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Eigenvalue analysis of structures with flexible random connections

E. E. Matheu†

Department of Engineering Science and Mechanics, Virginia Tech., Blacksburg, VA 24061, U.S.A.

L. E. Suarez‡

Department of General Engineering, University of Puerto Rico, Mayagüez, PR 00681, U.S.A.

Abstract. A finite element model of a beam element with flexible connections is used to investigate the effect of the randomness in the stiffness values on the modal properties of the structural system. The linear behavior of the connections is described by a set of random fixity factors. The element mass and stiffness matrices are function of these random parameters. The associated eigenvalue problem leads to eigenvalues and eigenvectors which are also random variables. A second order perturbation technique is used for the solution of this random eigenproblem. Closed form expressions for the 1st and 2nd order derivatives of the element matrices with respect to the fixity factors are presented. The mean and the variance of the eigenvalues and vibration modes are obtained in terms of these derivatives. Two numerical examples are presented and the results are validated with those obtained by a Monte-Carlo simulation. It is found that an almost linear statistical relation exists between the eigenproperties and the stiffness of the connections.

Key words: modal analysis; random eigenvalue problem; flexible connections; finite flement modeling.

1. Introduction

The importance of the joint flexibility effects on the behavior of structural frameworks has been recognized for many years. This problem is relevant to diverse areas of structural engineering. In civil engineering for example, the commonly used methods of analysis and design of steel structures are based on the assumption that the member connections behave as pinned or perfectly rigid joints. However, early experimental studies on steel frames revealed that few real connections behave according to this assumption, and thus extensive research has been performed to characterize the behavior of flexible riveted and bolted beam-to-column connections. Besides the conventional steel building frames there are other engineering structures in which the flexibility of the joints can be important. For example, in mechanical engineering, it is important to assess the influence of body connection flexibility in vehicle structures which consist of irregular and complicated members connected by overlapping sheet metals fastened by spot welds. This is also the case in piping systems, in which in order to calculate more accurately the bending moment acting at the joints it may be necessary to take into account the flexible behavior

† Graduate Research Assistant
‡ Associate Professor

of the connections.

Although using test results it is possible to assign a value to the rotational stiffness of a given connection configuration, one cannot guarantee that this value will be the same for all the connections of the same type in a given structure. The values of the connection stiffness obtained from a test should only be regarded as the mean value of the stiffness of connections with the same configuration. There will always be a degree of dispersion due to the variations in the several parameters that control the connection behavior. Therefore, it is logical to consider the connection stiffness as a random variable in the structural model. In this case both the stiffness and the mass matrix will have random coefficients, since both are function of the connection stiffness. The finite element equations of motion of the structure become a set of ordinary differential equations with time-independent random coefficients. Although the equations of motion are random they are still linear and hence they could, at least in principle, be decoupled and solved by modal analysis. However, the eigenvalue problem associated with the equations of motion also becomes random, and the eigenvalues and eigenvectors of the system end up being random variables as well.

The solution of random eigenvalue problems is a challenging problem in applied mathematics that has attracted the attention of both mathematicians and engineers (e.g., Boyce 1968, Collins and Thomson 1969, Hasselman and Hart 1972, Shinozuka and Astill 1972, Hart 1973). There are several solution methods available, such as asymptotic methods (Boyce 1968), integral equation methods (Boyce-Goodwin 1964), hierarchy methods (Haines 1965), and perturbation methods. A rigorous treatment of the subject is presented in the monograph by Scheidt and Purkert 1983. They only used the perturbation technique as the solution method because, as they pointed out, the other methods can only be used on a limited scale.

One of the first studies on the solution of the eigenvalue problems with random matrices is due to Collins and Thomson (1969). They obtained expressions to define the differentials of the eigenvalues and eigenvector components. Based on these equations, they obtained the eigenvalue and eigenvector statistics as linearized expressions in terms of small variations about their mean values. The validity of the formulation was confirmed by comparing the results with a Monte Carlo simulation. Shinozuka and Astill (1972) obtained estimates of the variance of the n^{th} natural frequency of vibration of a beam-column with random properties using a Monte-Carlo simulation. The results obtained using the MCS were compared with the corresponding results using the perturbation method. It was found that the perturbation method provides a reasonable solution over a much wider range of variation of the material and geometric properties than would be found in practice. Hasselman and Hart (1972) presented a method for computing the variance of the eigenproperties of large structural systems using a modal synthesis technique. The method is based on a first-order perturbation approach. The effects of modal truncation on the accuracy of the modal statistics were investigated and the results showed that component mode synthesis can be effectively used to compute the mean values and standard deviations of the eigenproperties.

Most of the studies on random eigenproblems are limited to the calculations of the first moments of the eigenvalues and eigenvectors. The calculation of the probability distribution of the eigenproperties is a very difficult problem that has been solved only for a few simple cases (e.g., Scheidt-Purkert 1983, Iyengar-Manohar 1989). Besides the mathematical complexity of the problem, it requires considerable information about the joint probabilistic behavior of the random coefficients of the equations. Quite often, however, the only information available on the probabilistic structure of the coefficients is the first two moments, i.e., the mean and correlation functions. Nevertheless, in many important cases, for example for reliability analysis, engineering decisions are also based upon the first two moments of the solution process.

The objective of this paper is to study the random algebraic eigenvalue problem associated with a structural system with random flexible end connections. In the first part of the paper a succinct account of the finite element formulation of a beam element with flexible connections is presented. This is followed by the derivation of the closed form expression for the statistics of the eigenvalues and eigenvectors via a perturbation approach. The derivatives of the element matrices with respect to the parameters used to model the connection behavior are also provided. Numerical examples are presented to validate the formulation and to assess the effect of the inherent uncertainties in the values of the stiffness of the connections on the dynamic properties of the structures.

2. Finite element model of a beam with flexible end connections

A summary of the formulation of a finite element (FE) model of a beam element that incorporates the effects of the flexibility of the end connections is presented here. The details of the development in which the finite size of the connections is also included can be found in a previous work by the authors (Suarez-Matheu 1992).

In the dynamic finite element formulation of beam elements, the displacement field $w(x, t)$ is usually expressed in terms of a shape function vector $\boldsymbol{n}(x)$ and a vector of end displacements $\boldsymbol{u}(t)$:

$$w(x, t)=\boldsymbol{n}(x)^T\boldsymbol{u}(t)=\{n_1(x)\,n_2(x)\,n_3(x)\,n_4(x)\}\begin{Bmatrix} w_1(t) \\ \theta_1(t) \\ w_2(t) \\ \theta_2(t) \end{Bmatrix} \tag{1}$$

where $n_i(x)$ are the standard cubic polynomials used as shape functions for beam elements.

The form in Eq. (1) of the displacement field is not the most convenient to develop the finite element model of a beam with flexible end connections. Since the end rotations of the two members concurring at a node are different, such a formulation would increase the number of degrees of freedom of the structural model. It is more convenient to formulate a FE model in which the rotational coordinates are the joint rigid body rotations ϕ_1 and ϕ_2 instead of the usual end member rotations θ_1 and θ_2. These rotations are related as follows (see Fig. 1):

$$\phi_i(t)=\theta_i(t)+\alpha_i(t)=\theta_i(t)-\frac{M_i(t)}{k_i}\ ;\ i=1,\ 2 \tag{2}$$

where α_i is the additional end rotation due to the flexibility of the joint. It is assumed that for small rotations the connections behave linearly and thus their effect can be included through rotational springs with constant k_i.

The moments M_1 and M_2 can be expressed in terms of the node displacements by substituting the assumed displacement field $w(x, t)$ in:

$$M_1(t)=EIw''(x,\ t)|_{x=0}\ ;\ M_2(t)=-EIw''(x,\ t)|_{x=L} \tag{3}$$

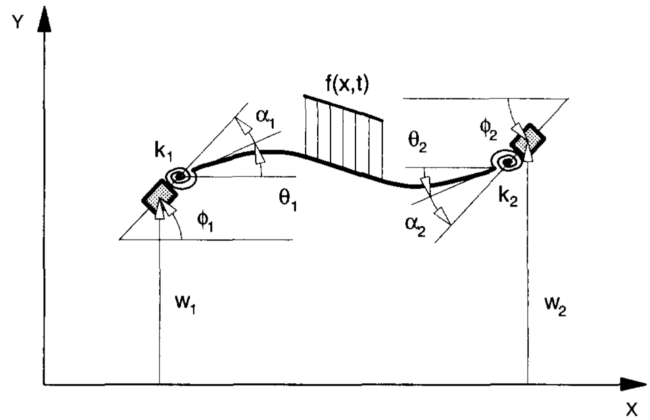


Fig. 1 Beam element with flexible joints.

where E is the Young's modulus and I is the cross-sectional area moment of inertia of the beam element.

Considering Eqs. (1), (2) and (3), the moments M_1 and M_2 can be written as:

$$M_1 = EI\boldsymbol{s}_1^T\boldsymbol{q}(t); \quad M_2 = EI\boldsymbol{s}_2^T\boldsymbol{q}(t) \tag{4}$$

where $\boldsymbol{q}(t)$ is the vector of joint displacements $\{w_1, \phi_1, w_2, \phi_2\}^T$. The vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are given by:

$$\boldsymbol{s}_1 = \frac{1}{D}\left\{\begin{array}{c} -\frac{6}{L}(1+2\gamma_2) \\ -4(1+3\gamma_2) \\ \frac{6}{L}(1+2\gamma_2) \\ -2 \end{array}\right\}; \quad \boldsymbol{s}_2 = \frac{1}{D}\left\{\begin{array}{c} -\frac{6}{L}(1+2\gamma_1) \\ -2 \\ \frac{6}{L}(1+2\gamma_1) \\ -4(1+3\gamma_2) \end{array}\right\} \tag{5}$$

in which:

$$D = (1+4\gamma_1)(1+4\gamma_2) - 4\gamma_1\gamma_2 \tag{6}$$

$$\gamma_1 = \frac{EI}{L}\frac{1}{k_1}; \quad \gamma_2 = \frac{EI}{L}\frac{1}{k_2} \tag{7}$$

The coefficients γ_1 and γ_2 are nondimensional parameters that characterize the linear behavior of the connection. Their values vary from zero for a perfectly rigid connection to infinity for a frictionless pin connection.

