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Large displacement Lagrangian mechanics Part II - Equilibrium principles

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Abstract. In Lagrangian mechanics, attention is directed at the body as it moves through space. Each body point is identified by the position it would have if the body were to occupy an arbitrary reference configuration. A result of this approach is that the analyst often describes the body by using quantities that may involve more than one configuration. This is particularly common in incremental calculations and in changes of the choice of reference configuration. With the rise of very powerful computing machinery, the popularity of numerical calculation has become great. Unfortunately, the mechanical theory has been evolved in a piecemeal fashion so that it has become a conglomeration of differently developed patches. The current work presents a unified development of the equilibrium principle. The starting point is the conservation of momentum. All details of configuration are shown. Finally, full dynamic and static forms are presented for total and incremental work.

Key words: large displacement; equilibrium principle; updated Lagrangian

1. Introduction

In Lagrangian (referential) mechanics a conceptual step is inserted which gives to many analytical and computational efforts much generality and convenience. This is that there is no attempt to deal directly with the body-points. Instead, we gain access to these body-points and whatever features of them that might interest us by consulting the configuration (placement) chosen to be reference. This is any region of space that the body could occupy. Each body-point is named by the position it would have, if the body were in the reference placement. A helpful but informal analogy for the reference placement is a fictitious blank copy of the body where we may make or consult notes about the body. This helps make the naming convention clear. When we wish to know something about body-point A, we consult the notes as found on the reference copy at location ${}^{R}\bar{x}(A)$. Now we have a way of describing a body that can be adapted in any way that expedites our current goals.

In the so-called total Lagrangian formulation, the initial configuration is used as the reference

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placement throughout. In the updated Lagrangian formulations, the choice of reference configuration is occasionally changed. In some of these it is "updated" to be the configuration currently occupied by the body after each increment of motion. It is not required by the theory that updating should be done in this way. Nonetheless, it has so been universally found in the literature of updated Lagrangian techniques. This is unfortunate since this restriction of updating practice to only one scheme has served to obscure a broad understanding of the roles played by various configurations in calculation. For example, it has been discovered that the linear part of the increment of strain has been mistakenly evaluated (details of correct forms are available in Underhill, *et al.* 1995).

In principle, any configuration could be chosen. Which placement is treated as the placement of reference and when this choice is changed is for the analyst to elect. For this freedom the analyst must pay a price in care. It is necessary to keep track of how various placements are used as a computation proceeds. The present exposition concerns the precise forms that arise when using equilibrium principles to study the motions of deformable bodies during large displacements. The notation used has these features:

Left superscripts

Left superscripts indicate in which configuration a quantity is defined. For simple properties, such as temperature, this is the configuration in which the property could be measured. For some quantities more than one configuration is needed.

Displacements have the "to" configuration as the first left superscript and the "from" configuration as the second left superscript. Other quantities will have their notation explained as need arises.

Left subscript

The left subscript indicates the placement of reference.

Gradients

One extra configuration is required to describe a gradient. That is the configuration, with respect to position in which, the gradient is taken. This must also be the gradient of reference of the quantity of which the gradient is desired. The notation which marks a gradient is in the left superscripts. To the right of whatever other superscripts describe a quantity there is a comma and then the name of the gradient configuration. For example, the gradient of a quantity, ${}^{n}_{}q$, measured in C_P taken with respect to position in C_Q and referred to C_R would be ${}^{P,Q}_{R}{}^{n+1}_{}q$. This may also be shown as

$$ {}^{P,Q}_{R}{}^{n+1}_{}q = \frac{\partial {}^{P}_{Q}{}^{n}_{}q}{\partial {}^{Q}_{R}\bar{x}} $$

Configurational derivative

Configurational derivatives are a special case of the calculus of variations. In particular, they

