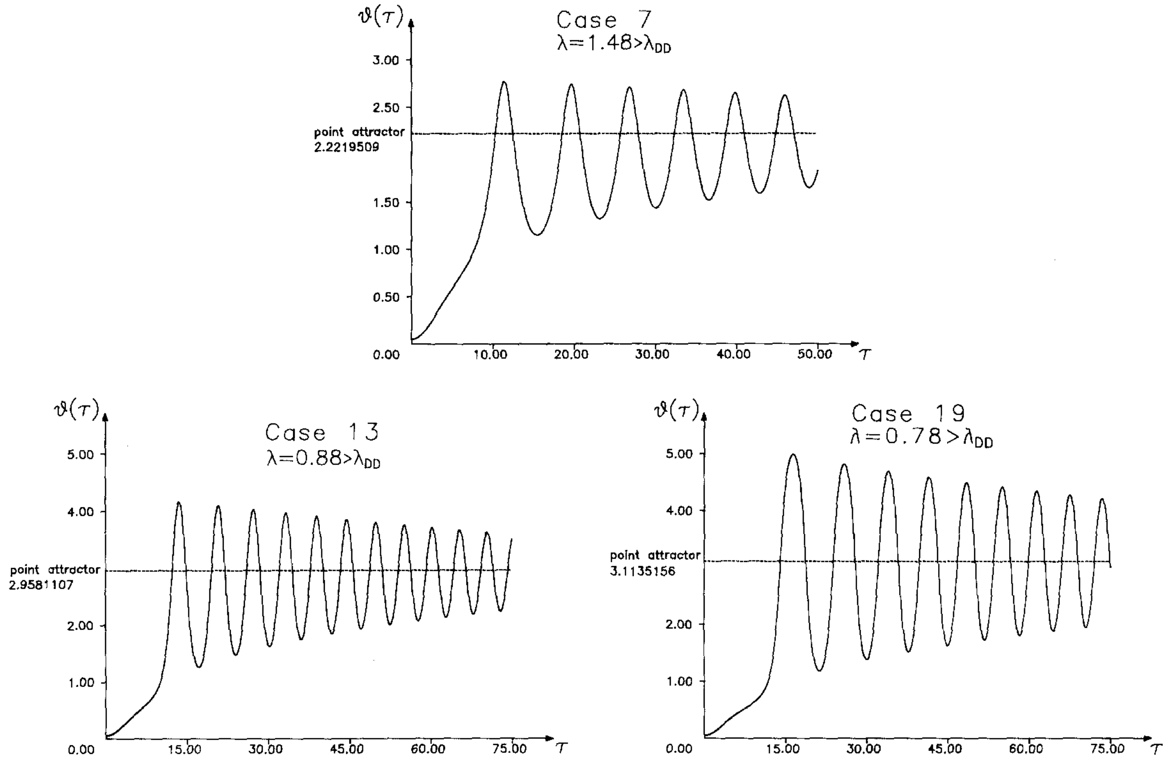

Fig. 6 Time series τ, $\vartheta(\tau)$ for three limit point systems exhibiting a point attractor response on their corresponding remote equilibrium position, belonging to the physical path.

Finally, when $\bar{c}=0.00$ (pure typical hinge) for imperfect systems a limit point instability occurs, while for perfect ones a behaviour similar with the one of the previous case. For all the above considered cases, one may observe also complementary equilibrium paths, physically not accepted, which as it will be shown below act as attractors.

3.2. Dynamic buckling analysis

Using the 7th order Runge-Kutta-Verner numerical scheme, with a suitably small integration step, in order to avoid an increased ratio of error, the differential equation of motion is solved, for damping coefficients equal to $\beta_1=\beta_2=0.02$. Of course, the inflection point criterion gives the exact values of λ_D and ϑ_D, clearly marked, when present in Figs. 2-5. When damping is accounted for, the following types of dynamic response are possible:

1) In cases of monotonically rising primary equilibrium paths, the system is dynamically stable, exhibiting an expected point attractor response.
2) Pure limit point systems, in the case where the primary path contains also a remote stable branch, buckle dynamically through a saddle, and the escaped motion initiated is captured by these remote stable states, leading to a point attractor (large amplitude motion), as shown in Fig. 6 for three characteristic examples.
3) On the other hand, when there is no remote stable branch of the primary path, for a