Combining Eqs. (1), (2) and (4), the displacement field $w(x, t)$ can be written as:

$$w(x, t) = \boldsymbol{n}(x)\ [[I] + [S]]\ \boldsymbol{q}(t) \tag{8}$$

where:

$$[S] = [0;\ \boldsymbol{s}_1;\ 0;\ \boldsymbol{s}_2]^T \tag{9}$$

The continuous system is then discretized by substituting the displacement field $w(x, t)$ from Eq. (8) in the kinetic and potential energy of a beam element:

$$T=\frac{1}{2}\int_0^L m(x)\ \dot{w}(x,\ t)^2\,dx \tag{10}$$

$$V=\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w(x,\ t)}{\partial x^2}\right)dx+\frac{1}{2}k_1\alpha_1^{\ 2}+\frac{1}{2}k_2\alpha_2^{\ 2} \tag{11}$$

Applying Lagrange's equations, the equations of motion can be expressed in the standard form:

$$[[M_0]+[M_1]]\ \ddot{\boldsymbol{q}}(t)+[[K_0]+[K_1]]\ \boldsymbol{q}(t)=\boldsymbol{f}_0(t)+\boldsymbol{f}_1(t) \tag{12}$$

The matrices $[M_0]$ and $[K_0]$ represent the standard mass and stiffness matrices of a uniform beam element, whereas $[M_1]$ and $[K_1]$ are correction matrices that account for the effect of flexible connections at the ends of the element. Similarly, $\boldsymbol{f}_0(t)$ and $\boldsymbol{f}_1(t)$ are, respectively, the standard vector of generalized forces and the correction term due to the flexibility of the connections.

Closed form expressions for these corrections terms are presented in the sequel. To obtain more compact expressions for these terms, the connection flexibility is characterized by a new set of nondimensional parameters. These parameters, referred to as the fixity factors of the element, are:

$$\mu_1=\frac{1}{1+3\gamma_1};\quad \mu_2=\frac{1}{1+3\gamma_2} \tag{13}$$

The correction matrices $[M_1]$ and $[K_1]$ will be only defined for the flexural effects (transverse displacements and rotations) since the connections are assumed to be rigid for the axial and torsional displacements. The matrices for a three-dimensional beam element can be obtained by combining the matrices associated with the flexural effects in the two principal planes of the cross section and adding the standard matrices for axial and torsional displacements. Thus, for the three-dimensional case it is necessary to define pairs of fixity factors at both ends of the element, corresponding to each principal plane of the cross-section. In practice, the axial flexibility of the most commonly used connections (welded, bolted) can be safely disregarded. The torsional flexibility can be important in mechanical and aeronautical engineering structures. The analysis of such systems is beyond the scope of this paper, although the same methodology introduced herein can be applied in their study.

The expression of $[M_1]$ is given by:

$$[M_1]=\frac{mL}{420}\begin{bmatrix} -156\beta_1(\mu_1,\ \mu_2) & & & \textit{symmetric} \\ -22L\beta_3(\mu_1,\ \mu_2) & -4L^2\beta_5(\mu_1,\ \mu_2) & & \\ 54\beta_2(\mu_1,\ \mu_2) & -13L^2\beta_4(\mu_1,\ \mu_2) & -156\beta_1(\mu_2,\ \mu_1) & \\ 13L\beta_4(\mu_2,\ \mu_1) & 3L^2\beta_6(\mu_1,\ \mu_2) & 22L\beta_3(\mu_2,\ \mu_1) & -4L^2\beta_5(\mu_2,\ \mu_1) \end{bmatrix} \tag{14}$$

in which m represents the distributed mass per unit length and L is the length of the element. The following auxiliary functions were introduced:

$$\begin{aligned}\beta_1(\mu_l,\ \mu_2)=(&64+7\mu_1^{\ 2}\mu_2^{\ 2}+55\mu_1^{\ 2}\mu_2-50\mu_1\mu_2^{\ 2}-32(\mu_1^{\ 2}+\mu_2^{\ 2})\\ &+16\mu_1\mu_2-224\mu_1+196\mu_2)/(39R(\mu_l,\ \mu_2)^2)\end{aligned} \tag{15}$$

$$\begin{aligned}\beta_2(\mu_l,\ \mu_2)=(&128+14\mu_1^{\ 2}\mu_2^{\ 2}+5(\mu_1^{\ 2}\mu_2+\mu_1\mu_2^{\ 2})-64(\mu_1^{\ 2}+\mu_2^{\ 2})\\ &+32\mu_1\mu_2-28(\mu_1+\mu_2))/(39R(\mu_l,\ \mu_2)^2)\end{aligned} \tag{16}$$

$$\beta_3(\mu_1, \mu_2)=2(88-7\mu_1^2\mu_2^2+43\mu_1^2\mu_2-16\mu_1\mu_2^2 \\ -32\mu_1^2+36\mu_1\mu_2-112\mu_1)/(11R(\mu_1, \mu_2)^2) \tag{17}$$

$$\beta_4(\mu_1, \mu_2)=2(104-21\mu_1^2\mu_2^2+32\mu_1^2\mu_2+19\mu_1\mu_2^2 \\ -2\mu_1\mu_2+64\mu_2^2-196\mu_2)/(13R(\mu_1, \mu_2)^2) \tag{18}$$

$$\beta_5(\mu_1, \mu_2)=2(16-7\mu_1^2\mu_2^2+31\mu_1^2\mu_2-32\mu_1^2-8\mu_1\mu_2)/R(\mu_1, \mu_2)^2 \tag{19}$$

$$\beta_6(\mu_1, \mu_2)=(48-28\mu_1^2\mu_2^2+64(\mu_1^2\mu_2+\mu_1\mu_2^2) \\ -148\mu_1\mu_2)/(3R(\mu_1, \mu_2)^2) \tag{20}$$

and the function $R(\mu_l, \mu_2)$ is given by:

$$R(\mu_l, \mu_2)=4-\mu_1\mu_2 \tag{21}$$

The expression of $[K_1]$ is given by:

$$[K_1]=\frac{2EI}{L^3}\begin{bmatrix} -6\zeta_1(\mu_1, \mu_2) & & & \textit{symmetric} \\ -3L\,\zeta_2(\mu_1, \mu_2) & -2L^2\zeta_3(\mu_1, \mu_2) & & \\ 6\zeta_1(\mu_1, \mu_2) & 3L\zeta_2(\mu_1, \mu_2) & -6\zeta_1(\mu_1, \mu_2) & \\ -3L\,\zeta_2(\mu_1, \mu_2) & -L^2\zeta_4(\mu_1, \mu_2) & 3L\,\zeta_2(\mu_2, \mu_1) & -2L^2\zeta_3(\mu_2, \mu_1) \end{bmatrix} \tag{22}$$

where the auxiliary functions ζ_i are defined as follows:

$$\zeta_1(\mu_l, \mu_2)=(4-2\mu_1\mu_2-\mu_1-\mu_2)/R(\mu_1, \mu_2) \tag{23}$$

$$\zeta_2(\mu_l, \mu_2)=2(2-\mu_1\mu_2-\mu_1)/R(\mu_l, \mu_2) \tag{24}$$

$$\zeta_3(\mu_l, \mu_2)=(4-\mu_1\mu_2-3\mu_1)/R(\mu_l, \mu_2) \tag{25}$$

$$\zeta_4(\mu_l, \mu_2)=4(1-\mu_1\mu_2)/R(\mu_l, \mu_2) \tag{26}$$

Finally, the expression of $f_1(t)$ is given by:

$$f_1(t)=\int_0^L f(x, t)\ [S]^T\boldsymbol{n}(x)dx \tag{27}$$

3. Effective connection stiffness as a random variable

We have assumed so far that the connections have a linear behavior, that is, the end moment M_i can be obtained by multiplying the additional end rotation α_i by the stiffness k_i. In general, the value assigned to this stiffness involves some degree of uncertainty. This variability can be caused by a number of factors, such as fluctuations in the material characteristics, the geometry of the connection, the assembling method or the quality control of the process. Consequently, for a given level of deformation, the ratio between the end moments and the additional end rotations has to defined in terms of a stiffness coefficient considered as a random variable.

The determination of the probability associated with a particular value of stiffness requires the knowledge of either its distribution function or its density function. In general, the choice of a distribution to describe the probabilistic behavior of a physical variable is governed by the nature of the phenomenon. In this case, since the connection stiffness represents the slope of the linear moment-rotation relationship, its distribution should be restricted to nonnegative values. On the other hand, the moment-rotation behavior of a connection can be affected by the presence of initial stresses, and the moments induced by these stresses at the ends of the beam can be either positive or negative. These initial moments modify the position of the linear relationship, which may be now in the positive or negative half-plane while still having a positive slope. The resulting end moment can be written as:

$$M_i = \hat{k}_i \alpha_i + \hat{M}_{ri} \tag{28}$$

where $\hat{k}_i$ and $\hat{M}_{ri}$ are random quantities representing the actual connection stiffness and the initial end moment. We can now define an effective stiffness coefficient as follows:

$$M_i = k_i \alpha_i \tag{29}$$

where k_i is a random variable that includes the uncertainty associated with the connection stiffness as well as the effect of the variability of the end moments produced by initial stresses. For our purposes, we assume that this effective stiffness can be modeled as a random variable with normal or Gaussian distribution. Furthermore, the value of this random parameter can be regarded as the sum of a deterministic component and a component representing random perturbations. Therefore, the effective stiffness value corresponding to the i^{th} structural joint can be represented as follows:

$$k_i = \bar{k}_i + \xi_i \tag{30}$$

where $\bar{k}_i$ is a deterministic constant and ξ_i is a random perturbation. In order to simplify the analysis the mean value of the random perturbation will be incorporated into the deterministic component so that ξ_i can be regarded as a zero-mean random variable. Moreover, if we assume that the probability distribution of the effective stiffness is normal, then we conclude that ξ_i is also normal, since normal random variables remain normal under linear transformations. With the above considerations it is straightforward to show that the first two moments of the random variables ξ_i and k_i are given by:

$$E\{\xi_i\} = 0; \quad E\{k_i\} = \bar{k}_i \tag{31}$$

$$E\{(\xi_i)^2\} = \sigma^2_{\xi_i} = \sigma^2_{k_i}; \quad E\{(k_i)^2\} = \sigma^2_{k_i} + \bar{k}_i^2 \tag{32}$$

Hence, the mean value of the stiffness is $\bar{k}_i$, and its variance $\sigma^2_{k_i}$ is equal to the variance $\sigma^2_{\xi_i}$ of the random perturbation ξ_i. These two quantities define completely the probability distribution.

