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Braced, partially braced and unbraced columns: complete set of classical stability equations

J. Dario Aristizabal-Ochoa†

School of Mines, National University, Medellin, Colombia

Abstract. Stability equations that evaluate the elastic critical axial load of columns in any type of construction with sidesway uninhibited, partially inhibited, and totally inhibited are derived in a classical manner. These equations can be applied to the stability of frames (unbraced, partially braced, and totally braced) with rigid, semirigid, and simple connections. The complete column classification and the corresponding three stability equations overcome the limitations and paradoxes of the well known alignment charts for braced and unbraced columns and frames. Simple criteria are presented that define the concept of partially braced columns and frames, as well as the minimum lateral bracing required by columns and frames to achieve non-sway buckling mode. Various examples are presented in detail that demonstrate the effectiveness and accuracy of the complete set of stability equations.

Key words: buckling; building codes; construction type; columns; computer applications; design; frames; loads; stability.

1. Introduction

The elastic stability equations of braced and unbraced columns are presented and discussed in the technical literature on structural design (Salmon and Johnson 1980, Chapter 14). Both transcendental equations and their corresponding alignment charts have been endorsed by most construction codes (ACI 1992, AISC-ASD 1989, AISC-LRFD 1993). Since the inception of the alignment charts in the design process of columns and frames (Kavanagh 1962), the case of partially braced columns and frames has not been mentioned in the technical literature nor in the construction codes. Because of their paradoxical results in nonsymmetrical frames and in frames with leaning columns, the validity of the alignment charts' K-factor has been questioned (Cheong-Siat-Moy 1986, 1991). Recently, the writer has proposed an approximate and nonparadoxical approach for the stability analysis and calculation of the effective length K-factor including the case of partially braced columns and frames of any type of construction (1994a-c, 1995).

The main objective of this publication is to present the complete set of classical stability equations for columns with semirigid connections, including the case of partially braced columns. It will be demonstrated that the stability equation of the partially braced column has been the "missing link" to fully understand the elastic stability behavior of columns and frames, and is the key to solve the paradoxes of the current alignment charts. As a consequence, the complete set of three stability equations, not only is more general than the current set of two stability equations (or the corresponding alignment charts), but also avoids any paradoxical results.

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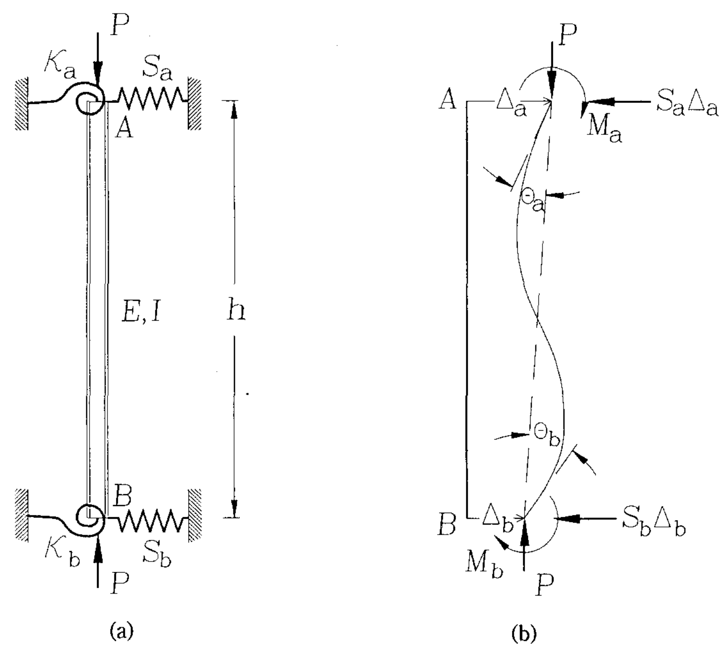

Fig. 1 Column with sidesway partially inhibited and with rotational and lateral end restraints.
a) Structural model;
b) End moments, forces, rotations and deflections

Their utilization will be illustrated with various examples, and criteria for minimum bracing required by frames to achieve non-sway buckling mode are presented.

2. Structural model

Assumptions. Consider a prismatic element that connects points A and B as shown in Fig. 1a. The element is made up of the column itself AB, the lumped flexural connectors κ_a and κ_b, and the lateral shear connectors S_a and S_b at the top and bottom ends, respectively. It is assumed that:

1) the column AB is made of a homogeneous linear elastic material with a modulus of elasticity E;
2) the centroidal axis of the member is a straight line;
3) the column is loaded with an end axial load P applied along the centroidal axis of the cross-section of area A and principal moments of inertia I_x and I_y; and
4) deformations are small so that the principle of superposition can be applied.

The lumped flexural connectors have stiffnesses κ_a and κ_b (whose units are in force-distance/radian), respectively. The units of lateral shear connectors S_a and S_b are in force/distance. The ratios $R_a = \kappa_a/(EI/h)$ and $R_b = \kappa_b/(EI/h)$ are denoted as the stiffness indices of the flexural connections. Where I = the column moment of inertia about the principal axis in question, and h = the column height. These indices vary from zero(i.e., $R_a = R_b = 0$) for simple connections (i.e., pinned) to infinity (i.e., $R_a = R_b = \infty$) for fully restrained connections (i.e., rigid). It is important to note that the proposed algorithm can be utilized in the inelastic analysis of framed structures when the nonlinear behavior is concentrated at the connections. This can be carried out by updating the flexural stiffness of the connections κ_a and κ_b for each load increment in a linear-incremental fashion. Gerstle (1988) has indicated lower and upper bounds for κ_a and κ_b. More recently, Xu and Grierson (1993) used these bounds in the design of frames with semirigid connections. Also Chen and Kishi (1989), Chen and Lui (1991) have presented data and modeling of semirigid connections in the stability of steel structures.

For convenience the following two parameters are introduced (Aristizabal-Ochoa 1994a):

$$\rho_a = \frac{1}{1+\dfrac{3}{R_a}} \tag{1a}$$

and

$$\rho_b = \frac{1}{1+\dfrac{3}{R_b}} \tag{1b}$$

where ρ_a and ρ_b are called the fixity factors. For hinged connnections, both the fixity factor ρ and the rigidity index R are zero; but for rigid connections, the fixity factor is 1 and the rigidity index is infinity. Since fixity factor can only vary from 0% to 100% (while the rigidity index R may vary form 0 to ∞), it is more convenient to use in the analysis of structures with semirigid connections (Cunningham 1990, Xu and Grierson 1993).

