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A spectrally formulated finite element method for vibration of a tubular structure

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Abstract. One of the major divisions in the mathematical modelling of a tubular structure is to include the effect of the transverse shear stress and rotary inertia in vibration of members. During the past three decades, problems of vibration of tubular structures have been considered by some authors, and special attention has been devoted to the Timoshenko theory. There have been considerable efforts, also, to apply the method of spectral analysis to vibration of a structure with rectangular section beams. The purpose of this paper is to compare the results of the spectrally formulated finite element analyses for the Timoshenko theory with those derived from the conventional finite element method for a tubular structure. The spectrally formulated finite element starts at the same starting point as the conventional finite element formulation. However, it works in the frequency domain. Using a computer program, the proposed formulation has been extended to derive the dynamic response of a tubular structure under an impact load.

Key words: spectral; finite element; vibration; tubular structure.

1. Introduction

In recent years, both researchers and practising engineers have recognised the efficiency of the spectral approach for solving a large vibrating structure. Most structures can be analysed and designed by using the conventional finite element method. However, in order to guarantee stability and accuracy of the solution, the number of elements used to model the structure may be very large indeed; more precisely, accurate results can be obtained after a substantial computational effort. As a consequence, it appears that for problems when the structure is large, it may be more effective to use alternative mathematical modelling. In this respect, attention is paid to the alternative spectral approach which works in the frequency domain, and draws its robustness from the speed and switching capabilities of the Fast Fourier Transform.

The conventional finite element method starts with the derivation of element matrices. Based on the Euler-Bernoulli theory for flexural vibration of the perfectly elastic undamped beam, the formulation can normally be started with the kinetic and strain energy considerations. However, in this formulation, the mass and inertial properties of the system are concentrated at the nodes. There are two popular ways for determining mass matrix of the system, the lumped mass method and the consistent mass method. In the lumped mass method, the mass matrix is a diagonal matrix in which all the terms in the diagonal directly represent the mass at the each degree of freedom. Another popular method is the consistent mass method, in which all

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the terms in the mass matrix are obtained from the kinetic energy of the element.

For the sake of clarity, the first part of the paper herein contains an analytical development of spectral method, while the second part deals with two examples which highlight the efficiency of the spectral method in the case of a beam and tubular structure. In the first part, the basic aspects of the spectral method are discussed, then its versatility in the case of the Timoshenko beam theory is presented and discussed. The first part ends with a presentation of a modified shear factor for the Timoshenko beam theory which takes advantage of the exact method proposed by Hutchinson (1986). In the second part, the vibration of a tubular clamped-free Timoshenko beam is investigated which has a practical interest. Finally, a fully documented example of a tubular structure is solved and discussed in order to show the efficiency and stability of the spectral method.

It is been mentioned that the efficiency and numerical stability of the spectral method arises directly from its mathematical modelling characteristics. These characteristics of the method can be shown using a program which has been written in the symbolic environment of the *Mathematica** computer package. The graphic capabilities of the *Mathematica* package allows investigation of the algorithmic features of the computed solution in relation to numerical analysis. In this way, the method feasibility can be shown in the form of flow charts, and it can be seen graphically.

The most practical difference between the spectral formulation and the conventional finite element method is that the number of elements needed in the spectral method to model a structure, which has no load or discontinuties between nodes, is much less than the number of elements in the conventional method to get the same accuracy. Thus only one element is needed to be used for a uniform segment. Using the spectral and conventional finite element methods, a comparative study has been carried out. One of the main contributions of this paper is to show the advantages of the spectral method. The length of the spectral element is not a limiting factor. and it allows a huge reduction in the number of elements needed for accurate results. The present method, which uses Timoshenko beam theory, reduces the subdivision required in a structure, and also, it treats the mass distribution exactly, which eliminates any additional effort to model the continuous mass distribution in a structure.

2. Conventional finite element review

Consider the conventional time domain method which is based on dynamic equilibrium satisfaction at selected time intervals. Based on the Euler-Bernoulli theory for flexural vibration of the perfectly elastic undamped three dimensional beam element (Fig. 1), the formulation can be started with the kinetic and strain energy consideration as follows:

$$T=\frac{1}{2}\int_0^L \rho A\dot{v}^2dx \quad \text{(kinetic energy)} \tag{1}$$

$$=\frac{1}{2}\int_0^L EIv''^2dx \quad \text{(strain energy)} \tag{2}$$

Where v is a lateral displacement. Note that the effects of shear deformation and rotary inertia have been neglected in the calculation of the kinetic and strain energy. Using Hamilton's principle

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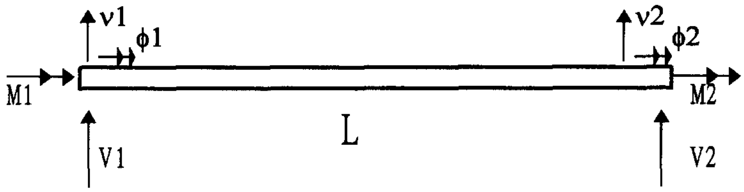


Fig. 1 Typical beam element.

$$EI\frac{\partial^4 v}{\partial x^4}+\rho A\frac{\partial^2 v}{\partial t^2}=0 \tag{3}$$

Defining the variable a as:

$$a=\sqrt{\frac{EI}{\rho A}} \tag{4}$$

the differential Eq. (3) can be written as:

$$a^2\frac{\partial^4 v}{\partial x^4}=-\frac{\partial^2 v}{\partial t^2} \tag{5}$$

If it is assumed that all of the mass is concentrated at the end nodes (lumped mass) and that there are no applied loads between these nodes. Eq. (5) can be rewritten as:

$$\frac{\partial^4 v}{\partial x^4}=0 \tag{6}$$

The simple solution of this equation can be

$$v(x)=a_0+a_1x+a_2x^2+a_3x^3 \tag{7}$$

where all coefficients are time dependent. The boundary conditions are:

$$v(0)=v_1 \qquad v(l)=v_2 \tag{8}$$

$$v'(0)=v'_1 \qquad v'(l)=v'_2 \tag{9}$$

Eq. (7) also can be written in terms of the nodal displacement as:

$$v(x)=\begin{bmatrix}\left[1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3\right]v_1+\frac{x}{L}\left[1-2\frac{x}{L}+\left(\frac{x}{L}\right)^2\right]Lv'_1+\left(\frac{x}{L}\right)^2\left[3-2\left(\frac{x}{L}\right)\right]v_2\\ +\left(\frac{x}{L}\right)^2\left[-1+\left(\frac{x}{L}\right)\right]Lv'_2\end{bmatrix} \tag{10}$$