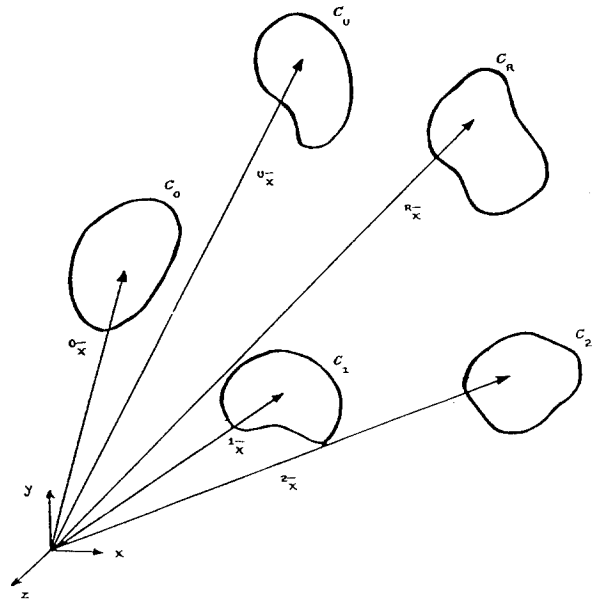

Fig. 1 Positions of a body-point X invarious configurations.

allow comparison of the value of a quantity that would pertain to a body-point if it were to occupy a nearby configuration. This is distinct from a gradient which compares the values pertaining to two distinct body-points in the same configuration. The notation for a configurational derivative uses a semi colon to separate the cofiguration about which variations are considered from the other left superscripts used to describe a quantity. Thus the configurational derivative that describes how $\frac{n}{q}$, measured in C_P, would change as the result of variations about C_Q, all referred to C_R, would be written as ${}^{P;Q}_{R}\frac{n+1}{q}$. Alternatively, we will also write

$$ {}^{P;Q}_{R}\frac{n+1}{q} = \frac{\partial {}^{P}_{Q}\frac{n}{q}}{\partial {}^{Q}_{R}\bar{x}} $$

where what is being varied is the choice of configuration, C_Q.

Some configurations are used frequently in the discussion to follow. Some of them are (see Fig. 1):

- C_0 Initial configuration.
- C_1 Present configuration. A placement achieved by motion from C_0. It may be translated, rotated and deformed.
- C_2 Next configuration. A placement later in the sequence of motion than C_1. It may be incrementally close to C_1.
- C_U Unstrained configuration. This could be any unstrained configuration. For bodies that are strained *ab initio*, such a placement may never be occupied.
- C_R Reference configuration. This is any configuration that is being used for the naming of body-points.

Other configurations may be used. These are named C_A, C_B and so on.

Another term that is frequently used as a primitive is "pseudo-". This is used with reference to quantities whose difinition involves placements that are not necessary to their measurement. The first and second Piola-Kirchhoff pseudo-stress tensors are examples. Another is the Green-Lagrange pseudo strain tensor, ${}^{U}_{R}\bar{\bar{E}}$. This is often expressed as

$$ {}^{U}_{R}\bar{\bar{E}}=\frac{1}{2}\left[{}^{U,U}_{R}\bar{\bar{u}}+{}^{U,U}_{R}\bar{\bar{u}}^{T}+{}^{U,U}_{R}\bar{\bar{u}}\cdot{}^{U,U}_{R}\bar{\bar{u}}^{T}\right] $$

where ${}^{U,U}_{R}\bar{\bar{u}}$ is the gradient of the displacement to the current placement from an unstrained placement. Since all qualities of strain are determined only by the configuration in which the body is placed and are independent of any unstrained configurations, the use of C_U introduces an extra configuration in the definition. Thus the Green-Lagrange strain is a pseudo-strain.

2. The linear momentum principle

The starting point for this discussion is the linear momentum principle. This we present without further proof or testing and accept as valid.

$$ -{}^{A}_{R}\rho\frac{d^{A}_{R}\bar{v}}{dt}+\frac{\partial\cdot{}^{A}_{A}\bar{\bar{\sigma}}}{\partial^{A}_{R}\bar{x}}+{}^{A}_{R}\bar{f}=\bar{0} \tag{1} $$

where ${}^{A}\rho$ is the mass density,

${}^{A}\bar{v}$ is the velocity,

${}^{A}\bar{\bar{\sigma}}$ is the Cauchy (true) stress tensor and

${}^{A}\bar{f}$ is the force density.

All quantities in Eq. (1) are measured in C_A, as the body passes through C_A, and referred to C_R. This principle we assume to reflect adequately a Law of Nature and to be capable of describing the actions within a small neighbourhood of material or any set of small neighbourhoods (body). With this much accepted we may proceed to use this relationship to study the motions and deformations of bodies.