The relationships between the fixity factors ρ_a, ρ_b and the alignment charts ratios ψ_a and ψ_b [i.e., $\psi=\sum(EI/h)_c/\sum(EI/L)_g$ at the top and bottom ends, respectively] of a column in a symmetrical rigid frame with sidesway uninhibited or partially inhibited are: $\rho_a=2/(2+\psi_a)$, and $\rho_b=2/(2+\psi_b)$ (Aristizabal-Ochoa 1994a). In symmetrical rigid frames with sidesway totally inhibited, the relationships are: $\rho_a=2/(2+3\psi_a)$, and $\rho_b=2/(2+3\psi_b)$. In symmetrical frames with semirigid beam-to-column connections, the fixity factors can be determined using structural principles as shown by the writer (1994a). Finally, for unsymmetrical frames with semirigid connections, the fixity factors can be determined using structural principles as shown by *example 2* of this publication.

2.1. Stability equations and the effective length K-factor

Stability criteria. In a frame with sidesway uninhibited or partially inhibited every column is defined as having reached its critical load when sidesway buckling of the frame occurs, with the distribution of load among the columns being as specified. The effective length K-factor of each column of the frame, then, is defined as that value that yields the appropriate critical load when applied to the classic Euler's formula, $\pi^2 EI/(Kh)^2$. The K-factor so obtained for each column must be greater than that calculated for the same column but assuming the frame with sidesway inhibited. Thus, the effective length K-factor would be infinity for unloaded columns, since critical load, as defined, is zero for those columns. Obviously, in this concept, the K-factor of each column is a function of its own properties, the properties of the entire frame (support and bracing conditions), and the distribution of load among the columns in the frame. The complete set of stability equations are included at the bottom of each case for easy reference

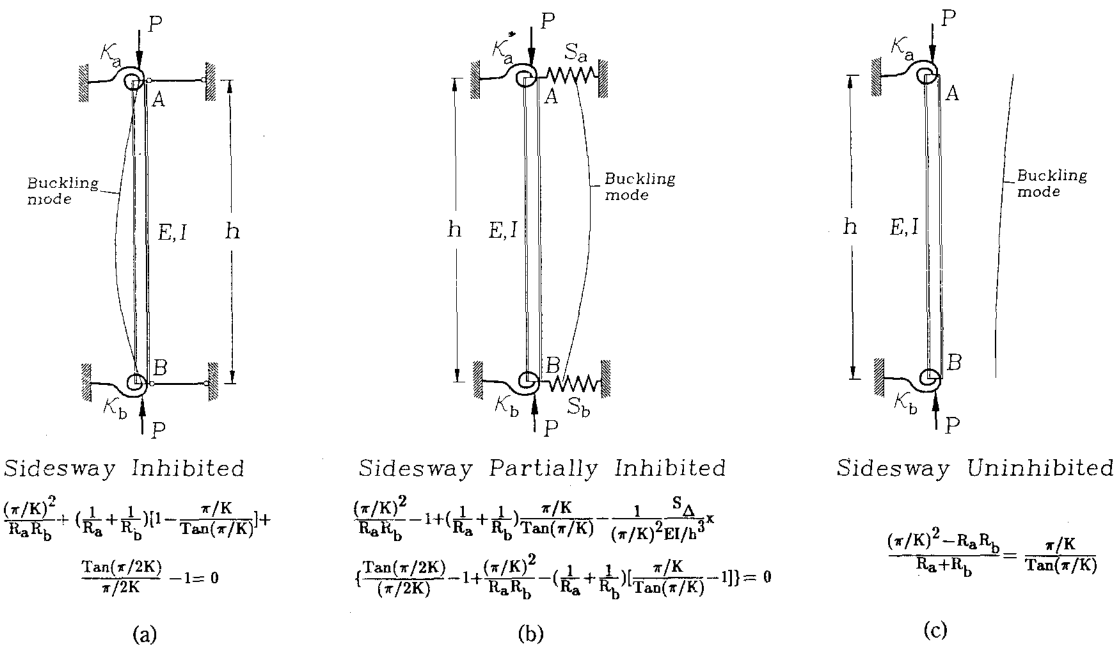

Fig. 2 Column classification and corresponding stability equation.
a) Column with sidesway totally inhibited;
b) Column with sidesway partially inhibited;
c) Column with sidesway uninhibited

and convenience (Figs. 2a-c). The stability equations are discussed and derived in the next sections.

2.1.1. Columns with sidesway totally inhibited

For columns in which the lateral sway between the two ends A and B is totally inhibited (Fig. 2a), the stability equation in terms of the stiffness indices is as follows:

$$\frac{(\pi/K)^2}{R_aR_b}+\left(\frac{1}{R_a}+\frac{1}{R_b}\right)\left[1-\frac{\pi/K}{\mathrm{Tan}(\pi/K)}\right]+\frac{\mathrm{Tan}(\pi/2K)}{\pi/2K}-1=0 \tag{2a}$$

which in terms of the fixity factors Eqs. (1a-b) into (2a) becomes:

$$(1-\rho_a)(1-\rho_b)(\pi/K)^2+3(\rho_a+\rho_b-2\rho_a\rho_b)\left[1-\frac{\pi/K}{\mathrm{Tan}(\pi/K)}\right]+9\rho_a\rho_b\left[\frac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)}-1\right]=0 \tag{2b}$$

Eq. (2a) is the classical stability equation for braced columns (Salmon and Johnson 1980).

2.1.2. Columns with sidesway partially inhibited

For columns in which the lateral sway between the two ends is partially inhibited by springs S_a and S_b (Figs. 1a or 2b), Eq. (3a) is proposed by the author:

$$\frac{(\pi/K)^2}{R_a R_b} - 1 + \left(\frac{1}{R_a} + \frac{1}{R_b}\right)\frac{\pi/K}{\mathrm{Tan}(\pi/K)} - \frac{1}{(\pi/K)^2}\frac{S_\Delta}{EI/h^3} \times$$

$$\left\{\frac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)} - 1 + \frac{(\pi/K)^2}{R_a R_b} - \left(\frac{1}{R_a} + \frac{1}{R_b}\right)\left[\frac{\pi/K}{\mathrm{Tan}(\pi/K)} - 1\right]\right\} = 0 \tag{3a}$$

which in terms of the fixity factors Eq. (1a−b) into Eq. (3a) becomes:

$$(1-\rho_a)(1-\rho_b)(\pi/K)^2 - 9\rho_a\rho_b - 3(\rho_a+\rho_b-2\rho_a\rho_b)\frac{\pi/K}{\mathrm{Tan}(\pi/K)} - \frac{1}{(\pi/K)^2}\frac{S_\Delta}{EI/h^3} \times \tag{3b}$$

$$\left\{9\rho_a\rho_b\left[\frac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)} - 1\right] + (1-\rho_a)(1-\rho_b)(\pi/K)^2 - 3(\rho_a+\rho_b-2\rho_a\rho_b)\left[\frac{\pi/K}{\mathrm{Tan}(\pi/K)} - 1\right]\right\} = 0 \tag{3b}$$

Where $1/S_\Delta = 1/S_a + 1/S_b$.