From elementary theory of structures we have:

$$M(x)=EI\frac{d^2 v}{dx^2}\ ; \qquad V(x)=-EI\frac{d^3 v}{dx^3} \tag{11}$$

For a four degree of freedom system, these equations can be written in matrix format as:

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1' \\ v_2 \\ v_2' \end{bmatrix}$$

3. Modified finite element formulation

To include the shear-deflection effect which is significant in the flexural vibration of a short member of a frame structure, the finite element formulation can be modified. This modification can be adapted into the finite element method through the stiffness matrix by introducing a shear coefficient. The shear coefficient is defined as the ratio of the actual beam cross-sectional area to the effective area resisting shear deformation (Przemieniecki 1968). The element shear stiffness generally decreases with increasing value of the shear coefficient. The significance of this coefficient decreases as the ratio of the radius of gyration of the beam cross-section to the beam length ($r/1$) becomes small compared with unity. As a result of this modification, some of the terms in the element stiffness matrix change. For a twelve degree of freedom 3-D tubular beam element (Fig. 2) the stiffness matrix in element coordinates can be drived as (Przemieniecki 1968).

$$K = \begin{bmatrix} AE/L & & & & & & & & & & & \\ 0 & X & & & & & & & \textit{Symmetric} & & & \\ 0 & 0 & X & & & & & & & & & \\ 0 & 0 & 0 & 2GI/L & & & & & & & & \\ 0 & 0 & Z & 0 & Y & & & & & & & \\ 0 & V & 0 & 0 & 0 & Y & & & & & & \\ -EA/L & 0 & 0 & 0 & 0 & 0 & EA/L & & & & & \\ 0 & W & 0 & 0 & 0 & Z & 0 & X & & & & \\ 0 & 0 & W & 0 & V & 0 & 0 & 0 & X & & & \\ 0 & 0 & 0 & -2GI/L & 0 & 0 & 0 & 0 & 0 & 2GI/L & & \\ 0 & 0 & Z & 0 & T & 0 & 0 & 0 & V & 0 & Y & \\ 0 & V & 0 & 0 & 0 & T & 0 & Z & 0 & 0 & 0 & Y \end{bmatrix} \quad (13)$$

where

$$X = \frac{12EI}{L^3(1+\lambda)}; \quad Y = \frac{EI(4+\lambda)}{L(1+\lambda)}; \quad Z = \frac{-6EI}{L^2(1+\lambda)}; \quad V = \frac{6EI}{L^2(1+\lambda)};$$

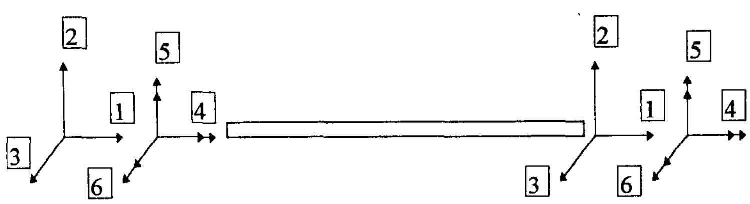


Fig. 2 Twelve degree of freedom tubular beam element.

$$W = \frac{-12EI}{L^3(1+\lambda)};\; T = \frac{EI(2-\lambda)}{L(1+\lambda)};\; \lambda = \frac{12EI}{GFL^2}$$

and F is the shear coefficient. Shear coefficient values for several common sections are as follows: rectangle 6/5, solid tubular 10/9, hollow thin tube 2. hollow thin walled square 12/5.

A popular method of deriving the mass matrix is the consistent mass method, it which all the terms in the mass matrix are obtained from the kinetic energy of the element.

$$T = \frac{1}{2}\int_0^L \rho A \dot{v}^2 dx \qquad \text{(kinetic energy)} \tag{14}$$

$$M_{ij} = \frac{\partial^2 T}{\partial \dot{v}_i \partial \dot{v}_j} \qquad \text{(mass matrix)} \tag{15}$$

The consistent mass matrix in the case of a tubular beam element with twelve degrees of freedom is:

$$M = \rho AL \begin{bmatrix} 1/3 & & & & & & & & & & & \\ 0 & X & & & & & & & & & & \\ 0 & 0 & X & & & & & & \textit{Symmetric} & & & \\ 0 & 0 & 0 & 2I/3A & & & & & & & & \\ 0 & 0 & Z & 0 & W & & & & & & & \\ 0 & Z & 0 & 0 & 0 & W & & & & & & \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 1/3 & & & & & \\ 0 & Y & 0 & 0 & 0 & V & 0 & X & & & & \\ 0 & 0 & Y & 0 & T & 0 & 0 & 0 & X & & & \\ 0 & 0 & 0 & 2I/6A & 0 & 0 & 0 & 0 & 0 & 2I/3A & & \\ 0 & 0 & T & 0 & P & 0 & 0 & 0 & Z & 0 & W & \\ 0 & T & 0 & 0 & 0 & P & 0 & Z & 0 & 0 & 0 & W \end{bmatrix} \tag{16}$$

where

$$X = \frac{1.2(r/L)^2 + 0.33\lambda^2 + 0.7\lambda + 0.37}{(1+\lambda)^2}$$

$$Y = \frac{-1.2(r/L)^2 + 0.17\lambda^2 + 0.3\lambda + 0.13}{(1+\lambda)^2}$$

$$Z = \frac{[(0.1-0.5\lambda)(r/L)^2 + 0.042\lambda^2 + 0.092\lambda + 0.052]L}{(1+\lambda)^2}$$

$$T = \frac{[-(0.1-0.5\lambda)(r/L)^2 + 0.042\lambda^2 + 0.075\lambda + 0.03]L}{(1+\lambda)^2}$$

$$W = \frac{[(0.13+0.17\lambda+0.33\lambda^2)(r/L)^2 + 0.0083\lambda^2 + 0.017\lambda + 0.0095]L^2}{(1+\lambda)^2}$$

$$V = \frac{-[(0.033+0.17\lambda-0.17\lambda^2)(r/L)^2 + 0.0083\lambda^2 + 0.017\lambda + 0.0071]L^2}{(1+\lambda)^2}$$

$$r = \sqrt{\frac{I}{A}}$$

In the next section, the spectral approach for the elementary theory of vibration is presented which can be extended to a higher vibration theory like the Timoshenko theory for any tubular structure.