Unfortunately, the linear momentum principle, as stated in Eq. (1), is not always suitable for calculation. A scalar equation would be more convenient. Such is part of the motivation behind the popular energy methods. These start by forming some kind of scalar from Eq. (1) and work with that scalar. A rather beautiful way to do this is the rate of work principle. This is found by taking the dot product of Eq. (1) with the velocity, ${}^{A}\bar{v}$. One of the advantages of this product is that it keeps all quantities measured in a single configuration. Another is that we will be able to find principles to describe both dynamic and static displacements from it.

2.1. The rate of work principle

Strart simplifying Eq. (1) by assuming that the configuration of measurement is the same as the configuration of reference, so that only one configuration is involved. Then multiply both sides by the velocity, ${}^{A}_{A}\bar{v}$. This gives

$$ -{}^{A}_{A}\rho\frac{d^{A}_{A}\bar{v}}{dt}\cdot{}^{A}_{A}\bar{v}+\frac{\partial\cdot{}^{A}_{A}\bar{\bar{\sigma}}}{\partial^{A}_{A}\bar{x}}\cdot{}^{A}_{A}\bar{v}+{}^{A}_{A}\bar{f}\cdot{}^{A}_{A}\bar{v}=\bar{0} \tag{2} $$

The terms of Eq. (2) can be manipulated separately to yield familiar forms.

2.2. Kinetic power term

A common manipulation improves the symmetry of this term. Use the rate of $({}^{A}\bar{v}\cdot{}^{A}\bar{v})$ in the first term of Eq. (2) to get

$$-{}^{A}_{A}\rho\frac{d^{A}_{A}\bar{v}}{dt}\cdot{}^{A}_{A}\bar{v}=-\frac{1}{2}{}^{A}_{A}\rho\frac{\partial}{\partial t}\left({}^{A}_{A}\bar{v}\cdot{}^{A}_{A}\bar{v}\right) \tag{3}$$

2.3. Internal power term

For this term another re-arrangement is convenient. Here consider the divergence of the product $({}^{A}_{A}\bar{\bar{\sigma}}\cdot{}^{A}_{A}\bar{v})$. We examine a combination of a derivative and a product. An underscore indicates that the derivative is not acting.

$$\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot\left({}^{A}_{A}\bar{\bar{\sigma}}\cdot{}^{A}_{A}\bar{v}\right)=\left(\frac{\partial\cdot{}^{A}_{A}\bar{\bar{\sigma}}}{\partial^{A}_{A}\bar{x}}\right)\cdot{}^{A}_{A}\bar{v}+\left(\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot{}^{A}_{A}\underline{\bar{\bar{\sigma}}}\cdot{}^{A}_{A}\bar{v}\right) \tag{4}$$

The last term can be rearranged to

$$\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot{}^{A}_{A}\underline{\bar{\bar{\sigma}}}\cdot{}^{A}_{A}\bar{v}=\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot\left({}^{A}_{A}\bar{v}\cdot{}^{A}_{A}\underline{\bar{\bar{\sigma}}}^{T}\right) \tag{5}$$

So,

$$\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot{}^{A}_{A}\underline{\bar{\bar{\sigma}}}\cdot{}^{A}_{A}\bar{v}=\frac{\partial^{A}_{A}\bar{v}}{\partial^{A}_{A}\bar{x}}:{}^{A}_{A}\bar{\bar{\sigma}}^{T} \tag{6}$$

The Angular Momentum Principle gives us that ${}^{A}\bar{\bar{\sigma}}$ is symmetric, and so the transpose mark can be dropped. Also, only the symmetric part of the velocity gradient participates in Eq. (6). Because it is in a double dot product with a symmetric dyadic, any antisymmetric part will contribute zero after the double dot product. We now restate Eq. (4) and recognize the first term on the right hand side as the internal power term of Eq. (2).

$$\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot\left({}^{A}_{A}\bar{\bar{\sigma}}\cdot{}^{A}_{A}\bar{v}\right)=\left(\frac{\partial\cdot{}^{A}_{A}\bar{\bar{\sigma}}}{\partial^{A}_{A}\bar{x}}\right)\cdot{}^{A}_{A}\bar{v}+\frac{\partial\cdot{}^{A}_{A}\bar{v}}{\partial^{A}_{A}\bar{x}}:{}^{A}_{A}\bar{\bar{\sigma}} \tag{7}$$

So the internal power is

$$\left(\frac{\partial\cdot{}^{A}_{A}\bar{\bar{\sigma}}}{\partial^{A}_{A}\bar{x}}\right)\cdot{}^{A}_{A}\bar{v}=\frac{\partial}{\partial^{A}_{A}\bar{x}}\cdot\left({}^{A}_{A}\bar{\bar{\sigma}}\cdot{}^{A}_{A}\bar{v}\right)-\frac{\partial^{A}_{A}\bar{v}}{\partial^{A}_{A}\bar{x}}:{}^{A}_{A}\bar{\bar{\sigma}} \tag{8}$$

2.4. External power term

$${}^{A}_{A}\bar{f}\cdot{}^{A}_{A}\bar{v}$$

This does not require any modification.