2.1.3. Columns with sidesway uninhibited

For columns in which the lateral sway between the two ends is uninhibited (Fig. 2c), the stability equation in terms of the stiffness indices is as follows:

$$\frac{(\pi/K)^2 - R_a R_b}{R_a + R_b} = \frac{\pi/K}{\mathrm{Tan}(\pi/K)} \tag{4a}$$

which in terms of the fixity factors Eq. (1a-b) into Eq. (3a) becomes:

$$\frac{(1-\rho_a)(1-\rho_b)(\pi/K)^2 - 9\rho_a\rho_b}{3(1+\rho_a+\rho_b-2\rho_a\rho_b)} = \frac{\pi/K}{\mathrm{Tan}(\pi/K)} \tag{4b}$$

Eq. (4a) is the classical stability equation for unbraced columns (Salmon and Johnson 1980) and is a particular case of Eq. (3a) when $S_\Delta = 0$. The derivations of Eq. (3a) and Eq. (3b) are presented next.

3. Derivation of the classical stability equations

The classical stability equations for a prismatic column in Fig. 1b are easier to formulate using the flexibility coefficients (Salmon and Johnson 1980) as follows:

$$\theta_a = \frac{M_a h}{EI}\frac{\mathrm{Sin}\phi - \phi\mathrm{Cos}\phi}{\phi^2\mathrm{Sin}\phi} + \frac{M_b h}{EI}\frac{\mathrm{Sin}\phi - \phi}{\phi^2\mathrm{Sin}\phi} = -\frac{\Delta_b - \Delta_a}{h} - \frac{M_a}{\kappa_a} \tag{5a}$$

$$\theta_b = \frac{M_b h}{EI}\frac{\mathrm{Sin}\phi - \phi\mathrm{Cos}\phi}{\phi^2\mathrm{Sin}\phi} + \frac{M_a h}{EI}\frac{\mathrm{Sin}\phi - \phi}{\phi^2\mathrm{Sin}\phi} = -\frac{\Delta_b - \Delta_a}{h} - \frac{M_b}{\kappa_b} \tag{5b}$$

Where: θ_a and θ_b = total rotations at A and B, respectively.
Δ_a and Δ_b = total lateral sway at A and B, respectively.

$$\phi = \frac{\pi}{K}$$

$$\Delta = \Delta_b - \Delta_a$$

Since four unknowns (M_a, M_b, Δ_a, and Δ_b) are involved, two more equations are required. These can be obtained applying static equilibrium (horizontal and rotational equilibrium of the column in Fig. 1b) as follows:

$$S_a\Delta_a + S_b\Delta_b = 0 \tag{6a}$$

$$M_a + M_b + P(\Delta_b - \Delta a) - S_b\Delta_b h = 0 \tag{6b}$$

Substituting Eq. (6a) into Eq. (6b) gives:

$$\frac{M_a}{h} + \frac{M_b}{h} + \left(P - \frac{S_a S_b}{S_a + S_b} h\right)(\Delta_b - \Delta_a)/h = 0 \tag{6c}$$

or simply

$$\frac{M_a}{h} + \frac{M_b}{h} + (P - S_\Delta h)(\Delta/h) = 0 \tag{7}$$

Where: $S_\Delta = \frac{S_a S_b}{S_a + S_b}$ = overall lateral stiffness provided to column AB;

$\Delta = \Delta_b - \Delta_a$ = relative lateral sway of B with respect to A; and

$P = \phi^2 \frac{EI}{h^2}$

Notice that the combined effect of the lateral shear connectors S_a and S_b is equivalent to a lateral spring in series S_Δ at either end of the column. The proper evaluation of S_Δ is vital to the stability analysis of framed structures. In general, there are three sources of S_Δ:

1) that provided by other columns that are part of the same story level from which column AB is located;
2) from diagonal bracings or shear walls within the same story level of the frame; and
3) from external bracing provided by other structures or structural elements connected to the column's top and bottom levels.

For instance, the lateral stiffness provided by a single column under axial compressive load N can be approximated by Eq. (8).

$$S_\Delta = \frac{12EI}{h^3}\frac{\rho_a + \rho_b + \rho_a\rho_b}{(4 - \rho_a\rho_b)} - \frac{2N}{5(4 - \rho_a\rho_b)^2}\frac{1}{h}[40 + 8(\rho_a^2 + \rho_b^2) + \rho_a\rho_b(\rho_a + \rho_b + 3\rho_a\rho_b - 34)] \tag{8}$$

Eq. (8) is derived by Aristizabal-Ochoa (1994a). The spring constant S_Δ provided by a single diagonal bracing of cross-sectional area A_b, horizontal length L and height h is as follows:

$$S_\Delta = \frac{A_b E L^2/h^3}{[1 + (L/h)^2]^{3/2}} \tag{9}$$

Eq. (9) is derived by Salmon & Johnston (1980). The utilization of Eq. (8) will be demonstrated in the next section of comparative examples.

Now, Eq. (5) and Eq. (7) can be represented in matrix form as follows:

$$\begin{bmatrix} \frac{h}{EI}\left(\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_a h}\right) & \frac{h}{EI}\;\frac{\text{Sin}\phi-\phi}{\phi^2\text{Sin}\phi} & 1 \\ \frac{h}{EI}\;\frac{\text{Sin}\phi-\phi}{\phi^2\text{Sin}\phi} & \frac{h}{EI}\left(\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_b h}\right) & 1 \\ \frac{1}{h} & \frac{1}{h} & \phi^2\frac{EI}{h^2}-S_\Delta h \end{bmatrix} \begin{bmatrix} M_a \\ M_b \\ \Delta/h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

Eq. (10) indicates that there are three major buckling modes in the elastic range of columns, they are:

1) buckling with relative sidesway totally inhibited (i.e., $\Delta=0$) corresponding to what is commonly referred to as "braced" columns;
2) buckling with relative sidesway partially inhibited (i.e., $\Delta\neq0$ and $S_\Delta>0$) corresponding to "partially braced" columns; and
3) buckling with sidesway totally uninhibited (i.e., $\Delta\neq0$ and $S_\Delta=0$) corresponding to what is commonly referred to as "unbraced" columns.

The current classical "braced" and "unbraced" cases, and the corresponding stability Eqs. (2) and (4), respectively and alignment charts are presented in the technical literature and are endorsed by most construction codes (Salmon and Johnson 1980, ACI 1992, AISC-ASD 1989, AISC-LRFD versions 1986 and 1993). However, partially braced columns were first introduced by the writer and approximate solutions for the effective length K-factor were proposed (Aristizabal-Ochoa 1994a-c).