4. Euler-Bernoulli spectral method

Consider a different type of solution (spectral solution) for the transverse motion of an elastic beam (Eq. (3)) as Doyle and Farris (1990):

$$v(x,\ t)=\hat{v}_1(x,\ \omega_1)e^{i\omega_1 t}+\hat{v}_2(x,\ \omega_2)e^{i\omega_2 t}+\cdots+\hat{v}_n(x,\ \omega_n)e^{i\omega_n t} \tag{17}$$

where $\hat{v}_n$ are essentially a set of discrete Fourier coefficients. In the spectral case, we obtain a method similar to classical modal superposition with some advantages. The stability of the Fast Fourier Transform allows use of a larger time step, and all frequency-dependent characteristics can be solved linearly. From a comparison between Eq. (17) and Eq. (10), it is clear that in the spectral formulation the time variational terms are formulated in the frequency domain instead of a direct formulation in the time domain. If the spectral method applies to the conventional method, it gives the following relationships:

$$[\hat{K}]=[K]-\omega^2[M] \tag{18}$$

$$[\hat{K}][U]=[F] \tag{19}$$

Where $[\hat{K}]$ is sometimes called the dynamic stiffness and is frequency dependent. At this stage, the flexural vibration equation can be written in spectral format as follows:

$$EI\frac{d^4\hat{v}_n}{dx^4}-\omega_n^2\rho A\hat{v}_n=0 \tag{20}$$

One of the solutions of this differential equation is:

$$\hat{v}_n(x)=A_n e^{-ik_n x}+B_n e^{-k_n x}+C_n e^{-ik_n(L-x)}+D_n e^{-k_n(L-x)} \tag{21}$$

where

$$k_n=\sqrt{\omega_n}\left[\frac{\rho A}{EI}\right]^{1/4} \tag{22}$$

Using Eq. (21) as a shape function, Eq. (12) can be rewritten in spectral format as:

$$[\hat{F}_n]=\frac{EI}{L^3}[\hat{K}_n][\hat{u}_n] \tag{23}$$

$$\begin{bmatrix}\hat{V}_1\\ \hat{M}_1\\ \hat{V}_2\\ \hat{M}_2\end{bmatrix}=\frac{EI}{L^3}[\hat{K}]\begin{bmatrix}\hat{v}_1\\ \hat{\phi}_1\\ \hat{v}_2\\ \hat{\phi}_2\end{bmatrix} \tag{24}$$

The $[\hat{k}]$ martrix can be determined using the boundary conditions (Doyle and Farris 1990).

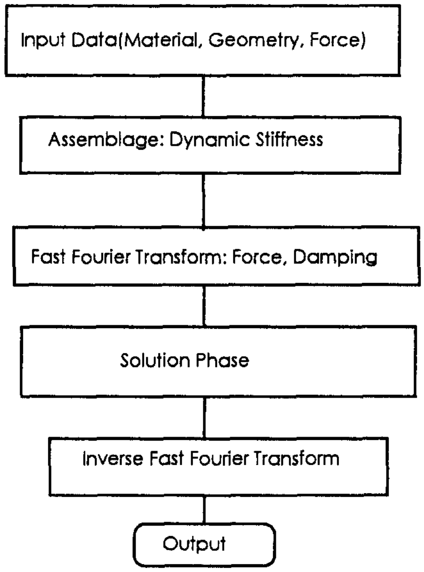


Fig. 3 Computer flow-chart for the spectral method.

$$
\begin{aligned}
\hat{k}_{11} &= (iz_{11}z_{22}-z_{12}z_{21})\zeta^3/det \\
\hat{k}_{12} &= 0.5(1+i)(z_{11}z_{22}-z_{12}z_{21})\zeta^3 L/det \\
\hat{k}_{13} &= i(z_{11}z_{21}-z_{12}z_{22})\zeta^3/det \\
\hat{k}_{14} &= -(1-i)(z_{11}z_{12})\zeta^2 L/det \\
\hat{k}_{22} &= (z_{11}z_{22}-iz_{12}z_{21})\zeta L^2/det \\
\hat{k}_{24} &= (-z_{11}z_{21}+iz_{12}z_{22})\zeta L^2/det
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
& z_{11}=1-e^{-i\zeta}e^{-\zeta};\ z_{12}=e^{-i\zeta}-e^{-\zeta} \\
& z_{21}=e^{-i\zeta}+e^{-\zeta};\ z_{22}=1+e^{-i\zeta}e^{-\zeta} \\
& det=(z_{11}^2+z_{12}^2)/(1+i);\ \zeta=kL
\end{aligned}
\tag{26}
$$

The stiffness matrix is symmetrical, as for the conventional finite element method, and terms are mostly complex. The spectral method can be constructed mainly by two concepts:

1) assembling of the dynamic stiffness matrix based on the spectral shape function for elements;
2) evaluation of the frquency-dependent quantities (damping, load) using the stable Fast Fourier Transform.

The computer algorithm for the spectral method is presented in Fig. 3 and it can be compared with the conventional finite element computer procedures.

5. Comparative study

For comparison between the results derived for the spectral Euler-Bernoulli beam element with those derived for the conventional finite element, the stiffness matrix can be closely examined. When the damping is zero, k is real and the first term in the dynamic stiffness matrix can

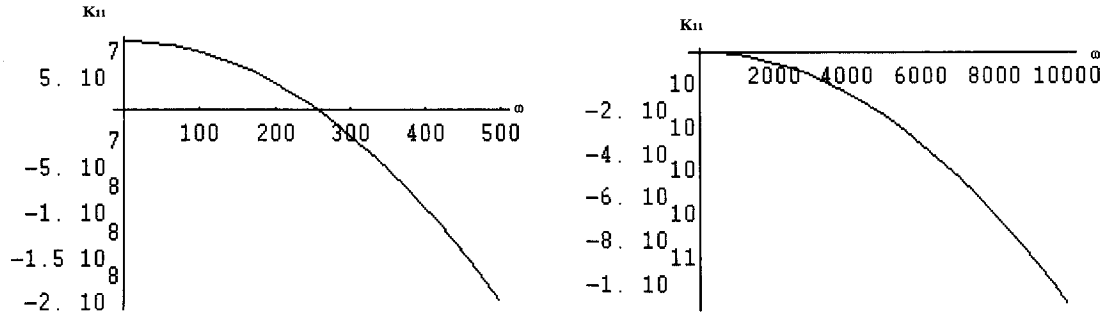


Fig. 4 Conventional finite element dynamic stiffness versus frequency.