4. Second order perturbation technique

In the formulation of a probabilistic finite element method based on the second order perturbation technique, each random variable is expanded about its mean value and terms up to second order are retained. The rates of change of the eigenproperties with respect to the fixity factors are used to obtain expressions for the mean and variances of the eigenproperties in terms of the first and second moments of the random stiffnesses. For this reason, this type of approach is called a second-moment analysis. The limitation of this formulation is that the statistical variations of the random variables have to be small in order to obtain acceptable accuracy, as it is the case with all perturbation-based methods.

If the random parameters of the structure are substituted by their mean or expected values in the analytical model, one obtains an averaged version of the structural system. In particular, if the mean values of the stiffness of the connections are used to define the system matrices, the solution of the associated eigenvalue problem will provide a deterministic set of eigenproperties. It will be shown later that these eigenproperties coincide with the mean values of those obtained through a perturbation method based on a first order expansion. Hence, to study the difference between the deterministic eigenproperties of the averaged system and the mean values of the eigenvalues and eigenvectors associated with the random system, it is necessary to include at least second order terms in the expansions.

The eigenvalue problem for a structural system with N unconstrained degrees of freedom and p random flexible joints is:

$$\{[K(\boldsymbol{k})]-\lambda_i(\boldsymbol{k})[M(\boldsymbol{k})]\}\ \phi_i(\boldsymbol{k})=\boldsymbol{0};\quad i=1,\ 2,\ \cdots,\ N \tag{33}$$

where $[K]$ and $[M]$ are the stiffness and mass matrices whose coefficients are function of the random variables $k_1,\ k_2,\ \cdots,\ k_p$. These variables are expressed in vector form as follows:

$$\boldsymbol{k}=\{k_1 k_2,\ \cdots,\ k_p\}^T \tag{34}$$

The eigenvalues λ_i and eigenvectors ϕ_i are nonlinear functions of the variables $k_1,\ k_2,\ \cdots,\ k_p$ and hence they are also random quantities. The specific form of the nonlinear functions cannot be determined, except for trivial cases. Nevertheless, assuming that the variables k_i are constrained to small fluctuations about their mean values, we can express the eigenproperties as Taylor series expansions in terms of the random stiffness parameters. The coefficients of the expansions are evaluated at the mean value vector $\bar{\boldsymbol{k}}$ defined as follows:

$$\bar{\boldsymbol{k}}=\{\bar{k}_1 \bar{k}_2,\ \cdots,\ \bar{k}_p\}^T \tag{35}$$

Therefore, the eigenvalues can be approximated as:

$$\begin{aligned}\lambda_i(\boldsymbol{k})=\lambda_i(\bar{\boldsymbol{k}})&+\sum_{m=1}^{p}\left.\frac{\partial\lambda_i}{\partial k_m}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}}(k_m-\bar{k}_m)\\&+\frac{1}{2}\sum_{m=1}^{p}\sum_{n=1}^{p}\left.\frac{\partial^2\lambda_i}{\partial k_m\partial k_n}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}}(k_m-\bar{k}_m)(k_n-\bar{k}_n)\end{aligned} \tag{36}$$

Introducing the notation:

$$\bar{\lambda}_i=\lambda_i(\bar{\boldsymbol{k}});\quad \hat{\lambda}^{I}_{i_m}=\left.\frac{\partial\lambda_i}{\partial k_m}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}};\quad \hat{\lambda}^{II}_{i_{mn}}=\left.\frac{\partial^2\lambda_i}{\partial k_m\partial k_n}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}} \tag{37}$$

and expressing the random perturbations from the mean values of the stiffness coefficients as:

$$\xi_m=k_m-\bar{k}_m \tag{38}$$

Eq. (36) can be written as:

$$\lambda_i=\bar{\lambda}_i+\sum_{m=1}^{p}\hat{\lambda}^{I}_{i_m}\xi_m+\frac{1}{2}\sum_{m=1}^{p}\sum_{n=1}^{p}\hat{\lambda}^{II}_{i_{mn}}\xi_m\xi_n \tag{39}$$

Proceeding in a similar way, the second order Taylor series expansion for the eigenvectors can be written as:

$$\phi_i=\bar{\phi}_i+\sum_{m=1}^{p}\hat{\phi}^I_{i_m}\xi_m+\sum_{m=1}^{p}\sum_{n=1}^{p}\hat{\phi}^{II}_{i_{mn}}\xi_m\xi_n \tag{40}$$

where

$$\bar{\phi}_i=\phi_i(\bar{\boldsymbol{k}});\quad \hat{\phi}^I_{i_m}=\left.\frac{\partial\phi_i}{\partial k_m}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}};\quad \hat{\phi}^{II}_{i_{mn}}=\left.\frac{\partial^2\phi_i}{\partial k_m\partial k_n}\right|_{\boldsymbol{k}=\bar{\boldsymbol{k}}} \tag{41}$$

To define the expressions for the eigenvalues and eigenvectors one needs their derivatives with respect to the stiffness coefficients k_i. These derivatives can be obtained in terms of the derivatives of the element matrices with respect to those coefficients. However, as shown by Eqs. (14) and (22), both the mass and stiffness matrices are defined in terms of the fixity factors μ_i. Based on this, one could be inclined to replace Eqs. (39) and (40) by the expansions with respect to μ_i. However, after inspecting the relationship between the fixity factors and the joint stiffness given by Eqs. (7) and (13), it is clear that if the variables k_i are considered to be normal random variables, the same assumption cannot be extended to the factors μ_i, since these equations do not define a linear transformation. Therefore, if we are interested in maintaining the Gaussian distribution assumption to take advantage of its properties, the expansions have to be done in terms of the stiffness coefficients k_i. Nevertheless, the rates of change of the eigenproperties with respect to those coefficients can be obtained in terms of the corresponding rates of change with respect to the fixity factors μ_i by making use of the chain rule as follows:

$$\bar{\lambda}^I_{i_m}=\frac{\partial\lambda_i}{\partial k_m}=\lambda^I_{i_m}\frac{\partial\mu_m}{\partial k_m};\quad \hat{\phi}^I_{i_m}=\frac{\partial\phi_i}{\partial k_m}=\phi^I_{i_m}\frac{\partial\mu_m}{\partial k_m} \tag{42}$$

$$\bar{\lambda}^{II}_{i_{mn}}=\frac{\partial^2\lambda_i}{\partial k_m\partial k_n}\begin{cases}=\lambda^{II}_{i_{mm}}\left(\dfrac{\partial\mu_m}{\partial k_m}\right)^2+\lambda^I_{i_m}\dfrac{\partial^2\mu_m}{\partial k_m^2} & \text{for } m=n\\[2ex] =\lambda^{II}_{i_{mm}}\dfrac{\partial\mu_m}{\partial k_m}\dfrac{\partial\mu_m}{\partial k_n} & \text{for } m\neq n\end{cases} \tag{43}$$

$$\hat{\phi}^{II}_{i_{mn}}=\frac{\partial^2\phi_i}{\partial k_m\partial k_n}\begin{cases}=\phi^{II}_{i\,mn}\left(\dfrac{\partial\mu_m}{\partial k_m}\right)^2+\phi^I_{i_m}\dfrac{\partial^2\mu_m}{\partial k_m^2} & \text{for } m=n\\[2ex] =\phi^{II}_{i\,mn}\dfrac{\partial\mu_m}{\partial k_m}\dfrac{\partial\mu_m}{\partial k_n} & \text{for } m\neq n\end{cases} \tag{44}$$

in which:

$$\frac{\partial\mu_m}{\partial k_m}=\frac{3\dfrac{EI}{L}}{\left(k_m+3\dfrac{EI}{L}\right)^2};\quad \frac{\partial^2\mu_m}{\partial k_m^2}=\frac{-6\dfrac{EI}{L}}{\left(k_m+3\dfrac{EI}{L}\right)^3} \tag{45}$$

The coefficients $\lambda^I_{i_m}$ and $\lambda^{II}_{i_{mn}}$ denote the first and second order derivatives of the eigenvalues with respect to the fixity factors, respectively. They are given by:

$$\lambda^I_{i_m}=\left.\frac{\partial\lambda_i}{\partial\mu_m}\right|_{\mu=\bar{\mu}}=\bar{\phi}_i^T\{[K^I_m]-\bar{\lambda}_i[M^I_m]\}\bar{\phi}_i \tag{46}$$

$$\lambda^{II}_{i_{mn}} = \frac{\partial^2 \lambda_i}{\partial \mu_m \partial \mu_n}\bigg|_{\mu=\bar{\mu}} = \bar{\phi}_i^T\{[K^I_m] - \bar{\lambda}_i[M^I_m]\}\bar{\phi}^I_{i_n} + \bar{\phi}_i^T\{[K^I_n] - \bar{\lambda}_i[M^I_n]\}\bar{\phi}^I_{i_m}$$
$$+ \bar{\phi}_i^T\{[K^{II}_{mn}] - \bar{\lambda}_i[M^{II}_{mn}]\}\ \bar{\phi}_i - \lambda^I_{i_m}\bar{\phi}_i^T[M^I_n]\,\bar{\phi}_i$$
$$- \lambda^I_{i_n}\bar{\phi}_i^T[M^I_m]\,\bar{\phi}_i - \lambda^I_{i_m}\bar{\phi}_i^T[\bar{M}]\,\phi^I_{i_n} - \lambda_{i_n}{}^I\bar{\phi}_i^T[\bar{M}]\,\phi^I_{i_m} \tag{47}$$