For columns with sidesway inhibited (i.e., $\Delta=0$), the stability equation is obtained by setting the determinant of the coefficients of the first two rows and columns in Eq. (10) equal to zero follows:

$$\left(\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_a h}\right)\left(\frac{Sin\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_b h}\right)-\left(\frac{Sin\phi-\phi}{\phi^2\text{Sin}\phi}\right)^2=0 \tag{11}$$

which may be simplified to become Eq. (2a). This buckling mode is depicted in Fig. 2a

For columns with sidesway partially inhibited or totally uninhibited (i.e., $\Delta\neq0$), there are no applied moments M_a, M_b and sidesway Δ can exist only after buckling occurs. Therefore, Eq. (10) may be satisfied when M_a, M_b and the relative sidesway Δ are zero (i.e., nobuckling) or the determinant of the coefficients must equal to zero.

$$(\phi^2\frac{EI}{h^2}-S_\Delta h)\left\{\left(\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_a h}\right)\left(\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_b h}\right)-\left(\frac{\text{Sin}\phi-\phi}{\phi^2\text{Sin}\,\phi}\right)^2\right\}-$$

$$\frac{1}{EI}\left[\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_a h}-\frac{\text{Sin}\phi-\phi}{\phi^2\text{Sin}\phi}+\frac{\text{Sin}\phi-\phi\text{Cos}\phi}{\phi^2\text{Sin}\phi}+\frac{EI}{\kappa_b h}-\frac{\text{Sin}\phi-\phi}{\phi^2\text{Sin}\phi}\right]=0 \tag{12}$$

which may be simplified to become Eq. (3a). These two buckling modes are depicted in Fig. 2b-c, respectively.

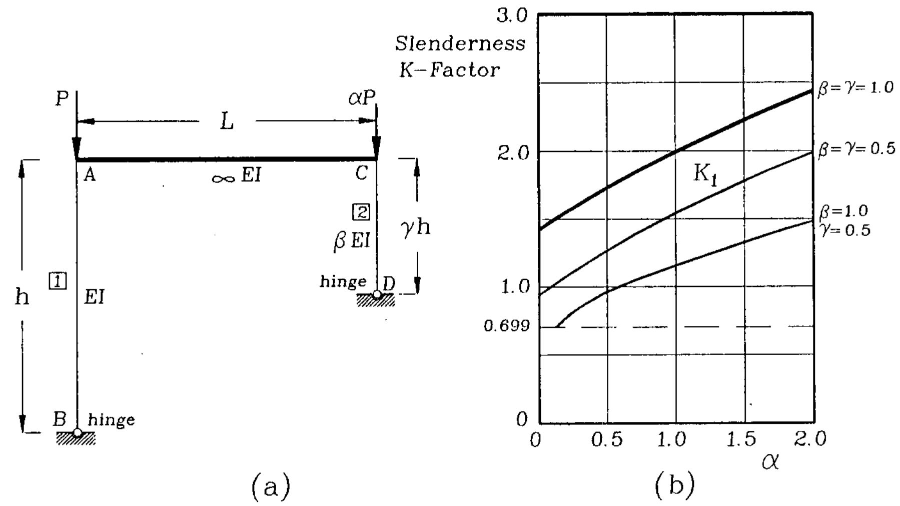

Fig. 3 Example 1: Rigidly connected frame.
a) Structural model;
b) Variation of the effective length K-factors with α

4. Comparative examples

4.1. Example 1: Rigidly connected frame

Consider the simple asymmetrical frame shown in Fig. 3a. To simplify calculations and for convenience, assume that the girder's flexural stiffness EI/L is infinitely larger than that of the columns. Determine using Fig. 3(b) the variation of the effective length K-factor of column AB and CD with α, β, and γ and compare the results with those presented by Cheong Siat-Moy (1986) for the particular case of $\beta=\gamma=1$ and $\alpha=1$, 0.25, and 0.

Solution: This frame appears to be a frame with sidesway-buckling uninhibited, however, this classification needs to be checked first. Assuming that column CD is partially restraining column AB, the K-factor of column AB must be determined first from Eq. (3b) taking into consideration that: $\rho_a=1.0$, $\rho_b=0$ for column AB, and $\rho_c=1.0$, $\rho_d=0$ for column CD, and from Eq. (8), $S_\Delta=\frac{3\beta EI}{(\gamma h)^3}-\frac{6}{5}\frac{\alpha P_{cr}}{\gamma h}$. Where the term $\frac{3\beta EI}{(\gamma h)^3}$ represents the lateral stiffness provided by column CD on column AB, and the term $\frac{6}{5}\frac{\alpha P_{cr}}{\gamma h}$ the approximate reduction (i.e., the geometric stiffness of column AB) caused by its compressive axial load αP_{cr}. Therefore, using Eq. (3b):

$$\frac{\pi/K}{\mathrm{Tan}(\pi/K)}-\frac{1}{(\pi/K)^2}\frac{S_\Delta}{EI/h^3}\left[\frac{\pi/K}{\mathrm{Tan}(\pi/K)}-1\right]=0 \tag{13}$$

which after substituting $S_\Delta=\frac{3\beta EI}{(\gamma h)^3}-\frac{6}{5}\frac{\alpha P_{cr}}{\gamma h}$ into Eq. (13), the stability equation of the frame in terms of the effective length K-factor of column AB can be approximated by Eq. (14).

$$\frac{\pi/K}{\mathrm{Tan}(\pi/K)}-3\left[\frac{\beta}{\gamma^3(\pi/K)^2}-\frac{2\alpha}{5\gamma}\right]\left[\frac{\pi/K}{\mathrm{Tan}(\pi/K)}-1\right]=0 \tag{14}$$

where $\alpha=P_2/P_1$
$\beta=(EI)_2/(EI)_1$
$\gamma=h_2/h_1$

The solution of Eq. (14) [the variation of the K-Factor with α] is presented in Fig. 3b for three particular cases: a) $\beta=\gamma=1$; b) $\beta=1$, $\gamma=0.5$; and c) $\beta=\gamma=0.5$. The K-factor of column CD can be obtained from the condition $P_2=\alpha P_1$ or $K_2^2=K_1^2\beta/(\alpha\gamma^2)$.

The solutions shown in Fig. 3b for case a) of: $K_1=2.0$, 1.585 and 1.425; $K_2=2.0$, 3.170 and ∞ for $\alpha=1$, 0.25 and 0, respectively, compare very well with those reported by Cheong Siat-Moy (1986) and Aristizabal-Ochoa (1994a). Notice that the K-factors of both columns vary, not only with the end flexural restraints (as it is usually specified by most construction codes), but with α, β and γ. That is, all columns in a particular story of a frame are coupled together and the stability of the entire story depends on:

1) the end flexural restraints of each column;
2) the load distribution among the columns;
3) the flexural stiffness of each column;
4) the unsupported lengths of each column that make up the story under consideration; and
5) coupling effect by which the stronger columns (i.e., those with large EI/h and strong end flexural restraints) and columns with low axial load provide lateral support or bracing to those heavily loaded and with soft end flexural restraints.