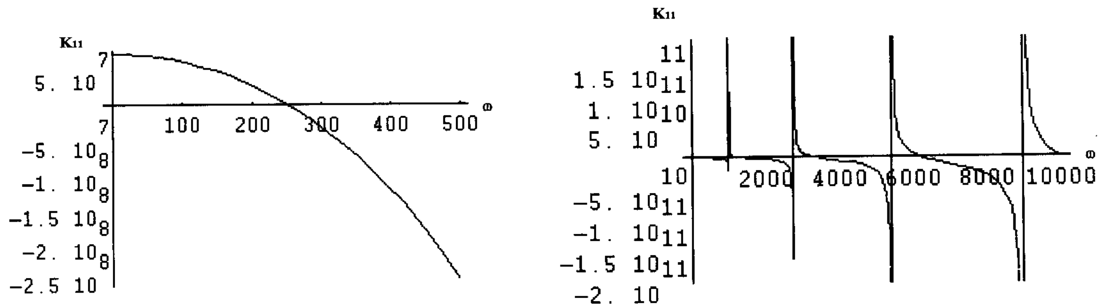


Fig. 5 Euler-Bernoulli dynamic stiffness versus frequency.

be written as Doyle and Farris (1990):

$$\bar{k}_{11}=\frac{EI}{L^3}\,\frac{(c(kL)sh(kL)+s(kL)ch(kL))(kL)^3}{1-c(kL)ch(kL)} \tag{27}$$

where

$$c=\cos;\quad s=\sin;\quad ch=\cosh;\quad sh=\sinh$$

The first term in the dynamic stiffness matrix for the conventional finite element formulation (Bathe 1982):

$$\bar{k}_{11}=k_{11}-\omega^2 m_{11}=\frac{12EI}{L^3}-\omega^2\frac{13\rho AL}{35} \tag{28}$$

Substituting Eq. (22) into Eq. (28), the spectral form of the first stiffness matrix term of the conventional finite element formulation can be written as:

$$\bar{k}_{11}=\frac{12EI}{L^3}-\frac{13(kL)^4 EI}{35L^3} \tag{29}$$

Figs. 4 and 5 show respectively, the first term of the dynamic stiffness matrix for the conventional finite element and the Euler-Bernoulli spectral method versus the frequency, using the graphical capability of the *Mathematica* package. A comparison between the spectral and conventional finite element methods shows that while the spectral dynamic stiffness intersects the zero axis

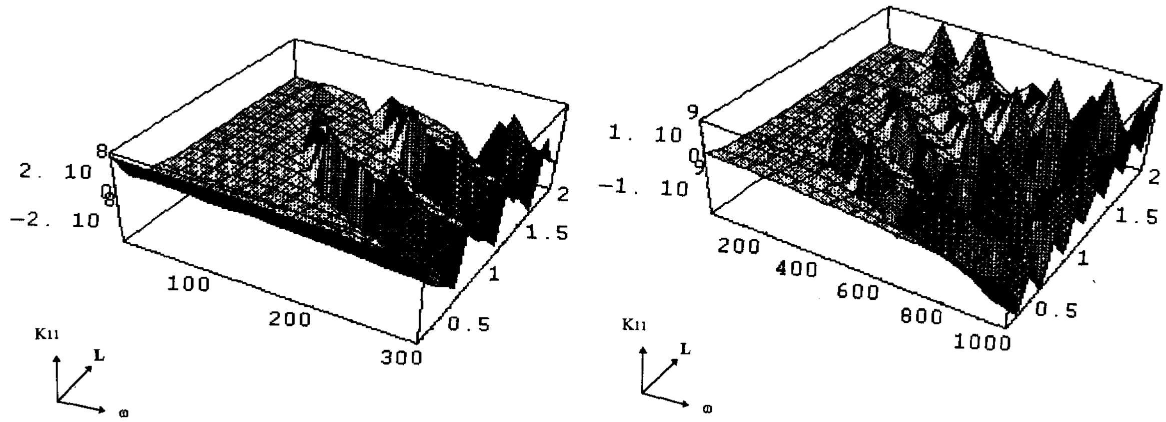


Fig. 6 Dynamic stiffness versus frequency and element length for the E-B method.

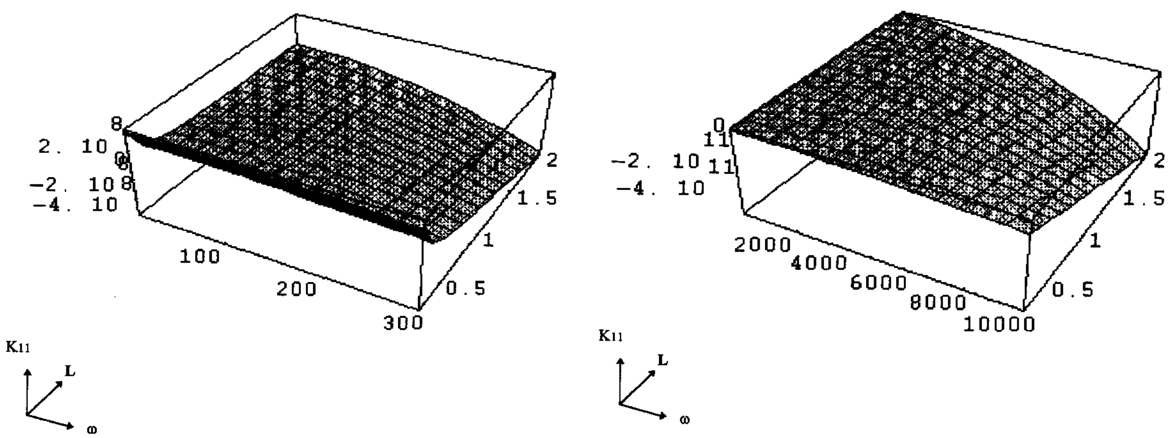


Fig. 7 Dynamic stiffness versus frequency and element length fot the FM method.

a number of times in a certain range of frequency, the conventional finite element formulation shows a single intersection with the zero axis only. (It should be noted that as the first term of the dynamic stiffness matrix is plotted against frequency, the frequencies at intersections are not resonant frequencies). Figs. 6 and 7 show the three dimensional plots for the first term of the dynamic stiffness matrix versus the frequency and element length, which have a significant physical meaning. For a constant element length in Fig. 6, as the frequency increases the number of intersections increase, and it is also true for a constant frequency and various lengths. The interaction between the frequency and element length in the first dynamic stiffness term is such that it behaves like a conventional dynamic stiffness term up to a certain limit of frequency and element length, and then it starts to behave differently. Thus, with the single formulation in the spectral method, it is possible to model a very large element in a given range of frequencies. In physical terms, it means that for the conventional finite element formulation tó have the same accuracy as the spectral method, it is necessary to subdivide a member. The degree of subdivision depends on the range of required frequencies. Only with an infinite number of elements would the conventional method match the spectral method. Thus, for a uniform beam which has no inter-span load, only one element need be used for modelling.