3. Transformation of rate of work principle

Now we can express the Rate of Work Principle in familiar terms. However there is a severe

restriction in the utility of this because only one configuration is involved. C_A is used for measurement, for gradients and for reference. The ability to use a combination of different placements for particular purposes is one of the strengths of the referential formulations of mechanics. So, now we examine how this principle may be transformed so that a placement other than the measurement configuration can be used.

First the principle is put into the integral form.

$$\int_{A\Omega}\left[-\frac{1}{2}{}^{A}_{A}\rho\frac{d}{dt}\left({}^{A}_{A}\bar{v}\cdot{}^{A}_{A}\bar{v}\right)+\frac{\partial}{\partial{}^{A}_{A}\bar{x}}\cdot\left({}^{A}_{A}\bar{\bar{\sigma}}\cdot{}^{A}_{A}\bar{v}\right)-\frac{\partial{}^{A}_{A}\bar{v}}{\partial{}^{A}_{A}\bar{x}}:{}^{A}_{A}\bar{\bar{\sigma}}+{}^{A}_{A}\bar{f}\cdot{}^{A}_{A}\bar{v}\right]{}^{A}_{A}dV=0 \tag{9}$$

Separate the second term and apply the divergence theorem. Refer the principle to C_B.

$$\int_{B\Omega}\left[-\frac{1}{2}{}^{A}_{B}\rho\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right)-\frac{\partial{}^{A}_{A}\bar{v}}{\partial{}^{A}_{B}\bar{x}}:{}^{A}_{B}\bar{\bar{\sigma}}+{}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v}\right]{}^{A}_{B}dV+\int_{B\partial\Omega}{}^{A}_{B}d\bar{a}\cdot({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v})=0 \tag{10}$$

Change the differentials, ${}^{A}_{B}dV$ and ${}^{A}_{B}d\bar{a}$, to those measured in C_B. This introduces the Jacobian. For volumes the Jacobian is the third scalar invariant of the deformation gradient, ${}^{AB}\bar{\bar{F}}$, where

$${}^{AB}\bar{\bar{F}}=\frac{\partial{}^{A}_{B}\bar{x}}{\partial{}^{B}\bar{x}} \tag{11}$$

and the third scalar invariant is

$${}^{AB}J=\frac{1}{3!}{}^{AB}\bar{\bar{F}}\overset{\times}{\times}{}^{AB}\bar{\bar{F}}:{}^{AB}\bar{\bar{F}} \tag{12}$$

This is well known as the ratio the mass densities.

$${}^{AB}J=\frac{{}^{B}\rho}{{}^{A}\rho} \tag{13}$$

The Jacobian for surface elements is the dyadic

$${}^{AB}\bar{\bar{J}}=\frac{1}{2!}{}^{AB}\bar{\bar{F}}\overset{\times}{\times}{}^{AB}\bar{\bar{F}}=\frac{1}{3!}({}^{AB}\bar{\bar{F}}\overset{\times}{\times}{}^{AB}\bar{\bar{F}}:{}^{AB}\bar{\bar{F}})\ {}^{AB}\bar{\bar{F}}^{-T} \tag{14}$$

This accomplishes the changes of scale and orientation of surface elements. In integral form we have

$$\int_{B\Omega}\left[-\frac{1}{2}{}^{A}_{B}\rho\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right)-\frac{\partial{}^{A}_{A}\bar{v}}{\partial{}^{A}_{B}\bar{x}}:{}^{A}_{B}\bar{\bar{\sigma}}+{}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v}\right]{}^{AB}_{B}J{}^{B}_{B}dV+\int_{B\partial\Omega}{}^{B}_{B}d\bar{a}\cdot{}^{AB}_{B}\bar{\bar{J}}\cdot({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v})=0 \tag{15}$$

Distribute ${}^{AB}_{B}J$ in the volume integral; apply the divergence theorem to the surface integral and recombine.