Similar procedure can be carried out for the frame shown in Fig. 4a (which is similar to that of Fig. 3. but with support D fixed). The stability equation of this frame in terms of the effective length K-factor of column AB can be approximated by Eq. (15).

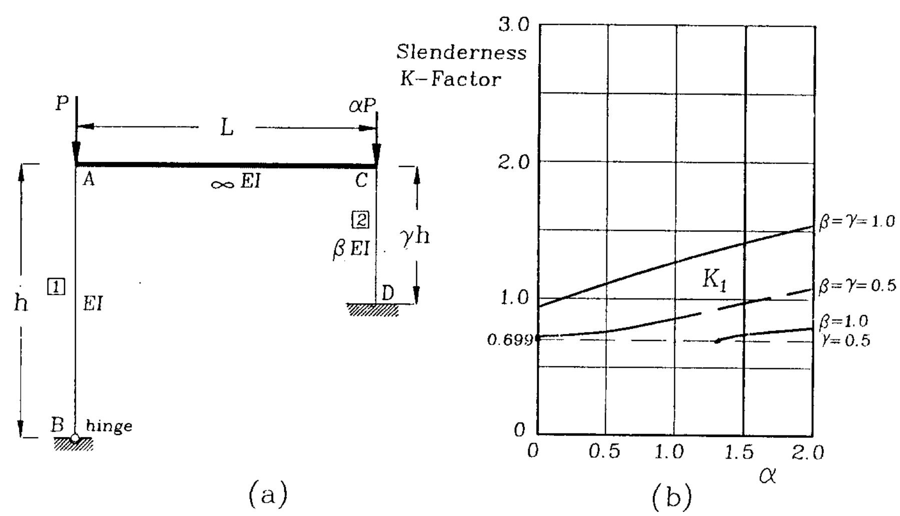

Fig. 4 Example 1 (Modified): Rigidly connected frame.
a) Structural model;
b) Variation of the effective length K-factors with α

$$\frac{\pi/K}{\operatorname{Tan}(\pi/K)}-3\left[\frac{4\beta}{\gamma^3(\pi/K)^2}-\frac{2\alpha}{5\gamma}\right]\left[\frac{\pi/K}{\operatorname{Tan}(\pi/K)}-1\right]=0 \tag{15}$$

The solution of Eq. (15) is presented in Fig. 4b for the same three cases: a) $\beta=\gamma=1$; b) $\beta=1$, $\gamma=0.5$; and c) $\beta=\gamma=0.5$. Notice the lower limit of $K=0.699$ (corresponding to a pinned-clamped column with sidesway inhibited) affects cases b) and c). Studying these two cases of this particular frame, one finds that stability analyses based on the current alignment charts give unsafe designs. For instance, consider case b) with identical axial loads in both columns (i.e., $\alpha=1$). From Fig. 4b the frame will become unstable by individual buckling of column AB without sidesway (i.e., $K=0.699$); therefore, $(P_1)_{cr}=(P_2)_{cr}=\pi^2 EI/(0.699h)^2=2.0467\pi^2 EI/h^2$. Had α been greater than 1.35, this frame case would have been classified as a frame with sidesway-buckling uninhibited (because $K>0.699$). Notice that $K<1.0$ in some ranges of α in all three frame cases under consideration.

It is interesting to note that utilizing the current alignment chart for unbraced columns the designer would obtain $K=2.0$ ($P_{1cr}=\pi^2 EI/4h^2$) and $K=1.0$($P_{2cr}=4\pi^2 EI/h^2$) or columns AB and CD, respectively, with a "predicted" total critical load for the frame of $P_{1cr}+P_{2cr}=4.25\pi^2 EI/h^2$. This would result in an extremely conservative design for column AB (overdesigned by a factor of $4\times 2.0467=8.187$), but unconservative for column CD (underdesigned by a factor of $4/2.0467=1.95$), and an overall underdesigned frame (since its total critical load capacity of $2\times 2.0467\pi^2 EI/h^2 <4.25\pi^2 EI/h^2$ predicted by the alignment chart). What makes this situation even worse is that by reducing the applied load on column CD, the total buckling capacity of the frame is reduced from $4.75\pi^2 EI/h^2$ (for $\alpha=2.25$) to $2.0468\pi^2 EI/h^2$(for $\alpha=0$). This large variation in the total critical load capacity in unsymmetrical frames, like the one shown in Figs. 3-4, makes nonsymmetrical structural configurations susceptible to instability at much lower loads than that predicted by the current alignment charts.

These results also indicate that the classification of frames as braced or unbraced, based purely on, the frame's geometry is not correct. In addition, the calculation of the K-factors for the columns based on the current alignment charts K-factors should be avoided because it might result in deficient designs.

4.2. Example 2: Bent frame

Consider the one-story frame shown in Fig. 5a. Assume $L=h=12.192$ m (40 ft), $I=41.6231\times 10^{-4}$ m^4 (10,000 in^4) and $E=20{,}684{,}272$ KPa (3000 Ksi). Neglect axial elongations of all three members. This frame would be considered a frame with sidesway-buckling uninhibited by most designers with K-factors obtained from the alignment chart for "unbraced" frames as follows:

1. For column AB: $\psi_{top}=0.4$; and $\psi_{bottom}=0$: then $K_{AB}=1.10$
2. For column CD: $\psi_{top}=0.4$; and $\psi_{bottom}=\infty$: then $K_{CD}=2.10$

According to Eq. (3b), however, the fixity factors must be established first in order to determine the K-factors. This can easily be carried out by applying a unit horizontal load at node A (or C) (i.e., 1 Kip=4.448 KN) and finding the moments and rotations at joints A and C of the frame with sidesway uninhibited. This can be accomplished using a standard structural engineering analysis for plane frames (like moment distribution) with the following results:

1. For column AB: Rotation at $A=0.000624$ Radians; and Moment at $A=19.30$ KN-m (170.84 Kip-in)

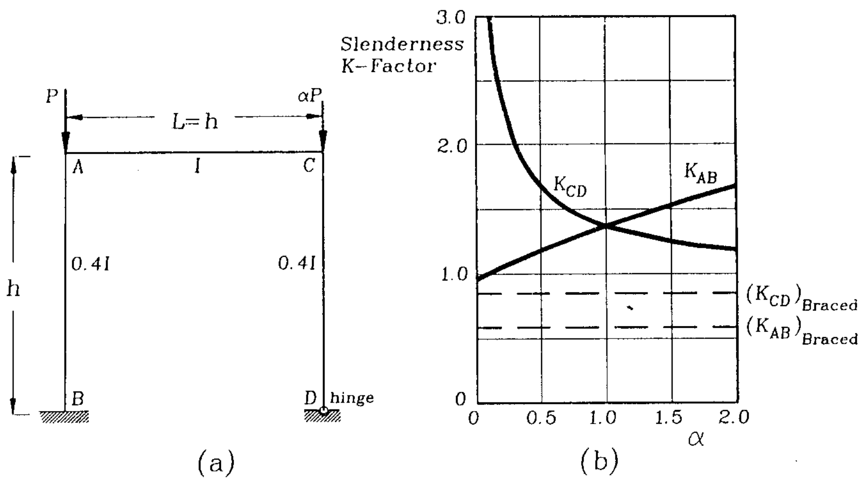

Fig. 5 Example 2: Bent frame.
a) Structural model;
b) Variation of the effective length K-factors with α

Therefore, the stiffness of the flexural connector at A is $\kappa_a = 19.30/0.000624 = 30{,}931$ KN-m/Radian; and the stiffness index at A is $R_a = \kappa_a/(EI/h)_{AB} = 30{,}931/(20{,}684{,}272 \times 0.4 \times 0.00416231/12.192) = 10.953$; then using Eq. (1a), the fixity factor at A becomes: $\rho_a = \dfrac{1}{1+3/10.953} = 0.785.$

The fixity factor at B: $\rho_b = \dfrac{1}{1+3/\infty} = 1.0$

2. For column CD: Rotation at $C = 0.000119$ Radians; and Moment at $C = 12.168$ KN-m (107,70 Kip-in). Therefore, the stiffness of the flexural connector at C is $\kappa_c = 12.168/0.000119 = 102{,}251$ KN-m/Radian; and the stiffness index at C is $R_c = \kappa_c/(EI/h)_{CD} = 102{,}251/(20{,}684{,}272 \times 0.4 \times 0.00416231/12.192) = 36.232$; then using Eq. (1a), the fixity factor at C becomes: $\rho_c = \dfrac{1}{1+3/36.232} = 0.924.$

   The fixity factor at D: $\rho_d = \dfrac{1}{1+3/0} = 0$

Once the fixity factors at the top and bottom of each column are determined and column AB is selected as being laterally restrained by column CD, the K-factor can be calculated using Eq. (3b) and Eq. (8) taking into consideration that: $S_\Delta = \dfrac{3 \times 0.4EI}{h^3}\rho_d - \dfrac{\alpha P_{cr}}{h}\left(1+\dfrac{\rho_d^2}{5}\right)$

The variation of the K-factor of both columns with α is shown in Fig. 5b. Each of these two K values has its own lower limit as indicated by the broken lines. These lower limits are obtained from Eq. (2b) assuming that the frame is "braced" (i.e., with sidesway totally inhibited at the top). To calculate these two limits, the fixity factors for each column must be determined again assuming that the frame is "braced" (i.e., zero sidesway at the top joints A and C). An additional structural analysis was carried out with a unit moment at C (1 Kip-in = 0.11298 KN-m), and the frame restrained along the horizontal direction AC at the top, yielding the following results:

1. For column AB: Rotation at $A=3.312\times10^{-6}$ Radians; and Moment at $A=0.03742$ KN-m (0.3312 Kip-in)
   Therefore, the stiffness of the flexural connector at A is $\kappa_a=0.03742/(3.312\times10^{-6})=11.298$ KN-m/Radian; and the stiffness index at A is $R_a=\kappa_a/(EI/h)_{AB}=11{,}298/(20{,}684{,}272\times0.4\times0.00416231/12.192)=4$; then using Eq. (1a), the fixity factor at A becomes: $\rho_a=\dfrac{1}{1+3/4}=4/7$. The fixity factor at B: $\rho_b=\dfrac{1}{1+3/\infty}=1.0$
   Therefore, using Eq. (2b):
$$\left[1-\frac{\pi/K}{\mathrm{Tan}(\pi/K)}\right]+4\left[\frac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)}-1\right]=0 \tag{16}$$
   Whose solution is: $(K_{AB})_{\text{with sway totally inhibited}}=0.5896$
2. For column CD: Rotation at $C=1.2733\times10^{-6}$ Radians; and Moment at $C=0.01079$ KN-m (0.0955 Kip-in).
   Therefore, the stiffness of the flexural connector at C is $\kappa_c=0.01079/(1.2733\times10^{-6})=8{,}474.1$ KN-m/Radian; and the stiffness index at C is $R_c=\kappa_c(EI/h)_{CD}=847.410/(20{,}684{,}272\times0.4\times0.00416231/12.192)=3$; then using Eq. (1a), the fixity factor at C becomes: $\rho_c=\dfrac{1}{1+3/3}=0.5$. The fixity factor at D: $\rho_d=\dfrac{1}{1+3/0}=0$
   Therefore, using Eq. (2b):
$$(\pi/K)^2+3\left[1-\frac{\pi/K}{\mathrm{Tan}(\pi/K)}\right]=0 \tag{17}$$
   Whose solution is: $(K_{CD})_{\text{with sway totally inhibited}}=0.8431$

The values of K_{AB} and K_{CD} and their lower limits are plotted in Fig. 5b. It can be concluded that: 1) this particular frame will buckle with sidesway under any compressive axial load combination (i.e., any α-value); 2) if the designer utilizes the alignment charts, columns AB and CD would be underdesigned for load combinations corresponding to $\alpha>0.3$ and $\alpha<0.25$, respectively. Now, by reducing the height of column CD to $0.5h$ as shown in Fig. 6a and maintaining everything equal, the K-factor of the short column CD is increased substantially, whereas that of the long column AB is reduced significantly as shown in Fig. 6b. Notice that the $K_{AB}<1$ for values of $\alpha<0.687$. These results, again, indicate that is unsafe to design unsymmetrical structures using the current alignment charts with the short columns being underdesigned and the long ones overdesigned. It is wrong to assume that each column has its own critical load. In reality, all columns are coupled in the stability of framed structures.