6. Timoshenko theory

In the previous section it was assumed that the cross-sectional dimensions of the beam element were small in comparison with its length (neglecting shear and rotary inertia effects). Corrections to the theory have been given by Timoshenko (1921) for the purpose of taking into account the effects of the cross-sectional dimensions on the frequency. With the introduction of shear deformation, the assumption of the Euler-Bernoulli theory that plane sections remain plane is no longer valid. It means that the slope θ of any section along the length of the beam simply cannot be obtained by differentiation of the transverse displacement v. Consequently, there are two independent motions θ, v. These corrections may be of considerable importance for studying the modes of vibration of higher frequencies (Horr and Schmidt 1955) when a vibrating beam is subdivided into comparatively short length portions. Timoshenko (1921) gave the equations of motion for a beam as:

$$\left(\frac{\rho A}{g}\right)\frac{\partial^2 v}{\partial t^2}-KAG\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial\theta}{\partial x}\right)=0 \tag{30}$$

$$EI\frac{\partial^2\theta}{\partial x^2}+KAG\left(\frac{\partial v}{\partial x}-\theta\right)-\left(\frac{I\rho}{g}\right)\frac{\partial^2\theta}{\partial t^2}=0 \tag{31}$$

Where K is a factor depending on the shape of the cross-section, and G is the modulus of shear rigidity. The shear force and bending moments acting on any section of beam are:

$$V=KAG\left(\frac{\partial v}{\partial x}-\theta\right)=-EI\frac{\partial^2\theta}{\partial x^2}+\left(\frac{I\rho}{g}\right)\frac{\partial^2\theta}{\partial t^2} \tag{32}$$

$$M=EI\frac{\partial\theta}{\partial x} \tag{33}$$

As shown in the previous section, the spectral solutions can be found as:

$$v(x,\ t)=\hat{v}_1(x,\ \omega_1)e^{-i(kx-\omega_1 t)}+\hat{v}_2(x,\ \omega_2)e^{-i(kx-\omega_2 t)}+\cdots+\hat{v}_n(x,\ \omega_n)e^{-i(kx-\omega_n t)} \tag{34}$$

$$\phi(x,\ t)=\hat{\phi}_1(x,\ \omega_1)e^{-i(kx-\omega_1 t)}+\hat{\phi}_2(x,\ \omega_2)e^{-i(kx-\omega_2 t)}+\cdots+\hat{\phi}_n(x,\ \omega_n)e^{-i(kx-\omega_n t)} \tag{35}$$

These solutions can be substituted into the equations of motion, which result in the following relationship between the motions for each mode (Gopalakrishnan, Martin and Doyle 1992):

$$\hat{v}=i\left[\frac{EIk^2+GAK-\rho I\omega^2}{GAKk}\right]\hat{\phi} \tag{36}$$

where

$$k=\pm\left[\frac{1}{2}\omega^2\left\{\left(\frac{1}{C_3}\right)^2+\left(\frac{C_1}{C_2}\right)^2\right\}\pm\sqrt{\left(\frac{\omega}{C_2}\right)^2+\frac{1}{4}\omega^4\left\{\left(\frac{1}{C_3}\right)-\left(\frac{C_1}{C_2}\right)^2\right\}^2}\right]^{1/2} \tag{37}$$

and

$$C_1=\sqrt{\frac{\rho I}{\rho A}};\qquad C_2=\sqrt{\frac{EI}{\rho A}};\qquad C_3=\sqrt{\frac{GAK}{\rho A}}$$

As the two motions are dependent, it is possible to take one of the motions as the unknown.

$$\hat{\phi}(x)=Ae^{-ik_1x}+Be^{-ik_2x}+Ce^{ik_1x}+De^{ik_2x}$$

$$\hat{v}(x)=P_1Ae^{-ik_1x}+P_2Be^{-ik_2x}-P_1Ce^{ik_1x}-P_2De^{ik_2x} \tag{38}$$

If it is assumed that $\rho I=0$ (no rotational inertia), and $K=\infty$ (no shear deformation), and substituting these two conditions into Eqs. (30) and (31).

$$k_1\approx\sqrt{\omega}\left[\frac{\rho A}{EI}\right]^{1/4}\qquad k_2\approx i\sqrt{\omega}\left[\frac{\rho A}{EI}\right]^{1/4} \tag{39}$$

which are the same as Eq. (22) in real and imaginary space. Using Eqs. (38) as a shape function which are the exact solutions to the governing Eqs. (30) and (31), the four unknown coefficients can be written in terms of nodal displacements. For the element of length L, with no load applied between nodes, the spectral shape function can be written as:

$$\hat{\phi}(x)=Ae^{-ik_1x}+Be^{-ik_2x}+Ce^{ik_1(L-x)}+De^{ik_2(L-x)} \tag{40}$$

$$\hat{v}(x)=P_1Ae^{-ik_1x}+P_2Be^{-ik_2x}-P_1Ce^{ik_1(L-x)}-P_2De^{ik_2(L-x)}$$

$$v_1=v(0);\ \phi_1=\phi(\theta);\ v_2=v(L);\ \phi_2=\phi(L) \tag{41}$$

It is noteworthy that there are four unknown coefficients as opposed to a minimum of six that are normally used in the conventional finite element method. Also these shape functions are different from interpolating functions as commonly used in the conventional finite element. The matrix format for the relation between the coefficients and the nodal degrees of freedom can be written as:

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=[\hat{Q}]\begin{bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{bmatrix} \tag{42}$$

After finding these coefficients, the end shear forces and end moments can be written in terms of the displacement matrix as:

$$\begin{bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{bmatrix}=\frac{EI}{L^3}[\hat{K}]\begin{bmatrix} \hat{v}_1 \\ \hat{\phi}_1 \\ \hat{v}_2 \\ \hat{\phi}_2 \end{bmatrix} \tag{43}$$

Where $\hat{K}$ is the stiffness matrix. The boundary conditions can be imposed exactly as for the conventional finite element method. These boundary conditions can be treated with the usual partitioning procedure of the system matrices.