$$\int_{B\Omega}\left[-\frac{1}{2}{}^{AB}_{B}J{}^{A}_{B}\rho\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right)+\frac{\partial}{\partial{}^{B}_{B}\bar{x}}\cdot{}^{AB}_{B}\bar{\bar{J}}\cdot\left({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v}\right)-{}^{AB}_{B}J\frac{\partial{}^{A}_{A}\bar{v}}{\partial{}^{A}_{B}\bar{x}}:{}^{A}_{B}\bar{\bar{\sigma}}+{}^{AB}_{B}J\,{}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v}\right]{}^{B}_{B}dV=0 \tag{16}$$

Invoke the arbitrary body argument to change from integral back to differential form. Finally, we have the Rate of Work Principle transformed so that gradients are taken in C_B and referred to C_B but still using values measured in C_A.

$$0=-\frac{1}{2}{}^{AB}_{B}J^{A}_{B}\rho\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right)+\frac{\partial}{\partial{}^{B}_{B}\bar{x}}\cdot{}^{AB}_{B}\bar{\bar{J}}\cdot\left({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v}\right)-{}^{AB}_{B}J\frac{\partial^{A}_{A}\bar{v}}{\partial^{A}_{B}\bar{x}}:{}^{A}_{B}\bar{\bar{\sigma}}+{}^{AB}_{B}J\,{}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v} \quad (17)$$

It is appropriate to perform some manipulations on the terms of Eq. (17) separately. These will put the individual terms into more useful forms.

3.1. Kinetic power term

$$-\frac{1}{2}{}^{AB}_{B}J^{A}_{B}\rho\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right)$$

Here we have the flavour of the referential description of motion. The first thing that we need to recognize is that we are concerned with the rate of kinetic energy of a small neighbourhood of material, not a small volume of space. Since we consider that mass is an adequate scalar measure of matter we represent the kinetic energy about a body-point as

$${}^{A}_{R}dK=\frac{1}{2}{}^{A}_{R}\bar{v}\cdot{}^{A}_{R}\bar{v}\;{}_{R}dm \quad (18)$$

Immediately the reader will notice that the quantity dm has no left superscript. This is because it is invariant with respect to configuration of measurement. The quantity of material within a small neighbourhood does not change simply because it moves. The space occupied by this material may change in size, shape and orientation, but the amount of matter does not change. Nonetheless we can refer this body-point to any configuration and keep our notes there. So, the use of a left subscript is entirely proper. This will also come to the fore in that rates of dm must vanish even though dm is expressed as a product of density and volume and possibly a Jacobian. Derivatives of these will give two or three terms which must exactly offset each other adding identically to zero.

Now we express the differential mass as a product of density and differential volume. Then transform the differential volume to C_B. The rate of kinetic energy of a body-point in general is

$$\frac{d^{A}_{R}dK}{dt}=\left\{{}^{AB}_{R}J\;{}^{A}_{R}\rho\;{}^{B}_{R}dV\left(\frac{d^{A}_{R}\bar{v}}{dt}\right)\cdot{}^{A}_{R}\bar{v}\right.$$

$$\left.+\frac{1}{2}\frac{d^{AB}_{R}J}{dt}\,{}^{A}_{R}\rho\;{}^{B}_{R}dV\,{}^{A}_{R}\bar{v}\cdot{}^{A}_{R}\bar{v}+\frac{1}{2}{}^{AB}_{R}J\,\frac{d^{B}_{R}\rho}{dt}\,{}^{B}_{R}dV\,{}^{A}_{R}\bar{v}\cdot{}^{A}_{R}\bar{v}+\frac{1}{2}{}^{AB}_{R}J^{B}_{R}\rho\,\frac{d^{B}_{R}dV}{dt}\,{}^{A}_{R}\bar{v}\cdot{}^{A}_{R}\bar{v}\right\} \quad (19)$$

Usually in practice we would choose C_B the same as C_R. The last term in Eq. (19) includes the rate of ${}^{B}dV$. While the volume occupied by dm may change as the body passes through the placement, the volume when actually in that placement is a constant. So the last term in Eq. (19) vanishes. A similar argument applies to the second and third terms. Between them they describe the rate of change of ${}^{B}\rho$. This follows from the nature of the volume Jacobian as the ratio of densities. So the sum of these terms comes identically to zero. In fact, these two are exactly equal in magnitude and opposite in sign. This leaves only the leading term.