5. Partially braced columns and minimum lateral bracing

5.1. Partially braced column criterion

A partially braced column is one whose effective length K-factor lies between $K_{\text{Unbraced Column}}$ and $K_{\text{Totally Braced Column}}$ or simply:

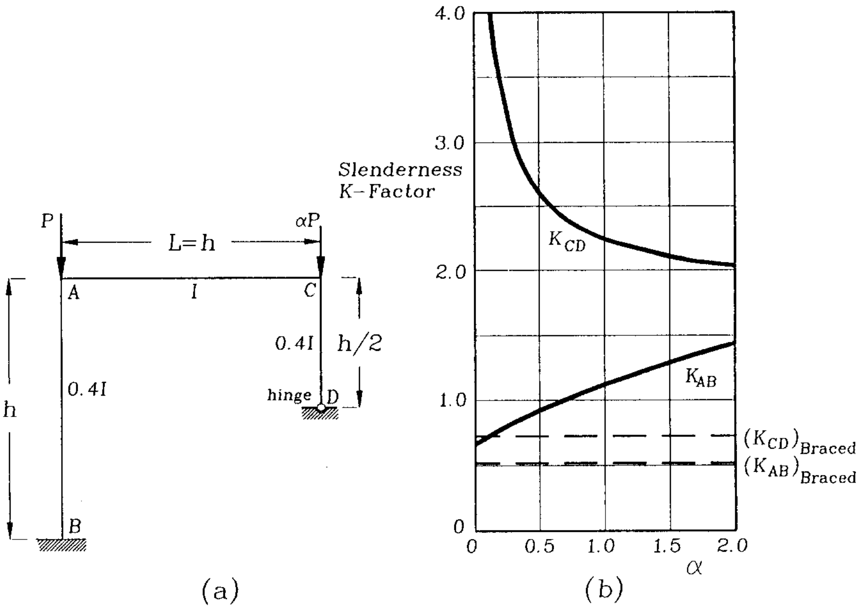

Fig. 6 Example 2 (Modified); Asymmetrical bent frame.
a) Structural model;
b) Variation of the effective length K-factors with α

$$\underset{\text{(from Eq. 2)}}{K_{\text{Totally Braced Column}}} \leq \underset{\text{(from Eq. 3)}}{K_{\text{Partially Braced Column}}} \leq \underset{\text{(from Eq. 4)}}{K_{\text{Unbraced Column}}} \tag{18}$$

This criterion is simple to apply and indicates that the K-factor of a partially braced column might be less than 1 but never less than that of the same columm but with sidesway totally inhibited. The lower bound of K for any column with no intermediate lateral support between its ends is 0.5 (this corresponds to a braced column with both ends clamped).

5.2. Minimum bracing criterion

The minimum bracing required to convert a frame with sidesway uninhibited or partially inhibited into a fully braced frame can be determined utilizing Eq. (18) by comparing Eqs. (2) and (3).

$$\underset{\text{(from Eq. 2)}}{K_{\text{Totally Braced Column}}} = \underset{\text{(from Eq. 3)}}{K_{\text{Partially Braced Column}}} \tag{19}$$

By combining Eqs. (2b) and (3b), for instance, the required S_Δ can be determined directly following the steps described below:

1) The fixity factors ρ_a and ρ_b must determined for both conditions braced and unbraced, as it was done in the frame of *Example 2;*
2) The K-factor for braced conditions is calculated from Eq. (2b) [utilizing, of course, the fixity factors ρ_a and ρ_b for the braced case];
3) The braced K-factor and ρ_a and ρ_b for unbraced conditions previously calculated are sub-

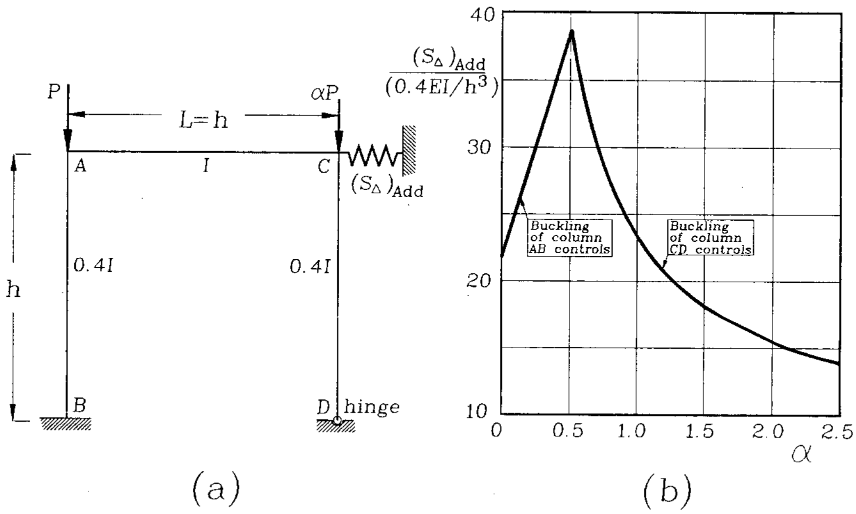

Fig. 7 Example 3: Minimum lateral bracing.
a) Structural model;
b) Variation of the minimum lateral bracing with α

stituted into Eq. (3b) from which the required minimum bracing S_Δ can be calculated directly as follows:

$$\frac{S_\Delta}{EI/h^3} = \frac{(\pi/K)^2\left\{(1-\rho_a)(1-\rho_b)(\pi/K)^2-9\rho_a\rho_b-3(\rho_a+\rho_b-2\rho_a\rho_b)\dfrac{\pi/K}{\mathrm{Tan}(\pi/K)}\right\}}{9\rho_a\rho_b\left[\dfrac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)}-1\right]+(1-\rho_a)(1-\rho_b)(\pi/K)^2-3(\rho_a+\rho_b-2\rho_a\rho_b)\left[\dfrac{\pi K}{\mathrm{Tan}(\pi/K)}-1\right]} \tag{20}$$

An example describing the steps for the calculation of $(S_\Delta)_{min}$. in a bent frame is presented below.

5.3. Example 3: Minimum lateral bracing

Utilizing Eqs. (3b), (8) and (20) determine the minimum bracing required to convert the bent frame of *Example 2* case a) shown in Fig. 7a into a fully braced frame for any value of α.

Solution: Assuming that column *CD* is restraining column *AB* laterally, and taking into consideration that: 1) $\rho_a=0.785$, $\rho_b=1$ for column *AB* and $\rho_c=0$, and $\rho_d=0.9235$ for column *CD* under unbraced conditions; and 2) $K=0.5896$ and 0.8431 for columns *AB* and *CD*, respectively, for braced conditions (already calculated in *Example 2*), then the required minimum bracing must be provided by at least two sources. First, by column *CD* itself [$(S_\Delta)_{CD}$ which can be obtained approximately from Eq. (8)], and second by an additional element or elements (like diagonal bracing) which will be denoted as to $(S_\Delta)_{add}$. Therefore:

$$S_\Delta=(S_\Delta)_{CD}+(S_\Delta)_{add} \tag{21}$$

From Eq. (8):

$$(S\Delta)_{CD}=\frac{12\beta EI}{(\gamma h)^3}\,\frac{\rho_c+\rho_d+\rho_c\rho_d}{(4-\rho_c\rho_d)}-\frac{2\alpha P}{5\gamma h(4-\rho_c\rho_d)^2}[40+8(\rho_c^2+\rho_d^2)+\rho_c\rho_d(\rho_c+\rho_d+3\rho_c\rho_d-34)] \quad (22)$$

or

$$\frac{(S\Delta)_{CD}}{EI/h^3}=\frac{12\beta}{\gamma^3}\,\frac{\rho_c+\rho_d+\rho_c\rho_d}{(4-\rho_c\rho_d)}-\frac{2\alpha(\pi/K)^2}{5\gamma(4-\rho_c\rho_d)^2}[40+8(\rho_c^2+\rho_d^2)+\rho_c\rho_d(\rho_c+\rho_d+3\rho_c\rho_d-34)] \quad (23)$$