Table 1 Timoshenko shear coefficient for a range of frequencies

Frequency[Hz]	Shear Coefficient K
0.0	0.9310
0.26	0.9293
0.52	0.9284
1.3	0.9259
2.6	0.9183
3.9	0.8931
5.2	0.7832

Table 2 Dynamic stiffness ($\bar{k}_{11}$) for range of wavenumbers

kL	F.E Dynamic Stiffness	E-B Dynamic Stifness	Timoshenko Dyamic Stiffness
0	12	12	12
0.5	11.977	11.977	11.975
1.0	11.629	11.628	11.626
1.5	10.12	10.11	10.08
2.0	6.057	5.961	5.324
2.5	−2.5	−3.11	−4.127
4	83.085	−130.733	−196.670

7. Calibration of shear coefficient for tubular elements

One of the main purposes of the development of a more refined theory is to check solutions based on elementary theories. One of the accurate theories of vibration is the Timoshenko theory, which was given in the spectral form earlier.

A shear coefficient based on Timoshenko (1921) was found best for long lengths of solid shafts and was given as:

$$K=\frac{(1+12\nu+6\nu^2)}{(7+12\nu+4\nu^2)} \tag{44}$$

where ν is Poisson's ratio. For $\nu=0.33$, the shear coefficient is $K=0.9314$. Cowper (1966) found shear coefficients for a wide range of shapes of cross-section based on comparisons with static three dimensional elasticity theory. Cowper's shear coefficient for beams of hollow circular cross-section is,

$$K=\frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2+(20+12\nu)m^2} \tag{45}$$

where m is the ratio of external to internal radius. For a thin-walled tubular member, the Cowper's shear coefficient is,

$$K=\frac{2(1+\nu)}{4+3\nu} \tag{46}$$

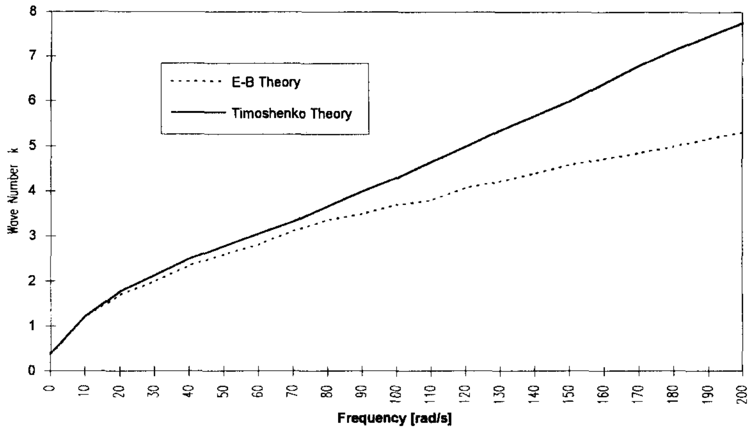


Fig. 8 k Versus frequency for E-B and Timoshenko theories.

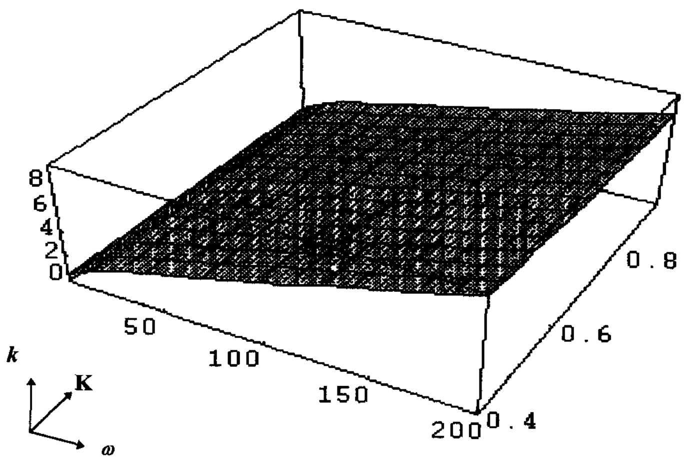


Fig. 9 k Versus frequency and shear coefficient for the Timoshenko method.

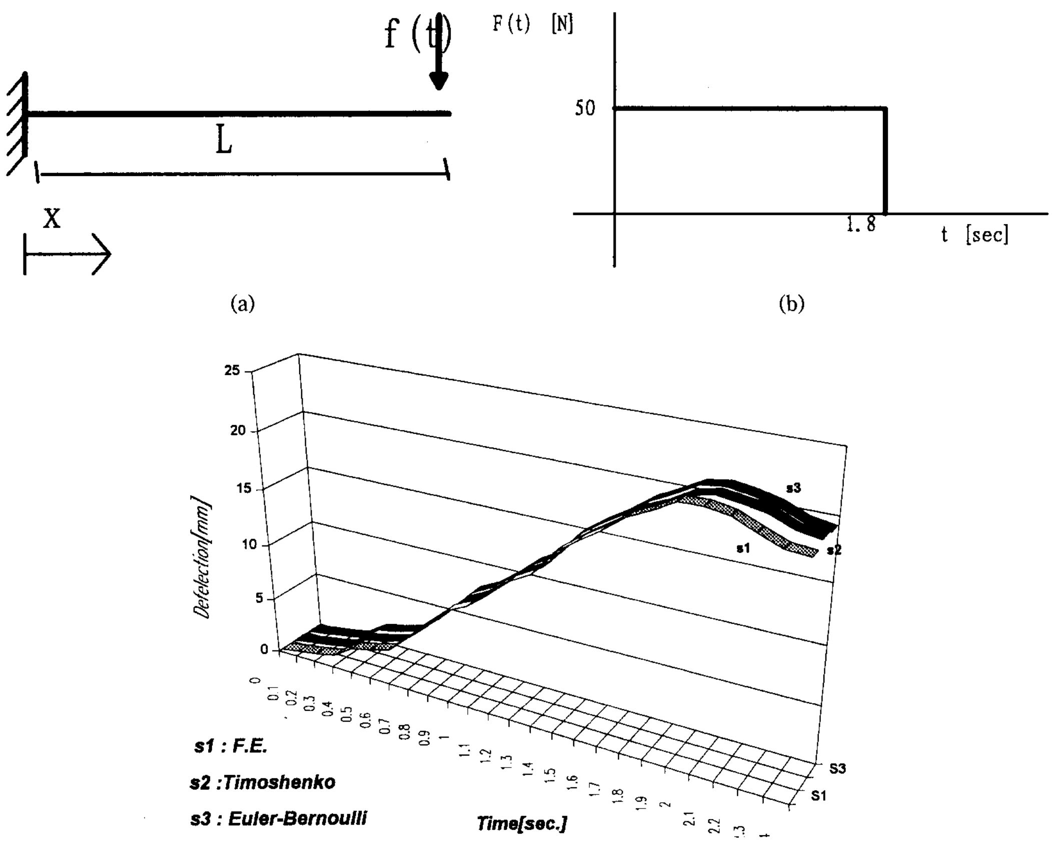
Timoshenko's shear coefficient was derived by matching with the plane stress solution for long wavelengths.