That is

$$\frac{d_R^A dK}{dt} = {}^{AB}_{B}J \; {}^{A}_{R}\rho \; {}^{B}_{R}dV \left(\frac{d_R^A \bar{v}}{dt} \right) \cdot {}^{A}_{R}\bar{v} \tag{20}$$

The discussion above defines a (pseudo-) kinetic power density, ${}^{AB}_{R}\kappa$, as

$$\frac{1}{2} {}^{AB}_{R}J \; {}^{A}_{R}\rho \frac{d}{dt} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) = \frac{d^{AB}_{B}\kappa}{dt} \tag{21}$$

In an Eulerian calculation the size of a neighbourhood is concerned only with the description of space. As the material moves and density changes the amount of material involved changes. So, for an Eulerian calculation of the rate of kinetic energy density, terms would arise that account for the changing density of material.

3.2. Stress divergence term

$$\frac{\partial}{\partial^B \bar{x}} \cdot {}^{AB}_{B}\bar{\bar{J}} \cdot \left({}^{A}_{B}\bar{\bar{\sigma}} \cdot {}^{A}_{B}\bar{v} \right)$$

Originally we had one term for internal power and one for external power. After some manipulation the internal power term was divided into two terms. The stress divergence term is one of these. The manipulations to be applied to this term are particularly simple at this stage. Later, the divergence theorem will be used to effectively change this to an expression for the power of tractions distributed over the surface of the body. For now all that is needed is to recognize the first Piola-Kirchhoff pseudo-stress tensor in the middle two factors. So, we can write

$$\frac{\partial}{\partial^B \bar{x}} \cdot \left({}^{AB}_{B}\bar{\bar{J}} \cdot {}^{A}_{B}\bar{\bar{\sigma}} \cdot {}^{A}_{B}\bar{v} \right) = \frac{\partial}{\partial^B \bar{x}} \cdot \left({}^{AB}_{B}\bar{\bar{T}} \cdot {}^{A}_{B}\bar{v} \right) \tag{22}$$

where ${}^{AB}_{B}\bar{\bar{T}}$ is the first Piola-Kirchhoff pseudo-stress tensor of C_A transformed to C_B and referred to C_R and

$${}^{AB}_{R}\bar{\bar{T}} = {}^{AB}_{R}\bar{\bar{J}} \cdot {}^{AB}_{R}\bar{\bar{\sigma}} \tag{23}$$

3.3. Velocity gradient term

$${}^{AB}_{B}\bar{\bar{J}} \frac{\partial^A_A \bar{v}}{\partial^A_B \bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}}$$

In this part we replace the velocity gradient itself with a product of the deformation gradient and the rate of pseudo-strain. To do this we take four steps:

(1) Find an expression relating the rate of the quadratic measure of deformation to the rate of pseudo-strain.
(2) Find an expression for the rate of the quadratic measure of deformation involving the velocity gradient,
(3) Use the first and second parts to identify and expression for the rate of pseudo-strain and
(4) Isolate the velocity gradient in the expression for the rate of pseudo-strain and replace.

Before beginning this process we derive a useful relation.

$$\frac{\partial {}^A_R d\bar{x}}{\partial t} = {}^A_B d\bar{x} \cdot \frac{\partial {}^A_A \bar{v}}{\partial {}^A_R \bar{x}}$$

Proof

Consider two body points X and Y, Let Y be differentially close to X, so

$${}^A\bar{x}(Y) = {}^A\bar{x}(X) + {}^A d\bar{x}(X, Y)$$

Simplify the notation so that ${}^A\bar{x}$ is ${}^A\bar{x}(X)$ and ${}^A d\bar{x}$ is ${}^A d\bar{x}(X, Y)$. Also let the velocity be $\bar{v}$ so

$${}^A\bar{v} = \frac{\partial {}^A\bar{x}}{\partial t}$$

Then we may write

$${}^A\bar{v}(X) = {}^A_A\bar{v}({}^A\bar{x})$$

$${}^A\bar{v}(Y) = {}^A_A\bar{v}({}^A\bar{x} + {}^A d\bar{x})$$

$${}^A\bar{v}(Y) = {}^A_A\bar{v} + {}^A_A d\bar{x} \cdot \frac{\partial {}^A_A \bar{v}}{\partial {}^A_A \bar{x}}$$