$(S\Delta)_{add}$ can be estimated combining Eqs. (20), (21) and (23) as follows:

$$\frac{S\Delta_{add}}{EI/h^3}=\frac{(\pi/K)^2\left\{(1-\rho_a)(1-\rho_b)(\pi/K)^2-9\rho_a\rho_b-3(\rho_a+\rho_b-2\rho_a\rho_b)\dfrac{\pi/K}{\mathrm{Tan}(\pi/K)}\right\}}{9\rho_a\rho_b\left[\dfrac{\mathrm{Tan}(\pi/2K)}{(\pi/2K)}-1\right]+(1-\rho_a)(1-\rho_b)(\pi/K)^2-3(\rho_a+\rho_b-2\rho_a\rho_b)\left[\dfrac{\pi K}{\mathrm{Tan}(\pi/K)}-1\right]}-$$

$$\frac{12\beta}{\gamma^3}\,\frac{\rho_c+\rho_d+\rho_c\rho_d}{(4-\rho_c\rho_d)}+\frac{2\alpha(\pi/K)^2}{5\gamma(4-\rho_c\rho_d)^2}[40+8(\rho_c^2+\rho_d^2)+\rho_c\rho_d(\rho_c+\rho_d+3\rho_c\rho_d-34)] \quad (24)$$

Eq. (24) represents the approximate solution for the minimum amount of lateral bracing required by a bend frame of any geometry, member sizes, support condition, and axial loadings in the columns. For the particular frame of Fig. 7a case a) for which $\beta=\gamma=1$, $h=L$, and $I_{\text{columns}}=0.4I$ and any α, the variation of $\frac{(S\Delta)_{add}}{EI/h^3}$ with α is shown in Fig. 7b.

Conclusion: Eq. (24) indicates the required additional bracing is a funtion of the degress of fixity of the columns (ρ's), the load distribution (α), the ratio of the columns' flexural stiffness (β) and height (γ), and the K-factor of the column that is first to buckle under braced conditions. For instance, for the frame shown in Fig. 7a, from $\alpha=0$ to approximately 0.50 the buckling under braced conditions is controlled by buckling of column AB, and for $\alpha>0.50$ by buckling of column CD. To guarantee braced buckling under any axial loading combination (i.e., for any α value), then $\frac{(S\Delta)_{add}}{EI/h^3}$ must be greater than 38.971, which is is the minimum lateral bracing required to achieve simultaneous buckling in columns AB and CD (which occurs at $\alpha\simeq0.50$)

6. Summary and conclusions

The complete set of three stability equations by which the effective length K-factor of columns in any type of construction can be evaluated is presented in a classical fashion. To understand the threeway classification of columns and the corresponding stability equations, three well documented examples are presented and the results compared to other researchers' work. The proposed column classification and the complete set of transcendental equations are more general than the current two classical stability equations and the corresponding alignment charts for braced and unbraced columns and frames. Definite criteria is given to determine the minimum amount of lateral bracing required in frames to achieve nonswaying buckling.

The studies carried out in this research indicate that:

1) the current stability equations and their corresponding alignment charts are limited to sym-

metrical frames under symmetrical loading; and

2) to avoid absurd results in unsymmetrical frames, partially braced frames, and frames with semirigid connections the utilization of Eq. (3a) and (3b) is recommended. Examples are presented that demonstrate the misconceptions of the current methods based on the alignment chart for unbraced columns and the accuracy and effectiveness of the poposed three-way classification method.

Analytical studies indicate that the utilization of the current column classification and their alignment charts in the design of framed structures with columns of different heights gives faulty and unsafe designs. This is vital in the design of highway bridges (at interchanges) and buildings on slopes (or with different column heights). While the alignment chart for unbraced cases indicates that the long columns have the lowest critical load and the short columns have the largest, the proposed model indicates opposite behavior. For systems with sidesway uninhibited and partially inhibited, all columns in the system are coupled together. It is wrong to assume that in a frame structure with side sway each column has its own critical load independently from those of the rest of the columns. In unsymmetrical frames this assumption gives unsafe designs. Frames and columns should be classified according to the buckling mode and the provided interstory lateral bracing as proposed by the writer and not according to frame "looks" or engineering "judgement".

According to the proposed classification, columns under light axial loads the effective length K-factor become very large ($K=\infty$ when the applied axial load is zero). Therefore, the limit imposed by most construction codes to the slenderness ratio Kh/r to all vertical members is not realistic. In a framed structures the slenderness ratio Kh/r for the column under the largest axial load shall not exceed a specified limit. However, the rest of the columns of the story system should have limited ratio of (h/r), instead.

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Notations

The following symbols are used in this paper

A_b	area of the truss bracing member
E	Young's modulus of the material
ψ_a and ψ_b	$\Sigma(I_c/h_c)/\Sigma(I_g/I_g)$ at ends A and B of column AB, respectively
h	column's height
I_g	girder moment of inertia
I or I_c	column's moment of the inertia
K	effective length factor of the representative column
L	girder span
N	applied axial load
P_{cr}	column's buckling load $[=\pi^2EI/(Kh)^2]$
$(P_{cr})_{CD}$	buckling load of column $CD[=\alpha(P_{cr})]$
S_a and S_b	lateral stiffnesses or bracings provided to column at A and B, respectively
$S_\Delta=\frac{S_aS_b}{S_a+S_b}$	resultant lateral stiffness or bracing provided to column at either end
R_a	stiffness index of the flexural connection at A $[\kappa_a/(EI/h)]$
R_b	stiffness index of the flexural connection at B $[\kappa_b/(EI/h)]$
α	ratio of axial load of column CD to that of representative column AB $[P_{CD}/P_{AB}]$
β	ratio of flexural stiffness of column CD to that of representative column AB $[=(EI)_{CD}/(EI)_{AB}]$
γ	ratio of height of column CD to that of representative column AB $[=h_{CD}/h_{AB}]$
κ_a and κ_b	the flexural stiffness of the end connections at A and B, respectively
ρ_a and ρ_b	fixity factors at A and B of column AB, respectively