However, If is interesting to note that for the Timoshenko model, the shear coefficient is also frequency-dependent. Table 1 shows the variation of shear coefficient with frequency, in which the value corresponding to the shear coefficient is derived so as to give the same result for the Timoshenko model as an exact series solution proposed by Hutchinson (1986).

Hutchinson's exact solution is a series solution in which each term of the series identically satisfies the linear elasticity equation of the flexural vibration of a beam. This solution leads to a matrix of coefficients whose determinant must be zero. The coefficents are transcendental functions of the natural frequency. For any discrete number of frequencies over the frequency range, K can be determined and substituted into Eq. (36).

Table 3 Beam data

E[GPa]	200
G[GPa]	116
L[mm]	400
D[mm]	50
ρ[kg/m^3]	7800
m	0.99

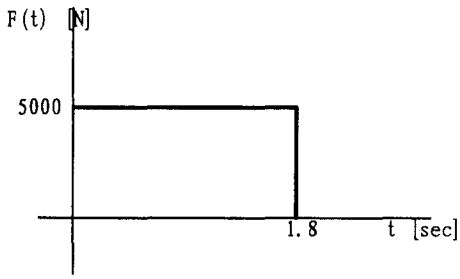


(c)

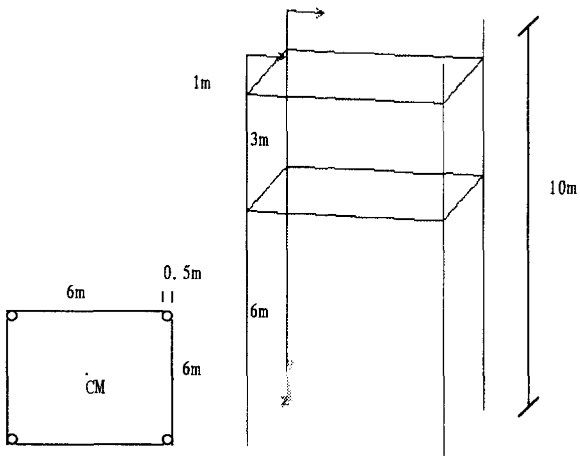
Fig. 10 (a) Cantilever beam: (b) Impulsive load; (c) Response of the beam.

8. Timoshenko dynamic stiffness matrix

For the Timoshenko theory, Eq. (37) can be substituted into Eq. (28) which gives a similar multiple zeros result as for the Euler-Bernoulli theory. Table 2 shows the values of the dynamic stiffness for the Timoshenko beam (with a frequency-dependent shear factor), the Euler-Bernoulli beam, and the conventional finite element model. They all have the same origin at $kL=0$ (low frequency), but at higher frequencies the behaviour is different. For the Timoshenko dynamic stiffness, there are a greater number of zeros in a given range than the Euler-Bernoulli dynamic



(a)



(b)

Fig. 11 (a) Impulsive load; (b) Tubular structure.

Table 4 Structure data

E[GPa]	200
G[GPa]	116
D[mm]	500
m	0.99
ρ[kg/m^3]	7800

stiffness, which means that it can give even more accurate results.

The variation of k with frequency for both of the Euler-Bernoulli and Timoshenko theories is shown in Fig. 8. It can be seen that there is a noticeable difference between these two theories as the frequency increases. If k is plotted against the frequency and shear factor (Fig. 9), it can easily be seen that the Euler-Bernoulli theory is a limit of the Timoshenko theory when $K \to 1$.

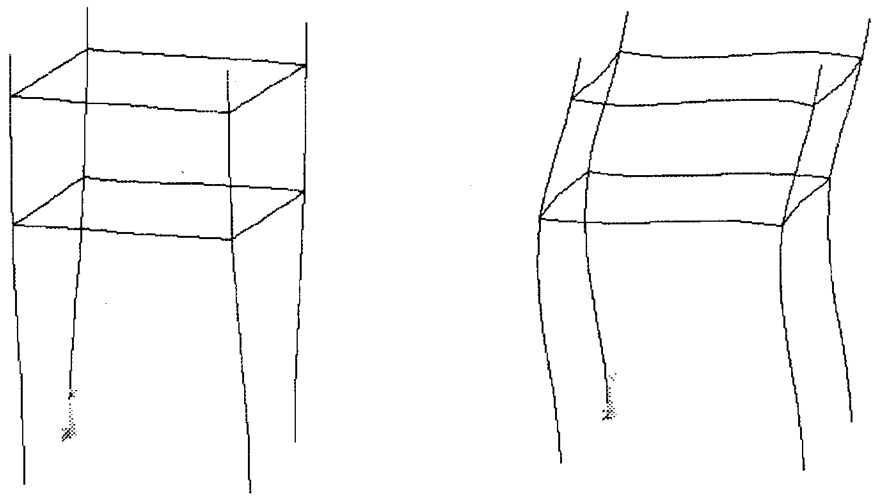


Fig. 12 Third (a) and sixth (b) mode shapes for the structure.