So the difference in velocity between the two body points is

$${}^A\bar{v}(Y) - {}^A\bar{v}(X) = {}^A_A\bar{v} + {}^A_A d\bar{x} \cdot \frac{\partial {}^A_A \bar{v}}{\partial {}^A_A \bar{x}} - {}^A_A\bar{v}$$

$${}^A\bar{v}(Y) - {}^A\bar{v}(X) = {}^A_A d\bar{x} \cdot \frac{\partial {}^A_A \bar{v}}{\partial {}^A_A \bar{x}}$$

Also

$${}^A\bar{v}(Y) = \frac{\partial}{\partial t}\left({}^A\bar{x} + {}^A d\bar{x}\right) = \frac{\partial {}^A\bar{x}}{\partial t} + \frac{\partial {}^A d\bar{x}}{\partial t}$$

$${}^A\bar{v}(Y) = \frac{\partial}{\partial t}\left({}^A\bar{x} + {}^A d\bar{x}\right) = {}^A\bar{v} + \frac{\partial {}^A d\bar{x}}{\partial t}$$

The differnce in the velocities of X and Y using this expression is

$${}^A\bar{v}(Y) - {}^A\bar{v}(X) = {}^A\bar{v} + \frac{\partial \bar{x}}{\partial t} - {}^A\bar{v} \frac{\partial {}^A d\bar{x}}{\partial t}$$

Now it is possible to subtract the two expressions for the difference in velocity

$$\frac{\partial {}^A d\bar{x}}{\partial t} - {}^A_A d\bar{x} \cdot \frac{\partial {}^A_A \bar{v}}{\partial {}^A_A \bar{x}} = \bar{0}$$

Since there are no geometric presuppositions, so it must be that the two terms are equal. That is

$$\frac{\partial {}^{A}_{R}d\bar{x}}{\partial t} = {}^{A}_{R}d\bar{x} \cdot \frac{\partial {}^{A}_{A}\bar{v}}{\partial {}^{A}_{R}\bar{x}}$$

As required

Now we begin. The quadratic measure of deformation measured in C_A and transformed and referred to C_R can be expressed using the pseudo-strain as

$${}^{A}_{R}D = {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2{}^{AR}_{R}\bar{\bar{E}} \tag{24}$$

where ${}^{AB}_{C}\bar{\bar{E}}$ is the Lagrangian pseudo-strain for a body in configuration C_A with the gradients taken in configuration C_B and referred to C_C. The form of this is

$${}^{AB}_{C}\bar{\bar{E}} = \frac{1}{2}\left[\left\{{}^{AB,B}_{C}\bar{\bar{u}} + {}^{AB,B}_{C}\bar{\bar{u}}^{T} + \left({}^{AB,B}_{C}\bar{\bar{u}} \cdot {}^{AB,B}_{C}\bar{\bar{u}}^{T}\right)\right\} + \left\{{}^{BU,B}_{C}\bar{\bar{u}} + {}^{BU,B}_{C}\bar{\bar{u}}^{T} - \left({}^{BU,B}_{C}\bar{\bar{u}} \cdot {}^{BU,B}_{C}\bar{\bar{u}}^{T}\right)\right\}\right] \tag{25}$$

and ${}^{AB,C}_{R}\bar{\bar{u}}$ is the gradient of the displacement from C_A to C_B.

$${}^{AB,C}_{R}\bar{\bar{u}} = \frac{\partial {}^{AB}_{C}\bar{u}}{\partial {}^{C}_{R}\bar{x}} \tag{26}$$

The rate of ${}^{A}_{R}D$ is

$$\frac{d\,{}^{A}_{R}D}{dt} = \frac{d}{dt}\left[{}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2{}^{AR}_{R}\bar{\bar{E}}\right] \tag{27}$$

The differential position vector in the reference configuration is a constant, so it has a rate of zero. So we are left with

$$\frac{d\,{}^{A}_{R}D}{dt} = {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2\left(\frac{d\,{}^{AR}_{R}\bar{\bar{E}}}{dt}\right) \tag{28}$$

We can also start from the definition of ${}^{A}D$ and differentiate with respect to time.

$${}^{A}_{R}D = {}^{A}_{R}d\bar{x} \cdot {}^{A}_{R}d\bar{x} - {}^{U}_{R}d\bar{x} \cdot {}^{U}_{R}d\bar{x} \tag{29}$$