Similarly, the coefficients $\phi^I_{i_m}$ and $\phi^{II}_{i_{mn}}$ denote the first and second order derivatives of the eigenvectors with respect to the fixity factors, respectively. They can be written as:

$$\phi^I_{i_m} = \frac{\partial \phi_i}{\partial \mu_m}\bigg|_{\mu=\bar{\mu}} = \gamma^I_{i_m}\bar{\phi}_i + \sum_{j=1,\, j\neq 1}^{N} \beta^I_{ij_m}\bar{\phi}_j \tag{48}$$

$$\phi^{II}_{i_{mn}} = \frac{\partial^2 \phi_i}{\partial \mu_m \partial \mu_n}\bigg|_{\mu=\bar{\mu}} = \gamma^{II}_{i_{mn}}\bar{\phi}_i + \sum_{j=1,\, j\neq 1}^{N} \beta^{II}_{ij_{mn}}\bar{\phi}_j \tag{49}$$

where:

$$\beta^I_{ij_m} = \left(\frac{1}{\lambda_j - \lambda_i}\right)\bar{\phi}_i^T\{\bar{\lambda}_i[M^I_m] - [K^I_m]\}\,\bar{\phi}_i; \quad i\neq j \tag{50}$$

$$\gamma^I_{i_m} = -\frac{1}{2}\bar{\phi}_i^T[M^I_m]\bar{\phi}_i \tag{51}$$

$$\beta^{II}_{ij_{mn}} = \left(\frac{1}{\lambda_j - \lambda_i}\right)(\bar{\phi}_j^T\{\bar{\lambda}[M^I_m] - [K^I_m]\}\,\phi^I_{i_n}$$
$$+ \bar{\phi}_j^T\{\bar{\lambda}_i[M^I_n] - [K^I_n]\}\,\phi^I_{i_m} + \bar{\phi}_j^T\{\bar{\lambda}_i[M^{II}_{mn}] - [K^{II}_{mn}]\}\bar{\phi}_i$$
$$+ \lambda^I_{i_m}\bar{\phi}_j^T[M^I_n]\,\bar{\phi}_i + \lambda^I_{i_n}\bar{\phi}_j^T[M^I_m]\,\bar{\phi}_i + \lambda^I_{i_m}\bar{\phi}_j^T[\bar{M}]\,\phi^I_{i_n} + \lambda^I_{i_n}\bar{\phi}_j^T[\bar{M}]\,\phi^I_{i_m}) \tag{52}$$

$$\gamma^{II}_{i_{mn}} = -\bar{\phi}_i^T[M^I_m]\,\phi^I_{i_n} - \bar{\phi}_i^T[M^I_n]\,\phi^I_{i_m} - \phi^{IT}_{i_m}[\bar{M}]\,\phi^I_{i_n} - \frac{1}{2}\bar{\phi}_i^T[M^{II}_{mn}]\,\bar{\phi}_i \tag{53}$$

in which the following notation was introduced:

$$[\bar{K}] = [K(\bar{\mu})]\,;\,[\bar{M}] = [M(\bar{\mu})] \tag{54}$$

$$[K_m{}^I] = \frac{\partial[K(\mu)]}{\partial \mu_m}\bigg|_{\mu=\bar{\mu}}\ ;\quad [M^I_m] = \frac{\partial[M(\mu)]}{\partial \mu_m}\bigg|_{\mu=\bar{\mu}} \tag{55}$$

$$[K^{II}_{mn}] = \frac{\partial^2[K(\mu)]}{\partial \mu_m \partial \mu_n}\bigg|_{\mu=\bar{\mu}}\ ;\quad [M^{II}_{mn}] = \frac{\partial^2[M(\mu)]}{\partial \mu_m \partial \mu_n}\bigg|_{\mu=\bar{\mu}} \tag{56}$$

Thus, the expressions for the rate of change of the eigenproperties with respect with the stiffness coefficients can be written in terms of the derivatives of the element matrices $[M]=[M_0]+[M_1]$ and $[K]=[K_0]+[K_1]$ with respect to the fixity factors. These derivatives can be obtained by considering the correction terms $[M_1]$ and $[K_1]$ defined in Eqs. (14) and (22). Closed form expressions for the derivatives are presented in Appendix A.

To obtain the expected values of the i^{th} eigenproperties, it is necessary to apply the expected value operator to the expansions in Eqs. (39) and (40). Considering Eq. (31), we can write:

$$E\{\lambda_i\}=\bar{\lambda}_i+\frac{1}{2}\sum_{m=1}^{p}\sum_{m=1}^{p}\hat{\lambda}^{II}_{i_{mn}}E\{\xi_m\xi_n\} \tag{57}$$

$$E\{\phi_i\}=\bar{\phi}_i+\frac{1}{2}\sum_{m=1}^{p}\sum_{n=1}^{p}\hat{\phi}^{II}_{i_{mn}}E\{\xi_m\xi_n\} \tag{58}$$

We can see from these equations that the expected values differ from the zero-order terms that are solution of the averaged problem, and the difference is a linear function of the covariances of the random variables.

If the random variables are considered to be uncorrelated, then it is possible to write:

$$E\{\xi_m\xi_n\}=\delta_{mn}\sigma^2_m \tag{59}$$

where δ_{mn} is the Kroenecker delta, and σ^2_m is the variance of the random variable ξ_m. With this assumption, Eqs. (57-58) take the form:

$$E\{\lambda_i\}=\bar{\lambda}_i+\frac{1}{2}\sum_{m=1}^{r}\hat{\lambda}^{II}_{i_{mm}}\sigma^2_m \tag{60}$$

$$E\{\phi_i\}=\bar{\phi}_i+\frac{1}{2}\sum_{m=1}^{R}\hat{\phi}^{II}_{i_{mm}}\sigma^2_m \tag{61}$$

The variance of the i^{th} eigenvalue is given by:

$$\sigma^2_{\lambda i}=E\{(\lambda_i-E\{\lambda_i\})^2\} \tag{62}$$

Substituting the corresponding expressions for λ_i and $E\{\lambda_i\}$ given by Eqs. (39) and (57), respectively, and considering the linearity of the expected value operator, we can write:

$$\begin{aligned}\sigma^2_{\lambda_i}=&E\left\{\left(\sum_{m=1}^{p}\hat{\lambda}^{I}_{i_m}\xi_m\right)^2\right\}+\frac{1}{4}E\left\{\left(\sum_{m=1}^{p}\sum_{n=1}^{p}\hat{\lambda}^{II}_{i_{mn}}\xi_m\xi_n\right)^2\right\}\\&+\frac{1}{4}E\left\{\left(\sum_{m=1}^{p}\sum_{n=1}^{p}\hat{\lambda}^{II}_{i_{mn}}E\,\xi_m\xi_n\}\right)^2\right\}\\&+E\left\{\left(\sum_{m=1}^{p}\sum_{r=1}^{p}\sum_{s=1}^{p}\hat{\lambda}^{I}_{i_m}\hat{\lambda}^{II}_{i_{rs}}\xi_m\xi_r\xi_s\right\}\right.\\&-E\left\{\left(\sum_{m=1}^{p}\sum_{r=1}^{p}\sum_{s=1}^{p}\hat{\lambda}^{I}_{i_m}\hat{\lambda}^{II}_{i_{rs}}\xi_m E\,\{\xi_r\xi_s\}\right\}\right.\\&-\frac{1}{2}E\left\{\left(\sum_{m=1}^{p}\sum_{n=1}^{p}\sum_{r=1}^{p}\sum_{s=1}^{p}\hat{\lambda}^{II}_{i_{mn}}\hat{\lambda}^{II}_{i_{rs}}\xi_m\xi_n E\,\{\xi_r\xi_s\}\right\}\right.\end{aligned} \tag{63}$$

Considering the properties of normal zero-mean random variables, it follows that the fourth and fifth terms in the above expression must vanish, since they involve odd moments:

$$E\,\{\xi_m\xi_r\xi_s\}=0 \tag{64}$$

$$E\,\{\xi_m E\,\{\xi_r\xi_s\}\}=E\,\{\xi_m\}\,E\,\{\xi_r\xi_s\}=0 \tag{65}$$

and Eq. (63) reduces to:

$$\sigma^2_{\lambda i} = \sum_{m=1}^{p} \sum_{n=1}^{p} \hat{\lambda}^{I}_{i_m} \hat{\lambda}^{I}_{i_n} E\{\xi_m \xi_n\}$$
$$- \frac{1}{4} \sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} \hat{\lambda}^{II}_{i_{mn}} \hat{\lambda}^{II}_{i_{rs}} E\{\xi_m \xi_n\} E\{\xi_r \xi_s\}$$
$$+ \frac{1}{4} \sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} \hat{\lambda}^{II}_{i_{mn}} \hat{\lambda}^{II}_{i_{rs}} E\{\xi_m \xi_n \xi_r \xi_s\} \quad (66)$$

The joint moment of four normal random variables can be written in terms of lower order moments as follows:

$$E\{\xi_m \xi_n \xi_r \xi_s\} = E\{\xi_m \xi_n\} E\{\xi_r \xi_s\} + E\{\xi_m \xi_r\} E\{\xi_n \xi_s\} + E\{\xi_m \xi_s\} E\{\xi_n \xi_r\} \quad (67)$$

and the multiple summation in the last term of Eq. (63) can be expressed in the following form:

$$\sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} \hat{\lambda}^{II}_{i_{mn}} \hat{\lambda}^{II}_{i_{rs}} E\{\xi_m \xi_n \xi_r \xi_s\}$$
$$= \sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} (\hat{\lambda}^{II}_{i_{mn}} \hat{\lambda}^{II}_{i_{rs}} + \hat{\lambda}^{II}_{i_{mr}} \hat{\lambda}^{II}_{i_{ns}} + \hat{\lambda}^{II}_{i_{ms}} \hat{\lambda}^{II}_{i_{nr}}) E\{\xi_m \xi_n\} E\{\xi_r \xi_s\} \quad (68)$$

Substituting Eq. (66) in (63), we finally obtain:

$$\sigma^2_{\lambda i} = \sum_{m=1}^{p} \sum_{n=1}^{p} \hat{\lambda}^{I}_{i_m} \hat{\lambda}^{I}_{i_n} E\{\xi_m \xi_n\}$$
$$+ \frac{1}{4} \sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} (\hat{\lambda}^{II}_{i_{mr}} \hat{\lambda}^{II}_{i_{ns}} + \hat{\lambda}^{II}_{i_{ms}} \hat{\lambda}^{II}_{i_{nr}}) E\{\xi_m \xi_n\} E\{\xi_r \xi_s\} \quad (69)$$