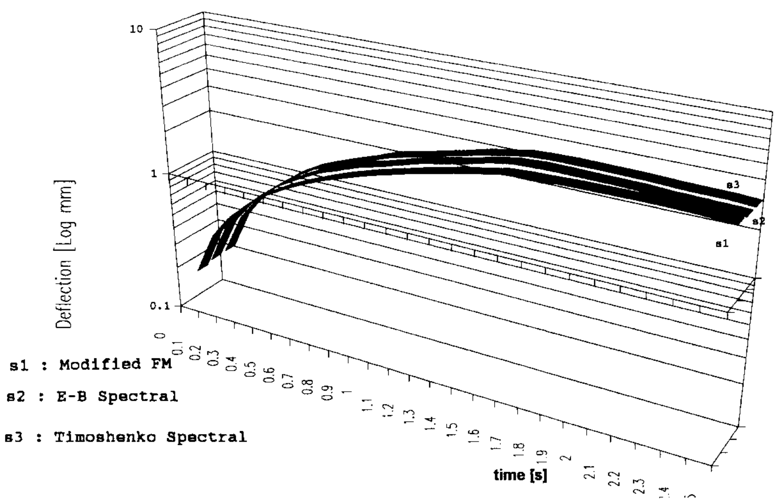


Fig. 13 Response of the structure.

9. Examples

9.1. Example I

In the first example, a clamped-free beam is considered and a simple impulsive load is applied at the tip of the beam (Fig. 10a). Using the numerical data given in Table 3, the three dimensional finite element model can be generated using three dimensional pipe elements in the ANSYS$^{\Phi}$ finite element program. The length of the beam was divided to 20 elements.

The spectral model for the beam consists of one spectral element of length 400 mm and at the fixed boundary all the DOF are constrained. The beam is impacted at the tip using a rectangular pulse (Fig. 10b), and the response is measured at the free end. Fig. 10c shows the response of the beam at the free end for the spectral and the conventional finite element methods.

9.2. Example II

The second example is a three dimensional tubular structure. Fig. 11 shows the finite element model of the structure which consists of 232 three dimensional pipe elements. The same simple impulsive load with 5000 N amplitude is applied as for the first example and the response evaluation is carried out using the conventional time domain transient analysis (Newmark method). The data for the structure are given in Table 4 and the maximum response in the x direction is measured using the ANSYS finite element program.

The spectral model consists of 20 finite spectral elements which benefits from the accuracy of the Timoshenko spectral shape function and the calibrated shear coefficient. The ratio of the internal to external radius, m, for all tubular members is 0.99 and the section is assumed to be a thin-walled section in this respect. It should be emphasised that in frequency analysis, a frequency-dependent shear coefficent requires only a single solution run and there is no need for a non-linear analysis. As already explained in the first example, as far as the computational time and memory allocation are concerned, the spectral approach is well ahead of the conventional finite element method for a large extended structure. Fig. 12 shows the third and sixth natural modes for the structure which is the result of a subspace eigenvalue analysis. The subspace method uses the subspace iteration method, which internally uses the generalised Jacobi iteration algorithm, (Bath 1982). Fig. 13 indicates and compares the response of the structure for all three methods in a logrithmic chart. The stability and accuracy of the proposed method provides a realistic solution without great computational effort.

10. Discussion and concluding remarks

A spectrally formulated finite element method of analysis has been presented which is capable of making accurate predictions of the dynamic response of tubular structures. The main features of the frequency domain spectral method were discussed in detail and two numerical examples have been solved. It was shown that the frequency-dependent characteristic of the shear coefficient can be modelled using the frequency domain spectral approach. The approach is unique, as it seems that there is no proper treatment of this problem.

The most practical difference between the spectral formulation and the conventional finite element method is the number of elements needed in the spectral method to model a structure, which is much less than the number of elements in the conventional method to obtain the same accuracy. The conventional finite element is relatively ideal for solving problems where the structure is small in extent (that is because of the differences in the number of elements needed to model the structure), whereas the spectral method is ideal when the structure is large in extent. As far as the stability and efficiency concerned, it can be assessed that even though the assemblage of global matrices has to be repeated for all frequency components in comparison with the single assemblage procedure for the conventional finite element method, the spectral approach outperforms the conventional method in the large structure cases. Furthermore, cases such as frequency-dependent damping (a common property of absorber materials) can be dealt with in a linear manner without need of iteration. The opportunity to investigate the consequences of the non-linear damping behaviour in damping devices with in a tubular structure will be taken, and this will be the subject of a subsequent paper.

References

Bathe, K. J. (1982), *Finite Element Procedures in Engineering Analysis.* New Jersey: Prentice-Hall.

Cowper, G. R. (1996), "The shear coefficient in Timoshenko's beam theory", *J. of Applied Mechanics,* 335-340.

Doyle, J. F. and Farris, T. N. (1990), "A spectrally formulated finite element for flexural wave propagation in beams", *Int. J. of Analytical And Experimental Modal Analysis,* 99-107.

Gopalarkishnan, S., Martin, M. and Doyle, J. E. (1992), "A matrix methodology for spectral analysis of wave propagation in multiple connected Timoshenko beams", *J. of Sound and Vibration,* 11-24.

Horr, A. M. and Schmidt, L. C. (1995), "Closed-form solution for the Timoshenko beam theory using a computer-based mathematical package", *Int. J. of Computers & Structures,* **55**, 405-412.

Hutchinson, J. R. (1986), "Vibration of free hollow circular cylinders", *J. of Applied Mechanics,* **53**, 641-646.

Przemieniecki, J. S. (1968), *Theory of Matrix Structrual Analysis,* McGraw-Hill, New York.

Timoshenko, S. P. (1921), "On the correction for shear of the differential equation for transverse vibrations of prismatic bars", *Philosophical Magazine,* **41**, 744-746.

Nomenclature

A	cross sectional area of beam
$[C]$	viscous damping matrix
D	beam diameter
E	Young's modulus of beam material
f	frequency
$[F]$	force vector
G	shear modulus
I	second moment of area of beam
J	torsional moment of inertia
K	cross sectional shape factor
$[K]$	stiffness matrix
L	length of the beam
m	ratio of the internal to external radius for tubular members
M_b	mass of the beam
M	mass at the free end
$[M]$	mass matrix
T	time
W	deformation energy
β	eigenvalue
ϕ	angle by which the stress leads the strain
ρ	density
ω	natural frequency
σ	stress
σ_y	yield stress
ε	strain
ν	Poisson's ratio