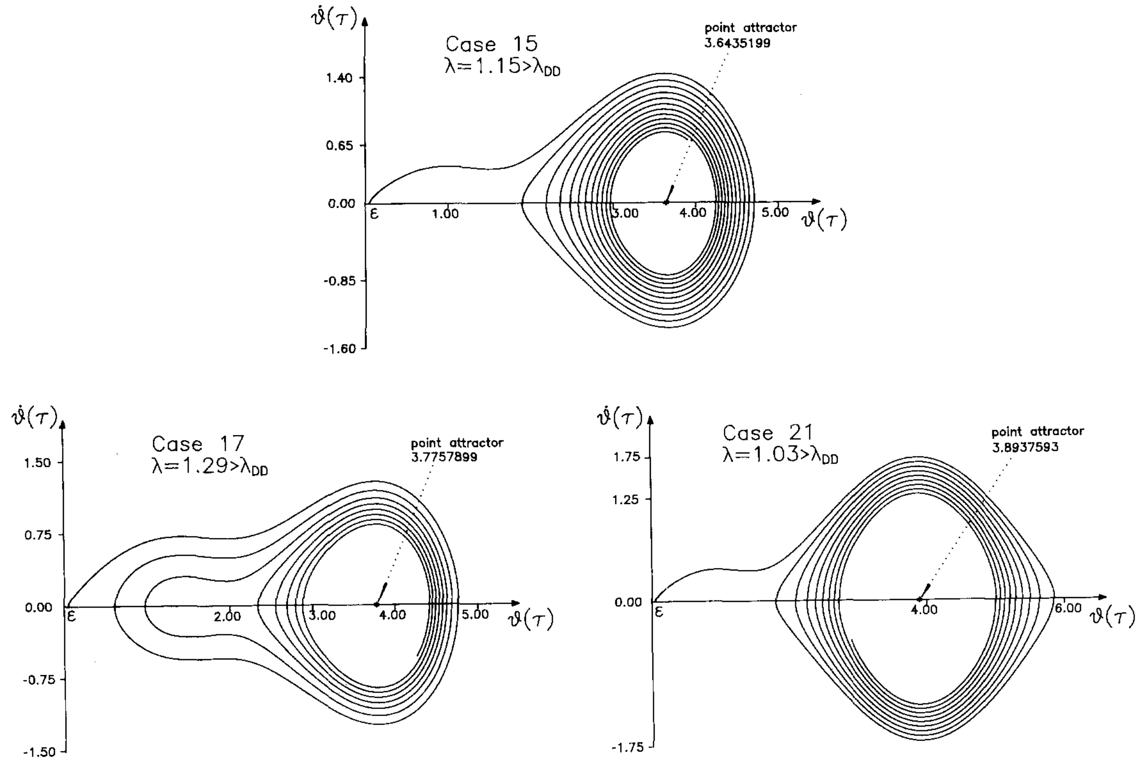

Fig. 7 Phase plane portraits $\vartheta(\tau)$, $\dot{\vartheta}(\tau$ of three characteristic limit point systems with a point attractor on a stable remote equilibrium position, belonging to complementary paths.

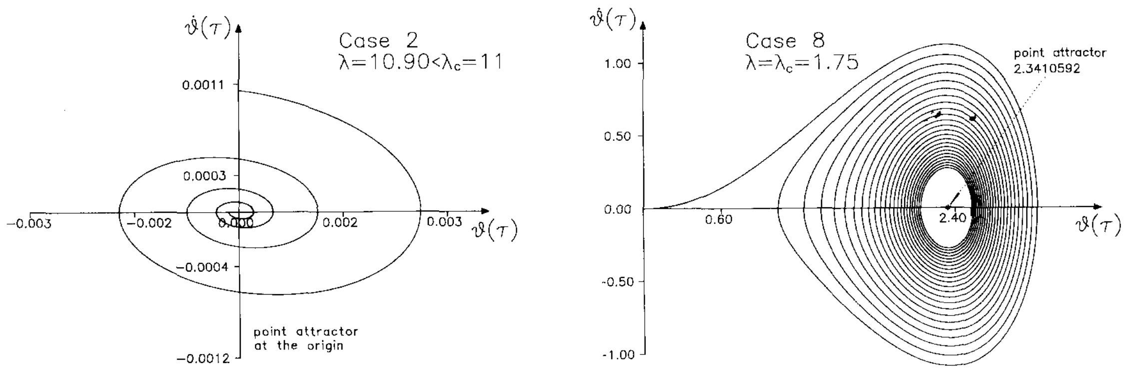

Fig. 8 Dynamic response of a model with a stable (case 2) and an unstable (case 8) symmetric branching point.

load greater than the dynamic buckling one the motion does not become unbounded (overflow), since stable branches of (always existing) complementary paths are now acting as attractors, and the system is again globally stable. This particular kind of dynamic response is represented in the phase plane portraits of Fig. 7.