The eigenvectors variance can be obtained following an analogous derivation. It can be shown that the variance of the eigenvectors is defined by the following expression:

$$\sigma^2_{\phi_i} = \sum_{m=1}^{p} \sum_{n=1}^{p} \hat{\phi}^{I}_{i_m} \hat{\phi}^{I}_{i_n} E\{\xi_m \xi_n\}$$
$$+ \frac{1}{4} \sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{r=1}^{p} \sum_{s=1}^{p} (\hat{\phi}^{II}_{i_{mr}} \hat{\phi}^{II}_{i_{ns}} + \hat{\phi}^{II}_{i_{ms}} \hat{\phi}^{II}_{i_{nr}}) E\{\xi_m \xi_n\} E\{\xi_r \xi_s\} \quad (70)$$

Furthermore, if we assume that the random variables are uncorrelated, then Eqs. (69) and (70) reduce to:

$$\sigma^2_{\lambda i} = \sum_{m=1}^{p} (\hat{\lambda}^{I}_{i_m})^2 \sigma^2_m + \frac{1}{2} \sum_{m=1}^{p} \sum_{n=1}^{p} (\hat{\lambda}^{II}_{i_{mn}})^2 \sigma^2_m \sigma^2_n \quad (71)$$

$$\sigma^2_{\varphi_i} = \sum_{m=1}^{p} (\hat{\phi}^{I}_{i_m})^2 \sigma^2_m + \frac{1}{2} \sum_{m=1}^{p} \sum_{n=1}^{p} (\hat{\phi}^{II}_{imn})^2 \sigma^2_m \sigma^2_n \quad (72)$$

5. Numerical examples

In order to validate the expressions obtained for the first and second order moments of the eigenproperties a Monte-Carlo Simulation will be implemented. For this we will consider the plane frame illustrated in Fig. 2(a) and we will compare the values of the moments obtained with the second order perturbation expansion method against the results of a Monte-Carlo simulation. The Monte-Carlo simulation (hereafter referred to as MCS) is a well known and powerful method to determine the performance of systems with random parameters. In a nutshell, the method consists in the generation of a set of systems derived from the original system by assigning values to the random parameters. All the systems are later statistically processed using techniques of sampling and parameter estimation (e.g., Soong 1981, Ross 1987). The values assigned to the parameters of the systems are obtained by means of an algorithm that generates random numbers. This requires to select the probability distribution function that governs the behavior of these parameters.

Once the accuracy of the expressions obtained with the Second Order Perturbation Method (from now on designated as SOPM) is established, they will be used to obtain the probability density function of the eigenvalues and elements of the eigenvector. This will allow us to assess the level of dispersion in the dynamic properties of the structure introduced by the random variation of the stiffness coefficients. The two-level frame in Fig. 2(b) will be used for this purpose.

5.1. Example 1: One-story frame

Two sets of random normal variables (k_{1_i}, k_{2_i}, $i=1, 2, \cdots, 50$) representing the effective stiffness coefficients were generated. The mean value of the these coefficients is such that the corresponding fixity factor is 0.50. Two different values of the coefficient of variation (c.o.v.) of the stiffness coefficients were considered: $\rho_k=0.10$ and $\rho_k=0.20$.

Tables 1-6 present the mean values and standard deviations for the eigenvalues and eigenvectors of the first two modes of the frame in Fig. 1 calculated with the two approaches. The frame was modelled with only three elements and 6 dof because the aim of the analysis is to compare the accuracy of the perturbation method without being concerned about the accuracy of the calculated eigenproperties. Tables 1 and 2 show the expected value and standard deviation of the lower eigenvalue of the frame, respectively. The results shown under the Monte-Carlo column are the sample mean and sample standard deviation. All the tables include the limiting values

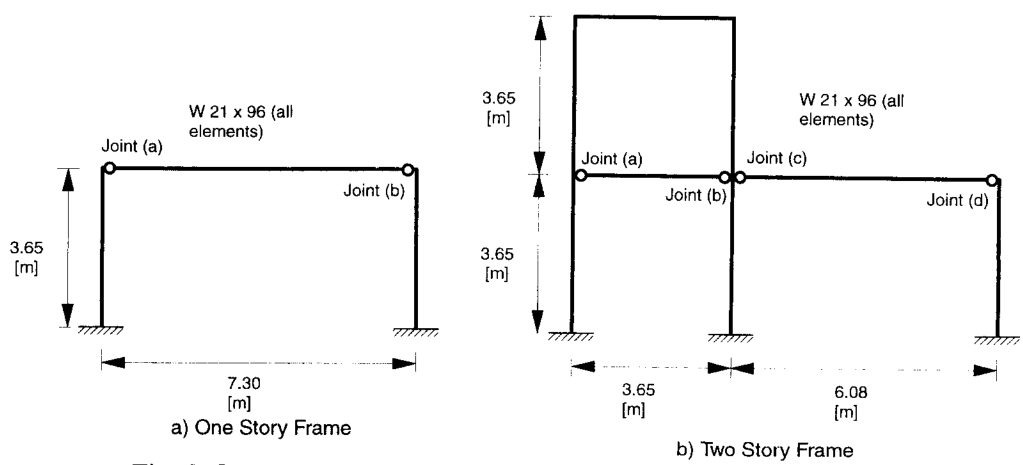


Fig. 2 One story and two story frames for numerical examples.

Table 1 Eigenvalue statistics: mean values

Eigenvalue #	2nd Order Perturbation	Monte-Carlo Mean	95% Confidence: Lower Limit	95% Confidence: Upper Limit
		a) Coefficient of Variation=0.10		
1	25906	25945	25843	26048
2	441812	439869	435146	444591
		b) Coefficient of Variation=0.10		
1	25816	25887	25678	26096
2	444647	440835	431219	450451

Table 2 Eigenvalue statistics: standard deviation

Eigenvalue #	2nd. Order Perturbation	Monte-Carlo St. Deviation	95% Confidence: Low Limit	95% Confidence: Upper Limit
		a) Coefficient of Variation=0.10		
1	343	360	300	448
2	15939	16616	13880	20706
		b) Coefficient of Variation=0.20		
1	695	736	615	917
2	32303	33835	28263	42162

corresponding to a confidence interval of 95%. It can be seen that the results obtained via the SOPM always lie within the confidence interval. Note also that according to the table, doubling the standard deviations of the connection stiffness produces the same effect on the standard deviations of the eigenvalues. The same phenomenon can be observed in the c.o.v. of the eigenvalues in Table 3. This reveals an almost linear probabilistic dependence between the stiffness of the connections and the eigenvalues of the structural system. Examining Table 3 it becomes apparent that the structural system filters out the effect that the uncertainties in the stiffness of the connections have on the eigenvalues of the system. For instance, when the c.o.v. of the stiffness of the joints is 0.10, the c.o.v. of the first and second eigenvalue is 0.0139 and 0.0378, respectively.

Tables 4-6 show the comparison in the statistics of the first eigenvector calculated using the SOPM and the MCS technique for two values of the c.o.v. of the connection stiffness. It is illustrative to discuss some characteristics of the results presented in these tables. The expression for the expected value of the i^{th} eigenvector given by Eq. (61), includes the second order derivatives of the eigenvector. These second order derivatives are obtained in terms of a linear combination of the eigenvectors of the deterministic system associated with the mean values of the stiffness coefficients. Due to the symmetric configuration of the structure under consideration, these deterministic eigenvectors (or modal shapes) are either symmetric or antisymmetric. Thus, the expected values of the eigenvectors of the random system are also either symmetric or antisymmetric.

Table 3 Eigenvalue statistics: coefficients of variation

Eigenvalue #	2nd Order Perturbation	Monte-Carlo
a) Coefficient of Variation=0.10		
1	0.0132	0.0139
2	0.0361	0.0378
b) Coefficient of Variation=0.20		
1	0.0269	0.0284
2	0.0726	0.0768

Table 4 First eigenvector statistics: mean values

DOF#	2nd. Order Perturbation	Monte-Carlo Mean	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation=0.10				
1	3.6399E-01	3.6396E-01	3.6391E-01	3.6401E-01
2	1.3977E-03	1.3990E-03	1.3768E-03	1.4212E-03
3	2.9631E-03	2.9607E-03	2.9511E-03	2.9704E-03
4	3.6399E-01	3.6395E-01	3.6390E-01	3.6400E-01
5	1.3977E-03	1.3899E-03	1.3692E-03	1.4105E-03
6	2.9631E-03	2.9584E-03	2.9482E-03	2.9686E-03
b) Coefficient of Variation=0.20				
1	3.6400E-01	3.6392E-01	3.6383E-01	3.6402E-01
2	1.4111E-03	1.3940E-03	1.3487E-03	1.4393E-03
3	2.9704E-03	2.9662E-03	2.9467E-03	2.9858E-03
4	3.6400E-01	3.6391E-01	3.6381E-01	3.6401E-01
5	1.4111E-03	1.3769E-03	1.3352E-03	1.4185E-03
6	2.9704E-03	2.9620E-03	2.9410E-03	2.9829E-03

The same situation occurs with the variance of the eigenvectors. The expression for the variance of the i^{th} eigenvector is given by Eq. (72), and it includes the squares of the first order derivatives, and the squares of the second order derivatives. Therefore, for this particular example the variance vectors are also symmetric. On the other hand, the results of MCS are obtained from the statistical processing of a set of structures which are not necessarily symmetric, since the stiffness coefficients are uncorrelated random variables. Therefore, for a sample of finite size the expected values and variances of the eigenvectors do not necessarily show a strict symmetry. In spite of this fact, the mean values and standard deviations of all the elements of the eigenvector calculated using the SOPM fall inside the 95% confidence interval. The c.o.v. for the first mode of vibration are presented in Table 6. The table shows that the modal dof associated with the horizontal displacement of the ends of the beam (1st and 4th elements) are practically insensitive to the uncertainties in the stiffness of the connections. Moreover, there is an almost linear statistical relation between the components of the eigenvector and the stiffness of the joints. The agreement

Table 5 First eigenvector statistics: standard deviations

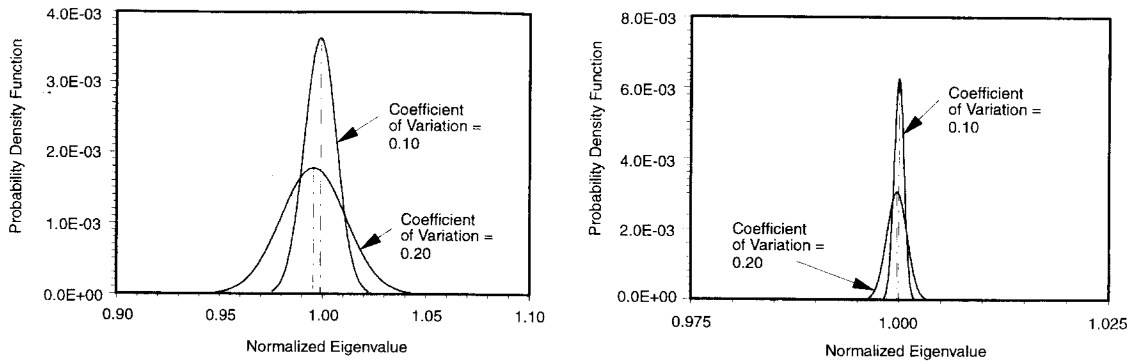
DOF #	2nd. Order Perturbation	Monte-Carlo St. Deviation	95% Confidence: Lower Limit	95% Confidence: Upper Limit
a) Coefficient of Variation=0.10				
1	1.6773E-04	1.7189E-04	1.4358E-04	2.1419E-04
2	7.0381E-05	7.7280E-05	6.4555E-05	9.6302E-05
3	3.2871E-05	3.3559E-05	2.8033E-05	4.1819E-05
4	1.6773E-04	1.7911E-04	1.4962E-04	2.2320E-04
5	7.0381E-05	7.1826E-05	5.9999E-05	8.9505E-05
6	3.2871E-05	3.5570E-05	2.9713E-05	4.4325E-05
b) Coefficient of Variation=0.20				
1	3.3603E-04	3.3784E-04	2.8221E-04	4.2099E-04
2	1.4205E-04	1.5776E-04	1.3178E-04	1.9659E-04
3	6.6444E-05	6.8073E-05	5.6864E-05	8.4828E-05
4	3.3603E-04	3.5319E-04	2.9504E-04	4.4013E-04
5	1.4205E-04	1.4521E-04	1.2130E-04	1.8095E-04
6	6.6444E-05	7.3115E-05	6.1076E-05	9.1112E-05

Table 6 First eigenvector statistics: coefficients of variation

DOF #	2nd Order Perturbation	Monte-Carlo
a) Coefficient of Variation=0.10		
1	0.0005	0.0005
2	0.0504	0.0552
3	0.0111	0.0113
4	0.0005	0.0005
5	0.0504	0.0517
6	0.0111	0.0120
b) Coefficient of Variation=0.20		
1	0.0009	0.0009
2	0.1007	0.1132
3	0.0224	0.0229
4	0.0009	0.0010
5	0.1007	0.1055
6	0.0224	0.0247

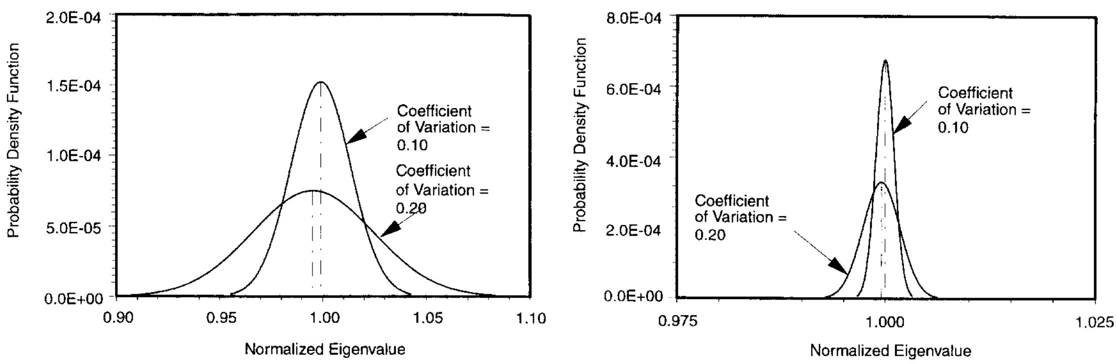
between the perturbation-based results and the Monte-Carlo technique in this as well as in the previous tables is remarkable.

The only objective for comparing the Monte-Carlo simulation with the perturbation-based results was to verify the accuracy of the latter method. Although it was not our intention to compare the computational efficiency of both methods, it is evident that the perturbation method, being a closed form solution, is simpler to use and requires much less computational effort.



(a) Probability Density Function of First Eigenvalue. (b) Probability Density Function of Second Eigenvalue.

Fig. 3 Probability density function of first and second eigenvalues.



(a) Probability Density Function of Third Eigenvalue. (b) Probability Density Function of Fourth Eigenvalue.

Fig. 4 Probability density function of third and fourth eigenvalues.

5.2. Example 2: Two-story plane frame

For the next example we will consider the two-story plane frame modelled with 42 dof shown in Fig. 2(b). Each beam and column was discretized with two frame elements, except the beam connecting the joints (c) and (d) which was divided into three elements. The stiffness coefficients of the connections at the ends of the beams at the first level, denoted as (a), (b), (c) and (d), are regarded as uncorrelated normal random variables. They have a distribution such that the mean value of the fixity factors is 0.70. Two different c.o.v., 0.10 and 0.20, will be used to describe the dispersion in the values of the stiffness coefficients.

It can be shown (Scheidt-Purkert 1983) that for the case in which the random quantities involved in the eigenproblem are weakly correlated, the PDF of the eigenproperties tends to be Gaussian. The PDF of the first eigenvalue is shown in Fig. 3(a). The mean value and the standard deviation to plot the curve were calculated with the SOPM. The eigenvalues in the horizontal axis were divided by the corresponding eigenvalue obtained from the deterministic eigenproblem using the mean value of the stiffness coefficients. The c.o.v. for the random eigenvalue are 0.0078 and 0.0158 when the c.o.v. of the stiffness coefficients are 0.10 and 0.20, respectively.

This indicates that, as in the previous example, the structure is not very sensitive to the uncertainties in the stiffness of the non-rigid joints. Figs. 3(b), 4(a) and 4(b) show the PDF for the 2nd, 3rd and 4th eigenvalues. The 3rd eigenvalue is the most sensitive to the random variability of the connections' stiffness. For c.o.v. of the stiffness equal to 0.10 and 0.20, the respective c.o.v. of the 3rd eigenvalue are 0.0146 and 0.0296. It can be noted that the 2nd and 4th eigenvalues do not have a significant dispersion meaning that they are not much affected by the uncertainties in the stiffness of the connections at the first floor.

The PDF for selected elements of the lower four eigenvectors are displayed in Fig. 5-8. The results for the 1st eigenvector are shown in Fig. 5. The random connections have a more pronounced effect in the modal dof associated with the vertical displacement of the node in the beam *c-d*. The two PDF's for this dof depicted in the figure have c.o.v. equal to 0.3072 and 0.6172 for c.o.v. of the stiffness equal 0.10 and 0.20, respectively. Although the effect is not as pronounced as in the previous case, the uncertainties in the connections also have an important influence in the rotation of joint (*d*). The PDF's associated with this dof plotted in the figure have c.o.v. equal to 0.0253 and 0.0517. On the contrary, the horizontal displacement of node (*d*) is relatively insensitive to the random connections, as evidenced by the sharp PDF for this dof shown in the figure.

Fig. 6 show similar results but this time for the 2nd eigenvector. The PDF's are similar to those for the 1st eigenvector. The PDF's for the modal displacements corresponding to the 3rd eigenvalue are presented in Fig. 7. In this case, the largest dispersion is associated with the horizontal displacement of node (*d*). The c.o.v. are 0.0239 and 0.0485 when the respective coefficients for the stiffness are 0.10 and 0.20. Finally, Fig. 8 displays the results for the 4th eigenvector.

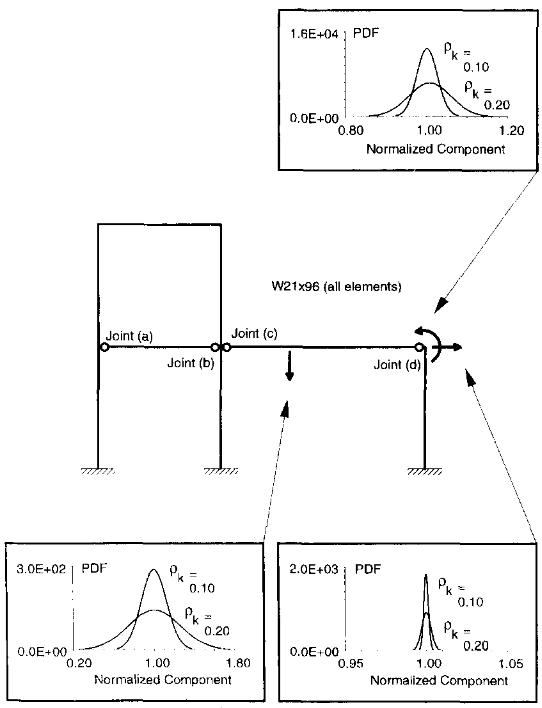


Fig. 5 Probability density function of selected Dof's of first eigenvector.

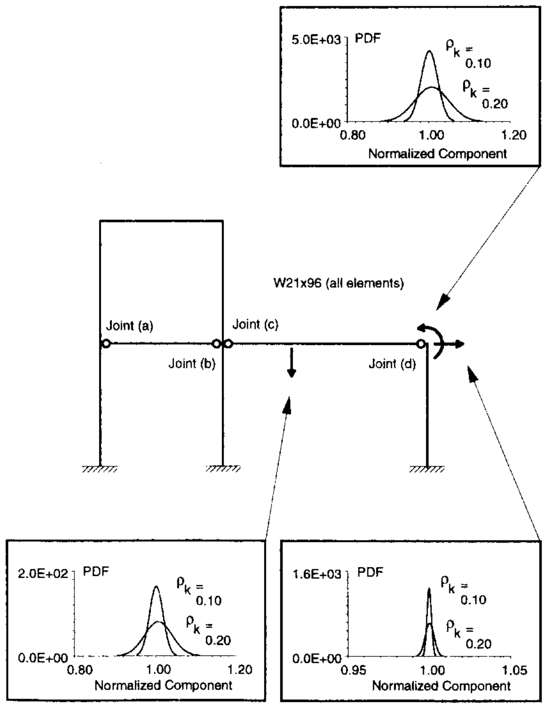


Fig. 6 Probability density function of selected Dof's of second eigenvector.