4) For stable symmetric branching points and $\lambda \leq \lambda_C$ the motion settles on a point attractor at the origin ($\vartheta = 0$), while for greater loads on a corresponding attractor on the stable

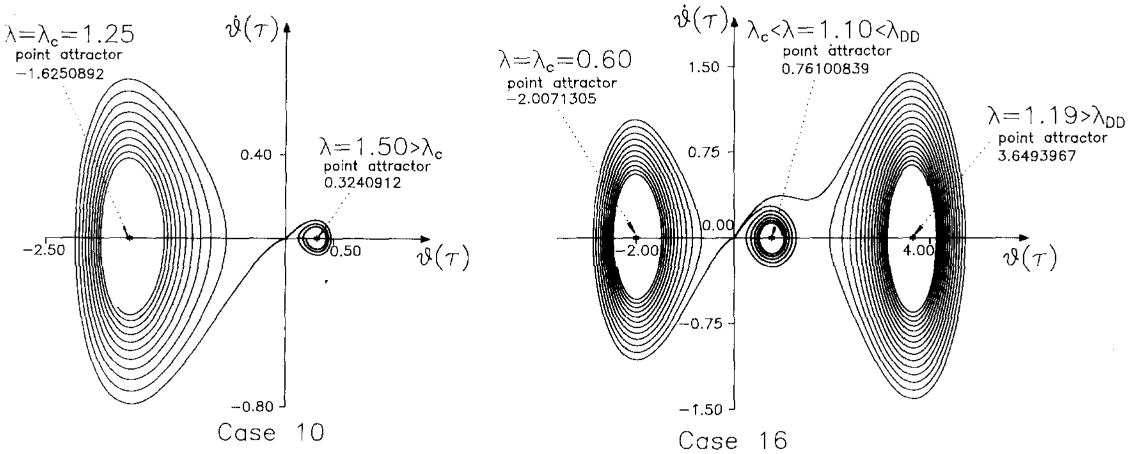

Fig. 9 Variation of the dynamic response of two characteristic models with an asymmetric point of bifurcation (cased 10 and 16), depending on the loading and the shape of the equilibrium paths.

primary path, Nevertheless, in the case when there exists an unstable symmetric point of bifurcation the previous phenomena occur for $\lambda < \lambda_C$ and $\lambda \geq \lambda_C$ respectively. These two particular kinds of dynamic response are shown in Fig. 8.

5) The most interesting case of dynamic behaviour is the one where asymmetric branching points are present. When these are not accompanied by limit points, for $\lambda < \lambda_C$ there is a point attractor at the origin, for $\lambda = \lambda_C$ a point attractor on the (only at this loading level existing) stable branch of the primary path corresponding to negative values of ϑ; for $\lambda > \lambda_C$ the motion is as expected attracted by stable equilibrium positions having positive ϑ.

   On the other hand, when a limit point S is also present (of course $\lambda_C < \lambda_S$) the dynamic response is as follows:

(1) $0 < \lambda < \lambda_C$: point attractor at the origin.
(2) $\lambda = \lambda_C$: point attractor with negative ϑ.
(3) $\lambda_C < \lambda < \lambda_{DD}$: point attractor on the primary stable equilibrium path.
(4) $\lambda \geq \lambda_{DD}$: escaped motion, leading to a point attractor on a remote stable equilibrium position belonging to complementary paths.

The latter characteristic response cases are presented graphically in Fig. 9.

4. Conclusions

The most important findings drawn from this study are:

1) The proposed model for suitable combinations of the foregoing parameters can very conveniently represent all kinds of distinct critical points.
2) The corresponding dynamic response is always stable (either locally or globally), and the escaped motion initiated through saddle points is always attracted by remote stable equilibria.
3) Complementary paths, and more specifically their stable branches, may also act as attractors.

as previously reported for multi-degree-of-freedom systems.

4) Finally all asymmetric points of bifurcation observed are unstable, although V_T and its two first derivatives are at these points equal to zero, and thus the motion does not return to the trivial state.

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