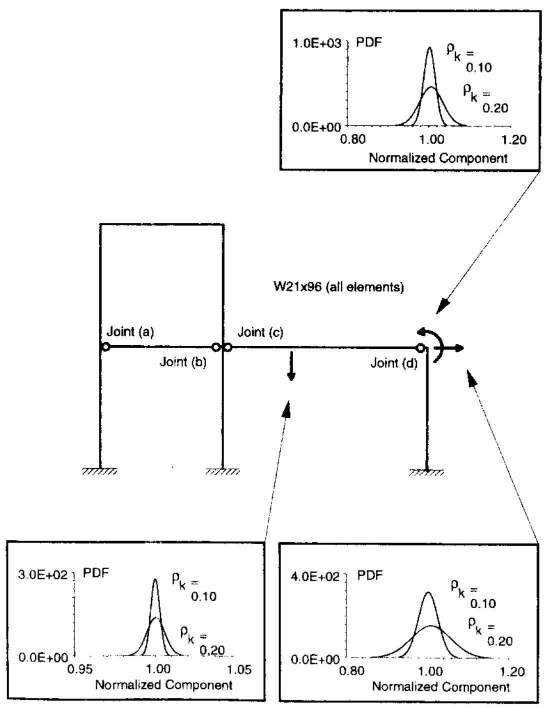


Fig. 7 Probability density function of selected Dof's of third eigenvector.

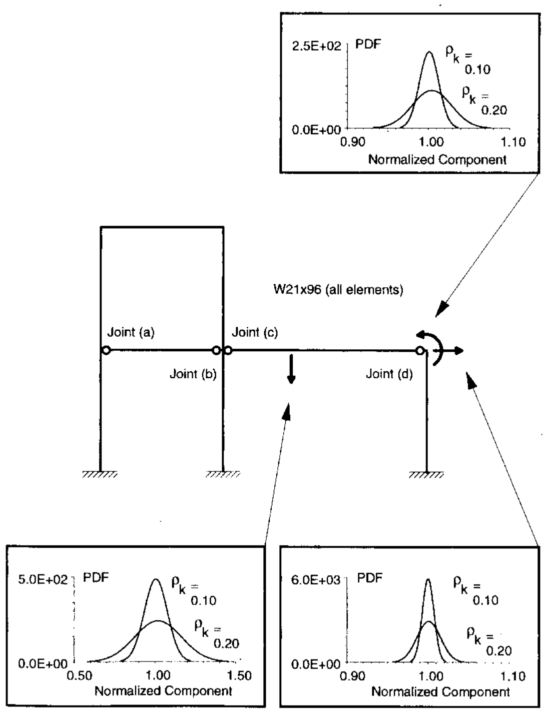


Fig. 8 Probability density function of selected Dof's of fourth eigenvector.

Here the vertical displacement of the interior node on beam (*c*)-(*d*) is the modal dof most affected by the randomness of the connections. The distributions have c.o.v. equal to 0.0712 and 0.1425 when the distribution of the stiffness have c.o.v. equal to 0.10 and 0.20, respectively.

6. Conclusions

The objective of the study presented in this paper was to quantify the effect of flexible connections with random stiffness on the modal properties of the structures. A brief description of the formulation of a finite element model of a beam element with flexible linear connections previously developed by the authors was presented. Since the mass and stiffness matrices contain random parameters, the modal decomposition solution of the equations of motion lead to a random eigenvalue problem. A second order perturbation expansion was used for the solution of the random eigenproblem. Closed form expressions for the mean and variance of the eigenproperties were provided. The method requires the derivatives of the eigenvalues and eigenvectors with respect to the stiffness coefficients. The expressions to calculate these derivatives in closed form were also provided in the paper.

Although, as it was mentioned in the introduction, perturbation methods have been applied before to the solution of the random eigenvalue problem, this paper adressed the specific case of a structure with random flexible connections. The analysis of these systems required, among other developments, the derivation of the proper mass and stiffness matrices as well as their first and second order derivatives.

The results were validated by comparing them against a Monte-Carlo simulation. An excellent agreement was observed. The numerical examples showed that the level of dispersion induced by the uncertainty of the connection stiffness varies with the eigenvalue number and is different for the different components of the eigenvectors. For the example structure considered, it was found that an almost linear statistical relation exists between the eigenproperties and the stiffness of the joints.

It is hoped that the work in this paper can lead to extensions that will further enhance the understanding of the phenomenon. Some of the possible extensions include the consideration of the energy dissipation in the joints by means of random rotational dampers and the calculation of the statistics of the time history response of structures with random connections.

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Appendix A

Derivatives of the Element Matrices

This Appendix presents the explicit expressions of the derivatives of the stiffiness and mass matrices of a flexural element with respect to the fixity factors μ_1 and μ_2

A.1. Mass Matrix:

a) First derivative with respect to μ_1:

$$\left[\frac{\partial M}{\partial \mu_1}\right]=\frac{mL}{420}\begin{bmatrix} -156\,\eta_1(\mu_1\ \mu_2) & & & \textit{symmetric} \\ -22L\,\eta_2(\mu_1\ \mu_2) & -4L^2\eta_5(\mu_1\ \mu_2) & & \\ 54\eta_3(\mu_1\ \mu_2) & -13L\,\eta_6(\mu_1\ \mu_2) & -156\,\eta_8(\mu_1\ \mu_2) & \\ 13L\,\eta_4(\mu_1,\ \mu_2) & 3L^2\eta_7(\mu_1\ \mu_2) & 22L\,\eta_9(\mu_1\ \mu_2) & -4L^2\eta_{10}(\mu_1\ \mu_2) \end{bmatrix} \tag{A.1}$$

b) First derivative with respect to μ_2:

$$\left[\frac{\partial M}{\partial \mu_2}\right]=\frac{mL}{420}\begin{bmatrix} -156\eta_8(\mu_1\,\mu_2) & & & \textit{symmetric} \\ -22L\,\eta_9(\mu_1,\ \mu_2) & -4L^2\eta_{10}(\mu_1,\ \mu_2) & & \\ 54\eta_3(\mu_2,\ \mu_1) & -13L\,\eta_4(\mu_1,\mu_2) & -156\eta_1(\mu_2,\ \mu_1) & \\ 13L\eta_6(\mu_1,\ \mu_2) & 3L^2\eta_7(\mu_2,\ \mu_1) & 22L\,\eta_2(\mu_2,\ \mu_1) & -4L^2\eta_5(\mu_2,\ \mu_1) \end{bmatrix} \tag{A.2}$$

c) Second derivative with respect to μ_1:

$$\left[\frac{\partial^2 M}{\partial \mu_1^2}\right]=\frac{mL}{420}\begin{bmatrix} -156\tau_1(\mu_1,\ \mu_2) & & & \textit{symmetric} \\ -22L\,\tau_2(\mu_1,\ \mu_2) & -4L^2\tau_5(\mu_1,\ \mu_2) & & \\ 54\tau_3(\mu_1,\ \mu_2) & -13L\tau_6(\mu_1,\ \mu_2) & -156\tau_8(\mu_1,\ \mu_2) & \\ 13L\tau_4(\mu_1,\ \mu_2) & 3L^2\tau_7(\mu_1,\ \mu_2) & 22L\,\tau_9(\mu_1,\ \mu_2) & -4L^2\tau_{10}(\mu_1,\ \mu_2) \end{bmatrix} \tag{A.3}$$

d) Second derivative with respect to μ_1 and μ_2:

$$\left[\frac{\partial\ \ M}{\partial \mu_1\,\partial\mu_2}\right]=\frac{mL}{420}\begin{bmatrix} -156\tau_{11}(\mu_1,\ \mu_2) & & & \textit{symmetric} \\ -22L\,\tau_{12}(\mu_1,\ \mu_2) & -4L^2\tau_{15}(\mu_1,\ \mu_2) & & \\ 54\tau_{13}(\mu_1,\ \mu_2) & -13L\,\tau_{14}(\mu_2,\ \mu_1) & -156\tau_{11}(\mu_2,\ \mu_1) & \\ 13L\,\tau_{14}(\mu_1,\ \mu_2) & 3L^2\tau_{16}(\mu_1,\ \mu_2) & 22L\,\tau_{12}(\mu_2,\ \mu_1) & -4L^2\tau_{15}(\mu_2,\ \mu_1) \end{bmatrix} \tag{A.4}$$

e) Second derivative with respect to μ_2:

$$\left[\frac{\partial^2 M}{\partial \mu_2^2}\right]=\frac{mL}{420}\begin{bmatrix} -156\tau_{11}(\mu_2,\ \mu_1) & & & \textit{symmetric} \\ -22L\,\tau_{12}(\mu_2,\ \mu_1) & -4L^2\tau_{15}(\mu_2,\ \mu_1) & & \\ 54\tau_3(\mu_2,\ \mu_1) & -13L\,\tau_4(\mu_2,\ \mu_1) & -156\tau_1(\mu_2,\ \mu_1) & \\ 13L\,\tau_{14}(\mu_2,\ \mu_1) & 3L^2\tau_7(\mu_2,\ \mu_1) & 22L\,\tau_2(\mu_2,\ \mu_1) & -4L^2\tau_5(\mu_2,\ \mu_1) \end{bmatrix} \tag{A.5}$$

where the following auxiliary functions were introduced for the definition of the first derivatives:

$$\eta_1(m,\ n)=-(50mn^3-72mn^2-216mn+256m+64n^3 -192n^2-192n+896)/(39R(m,\ n)^3) \tag{A.6}$$

$$\eta_2(m,\ n)=(-2)(16mn^3+20mn^2-232mn+256m+64n^2 -320n+448)/(11R(m,\ n)^3) \tag{A.7}$$

$$\eta_3(m,\ n)=(5mn^3+144mn^2+12mn-512m+128n^3 -36n^2+384n-112)/(27R(m,\ n)^3) \tag{A.8}$$

$$\eta_4(m,\ n)=2(19mn^3-170mn^2+256mn+128n^3 -316n^2+200)/(13R(m,\ n)^3) \tag{A.9}$$

$$\eta_5(m,\ n)=-(64mn^2-248mn+256m)/(R(m,\ n)^3) \tag{A.10}$$

$$\eta_6(m,n)=2(32mn^3-170mn^2-44mn+512m+128n^2 +200n-784)/(13R(m,\ n)^3) \tag{A.11}$$

$$\eta_7(m,\ n)=(64mn^3-372mn^2+512mn+256n^2 -496n)/(3R(m,\ n)^3) \tag{A.12}$$

$$\eta_8(m,\ n)=(55mn^3+72mn^2-204mn-256m-64n^3 -228n^2+192n+784)/(39R(m,\ n)^3) \tag{A.13}$$

$$\eta_9(m,\ n)=2(43mn^3-20mn^2-128mn-64n^3 -52n^2+320n)/(11R(m,\ n)^{3)} \tag{A.14}$$

$$\eta_{10}(m,\ n)=(31mn^3-64mn^2-64n^3+128n^2)R(m,n)^3 \tag{A.15}$$

The following auxiliary functions were used to facilitate the definition of the second derivatives:

$$\tau_1(m,\ n)=-(100mn^4-144mn^3-432mn^2+512mn+192n^4 -376n^3-864n^2+1824n+1024)/(39R(m,\ n)^4) \tag{A.16}$$

$$\tau_2(m,\ n)=(-2)(32mn^4+40mn^3-464mn^2+512mn+256n^3 -880n^2+416n+1024)/(11R(m,\ n)^4) \tag{A.17}$$

$$\tau_3(m,\ n)=(10mn^4+288mn^3+24mn^2-1024mn-384n^4 -88n^3+1728n^2-288n+2048)/(27R(m,\ n)^4) \tag{A.18}$$

$$\tau_4(m,\ n)=2(38mn^4-340mn^3+512mn^2+384n^4-872n^3 -80n^2+1024n)/(13R(m,\ n)^4) \tag{A.19}$$

$$\tau_5(m,\ n)=-(128mn^3-496mn^2+512mn+256n^2 -992n+1024)/R(m,\ n)^4 \tag{A.20}$$

$$\tau_6(m,\ n)=2(64mn^4-340mn^3-88mn^2+1024mn+512n^3 -80n^2-2528n+2048)(13R(m,\ n)^4) \tag{A.21}$$

$$\tau_7(m,\ n)=(128mn^4-744mn^3+1024mn^2+1024n^3 -2976n^2+2048)/(3R(m,n)^4) \tag{A.22}$$

$$\tau_8(m,\ n)=(110mn^4+144mn^3-408mn^2-512mn-192n^4 -464n^3+864n^2+1536n-1024)/(39R(m,\ n)^4) \tag{A.23}$$

$$\tau_9(m,\ n)=2(86mn^4-40mn^3-256mn^2-192n^4+16n^3 -880n^2-512n)/(11R(m,\ n)^4) \tag{A.24}$$

$$\tau_{10}(m,\ n)=(62mn^4-128mn^3-192n^4+496n^3-256n^2)/R(m,\ n)^4 \tag{A.25}$$

$$\tau_{11}(m,\ n)=(72m^2n^2+432m^2n-768m^2-408mn^2+960mn -1824m-768n^2+1536n+768)/(39R(m,\ n)^4) \tag{A.26}$$

$$\tau_{12}(m,\ n)=(-2)(20m^2n^2-464m^2n+768m^2+256mn +416m+512n-1280)/(11R(m,\ n)^4) \tag{A.27}$$

$$\tau_{13}(m,\ n)=(144m^2n^2+24m^2n-1536m^2+24mn^2+1920mn -288m-1536n^2-288n+1536)/(27R(m,\ n)^4) \tag{A.28}$$

$$\tau_{14}(m,\ n)=(-2)(170m^2n^2-512m^2n+88mn^2+960mn -1024m-1536n^2+2528n-800)/(13R(m,\ n)^4) \tag{A.29}$$

$$\tau_{15}(m,\ n)=-(64m^2n^2-496m^2n+768m^2+512mn-992m)/R(m,\ n)^4 \tag{A.30}$$

$$\tau_{16}(m,\ n)=-(372m^2n^2-1024m^2n-1024mn^2+3968mn -2048m-2048n+1984)/(3R(m,\ n)^4) \tag{A.31}$$

in which $R(m,\ n)$ is given by Eq. (21)

A.2 Stiffness Matrix:

a) First derivative with respect to μ_1:

$$\left[\frac{\partial K}{\partial \mu_1}\right]=\frac{2EI}{L^3}\begin{bmatrix} -6\delta_1(\mu_1,\ \mu_2) & & & symmetric \\ -3L\delta_2(\mu_1,\ \mu_2) & -2L^2\delta_3(\mu_1,\ \mu_2) & & \\ 6\delta_1(\mu_1,\ \mu_2) & 3L\delta_2(\mu_1,\ \mu_2) & -6\delta_1(\mu_1,\ \mu_2) & \\ 3L\delta_2(\mu_1,\ \mu_2) & -L^2\delta_4(\mu_1,\ \mu_2) & 3L\delta_2(\mu_1,\ \mu_2) & -2L^2\delta_3(\mu_1,\ \mu_2) \end{bmatrix} \tag{A.32}$$

b) First derivative with respect to μ_2:

$$\left[\frac{\partial K}{\partial \mu_2}\right]=\frac{2EI}{L^3}\begin{bmatrix} -6\delta_1(\mu_2,\ \mu_1) & & & symmetric \\ -3L\delta_3(\mu_2,\ \mu_1) & -2L^2\delta_6(\mu_2,\ \mu_1) & & \\ 6\delta_1(\mu_2,\ \mu_1) & 3L\delta_3(\mu_2,\ \mu_1) & -6\delta_1(\mu_2,\ \mu_1) & \\ -3L\delta_2(\mu_2,\ \mu_1) & -L^2\delta_5(\mu_2,\ \mu_1) & 3L\delta_2(\mu_2,\ \mu_1) & -2L^2\delta_4(\mu_2,\ \mu_1) \end{bmatrix} \tag{A.33}$$

c) Second derivative with respect to μ_1:

$$\left[\frac{\partial^2 K}{\partial \mu_1^2}\right]=\frac{2EI}{L^3}\begin{bmatrix} -6\rho_1(\mu_1,\ \mu_2) & & & symmetric \\ -3L\rho_2(\mu_1,\ \mu_2) & -2L^2\rho_4(\mu_1,\ \mu_2) & & \\ 6\rho_1(\mu_1,\ \mu_2) & 3L\rho_2(\mu_1,\ \mu_2) & -6\rho_1(\mu_1,\ \mu_2) & \\ 3L\rho_3(\mu_1,\ \mu_2) & -L^2\rho_5(\mu_1,\ \mu_2) & 3L\rho_3(\mu_1,\ \mu_2) & -2L^2\rho_6(\mu_1,\ \mu_2) \end{bmatrix} \tag{A.34}$$

d) Second derivative with respect to μ_1 and μ_2:

$$\left[\frac{\partial^2 K}{\partial \mu_1 \partial \mu_2}\right]=\frac{2EI}{L^3}\begin{bmatrix} -6\rho_7(\mu_1,\ \mu_2) & & & symmetric \\ -3L\rho_8(\mu_1,\ \mu_2) & -2L^2\rho_4(\mu_2,\ \mu_1) & & \\ 6\rho_7(\mu_1,\ \mu_2) & 3L\rho_8(\mu_1,\ \mu_2) & -6\rho_7(\mu_1,\ \mu_2) & \\ 3L\rho_8(\mu_2,\ \mu_1) & -L^2\rho_9(\mu_1,\ \mu_2) & 3L\rho_8(\mu_2,\ \mu_1) & -2L^2\rho_4(\mu_1,\ \mu_2) \end{bmatrix} \tag{A.35}$$

e) Second derivative with respect to μ_2:

$$\left[\frac{\partial^2 K}{\partial \mu_2^2}\right]=\frac{2EI}{L^3}\begin{bmatrix} -6\rho_1(\mu_2,\ \mu_1) & & & symmetric \\ -3L\rho_3(\mu_2,\ \mu_1) & -2L^2\rho_6(\mu_2,\ \mu_1) & & \\ 6\rho_1(\mu_2,\ \mu_1) & 3L\rho_3(\mu_2,\ \mu_1) & -6\rho_1(\mu_2,\ \mu_1) & \\ -3L\rho_2(\mu_2,\ \mu_1) & -L^2\rho_5(\mu_2,\ \mu_1) & 3L\rho_2(\mu_2,\ \mu_1) & -2L^2\rho_4(\mu_2,\ \mu_1) \end{bmatrix} \tag{A.36}$$

where the following auxiliary functions were introduced for the definition of the first derivatives:

$$\delta_1(m,\ n)=-(n^2+4n+4)/R(m,\ n)^2 \tag{A.37}$$

$$\delta_2(m,\ n)=(-4)(n+2)/R(m,\ n)^2 \tag{A.38}$$

$$\delta_3(m,\ n)=(-4)(n^2+2n)/R(m,\ n)^2 \tag{A.39}$$

$$\delta_4(m,n)=(-12)/R(m,\ n)^2 \tag{A.40}$$

$$\delta_5(m,\ n)=(-12)n/R(m,\ n)^2 \tag{A.41}$$

$$\delta_6(m,\ n)=(-3)n^2/R(m,\ n)^2 \tag{A.42}$$

The following auxiliary functions were used to facilitate the definition of the second derivatives:

$$\rho_1(m,\ n)=(-2)n(n^2+4n4)/R(m,\ n)^3 \tag{A.43}$$

$$\rho_2(m,\ n)=(-8)n(n+2)/R(m,\ n)^3 \tag{A.44}$$

$$\rho_3(m,\ n)=(-4)(n^3+2n^2)/R(m,\ n)^3 \tag{A.45}$$

$$\rho_4(m,\ n)=(-24)n/R(m,\ n)^3 \tag{A.46}$$

$$\rho_5(m,\ n)=(-24)n^2/R(m,\ n)^3 \tag{A.47}$$

$$\rho_6(m,\ n)=(-6)n^3/R(m,\ n)^3 \tag{A.48}$$

$$\rho_7(m,\ n)=(-4)(mn+2mn+2n+4)/R(m,\ n)^3 \tag{A.49}$$

$$\rho_8(m,\ n)=(-4)(mn+4m+4)/R(m,\ n)^3 \tag{A.50}$$

$$\rho_9(m,\ n)=(-12)(mn+4)/R(m,\ n)^3 \tag{A.51}$$