The unstrained configuration is a fixed chosen one. Therefore the rate of any properties of this configuration vanishes. So, the terms involving the rate of the differential position vector in the unstrained configuration vanish. So we have

$$\frac{d\,{}^{A}_{R}D}{dt} = 2{}^{A}_{R}d\bar{x} \cdot \frac{d\,{}^{A}_{R}d\bar{x}}{dt} \tag{30}$$

Invoke the preliminary result above to change Eq. (30) to

$$\frac{d\,{}^{A}_{R}D}{dt} = 2{}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \frac{d\,{}^{A}_{A}\bar{v}}{d\,{}^{A}_{R}\bar{x}} \tag{31}$$

An effect of the double dot product with the same vector is that any antisymmetric part of the velocity gradient makes no contribution. So, one could write

$$\frac{d\,{}^{A}_{R}D}{dt} = 2\,{}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} : \left(\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}}\right)^{sym} \tag{32}$$

in order to indicate that only the symmetric part survives.

Now change the differential position vectors so that they are measured in C_R. Eq. (32) becomes

$$\frac{d\,{}^{A}_{R}D}{dt} = {}^{R}_{R}d\bar{x}\;{}^{R}_{R}d\bar{x} : 2\left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}}\right)^{sym}\cdot\left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)^{T} \tag{33}$$

As before the symmetry marked on the velocity gradient is optional since any antisymmetric part vanishes in the double dot product.

The two expressions, Eqs. (28) and (33), for the rate of ${}^{A}_{R}D$ can be compared and it must be concluded that

$$\frac{d\,{}^{AR}_{R}\bar{\bar{E}}}{dt} = \left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}}\right)^{sym}\cdot\left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)^{T} \tag{34}$$

This we will put to use by isolating the velocity gradient factor as

$$\left(\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}}\right)^{sym} = \left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)^{-1}\cdot\left(\frac{d\,{}^{AR}_{R}\bar{\bar{E}}}{dt}\right)\cdot\left(\frac{\partial\,{}^{A}_{R}\bar{x}}{\partial\,{}^{R}_{R}\bar{x}}\right)^{-T} \tag{35}$$

Now we have the expression to substritute for the velocity gradient. So

$${}^{AB}_{B}J\;\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\sigma} = {}^{AB}_{B}J\left({}^{AB}_{B}\bar{\bar{F}}^{-1}\cdot\frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-T}\right) : {}^{A}_{B}\bar{\bar{\sigma}} \tag{36}$$

Note that both the Cauchy (true) stress and the (pseudo-) strain are symmetric second order tensors. Since the (pseudo-) strain is a symmetric tensor, the rate of the (pseudo-) strain must also be symmetric. So the transpose markers on the stress and strain factors may be dropped. We use two vector identities:

$$\bar{\bar{C}}:\bar{\bar{D}} = \bar{\bar{D}}:\bar{\bar{C}}$$

$$\bar{\bar{C}}:\bar{\bar{D}} = (\bar{\bar{C}}^{T}\cdot\bar{\bar{D}}):\bar{\bar{1}} = (\bar{\bar{C}}\cdot\bar{\bar{D}}^{T}):\bar{\bar{1}}$$

to help us re-arrange the terms in Eq. (36) to

$$\left({}^{AB}_{B}\bar{\bar{F}}^{-1}\cdot\frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-T}\right) : {}^{A}_{B}\bar{\bar{\sigma}} = \left({}^{AB}_{B}\bar{\bar{F}}^{-T}\cdot{}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-1}\right) : \frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt} \tag{37}$$

This can be put into the original term

$${}^{AB}_{B}J\;\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\sigma} = {}^{AB}_{B}J\left({}^{AB}_{B}\bar{\bar{F}}^{-T}\cdot{}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-1}\right) : \frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt} \tag{38}$$

Finally we recognize that all the factors on the right hand side before the double dot product taken together are the second Piola-Kirchhoff pseudo-stress tensor, ${}^{AB}_{B}\bar{\bar{S}}$. So

$${}^{AB}_{B}J\;\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} = {}^{AB}_{B}\bar{\bar{S}} : \frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt} \tag{39}$$

This is the form sought.

3.4. Body force term

$$ {}^{AB}_{B}J\ {}^{A}_{B}\overline{f} \cdot {}^{A}_{B}\overline{v} $$

It is practical to lump the first two factors together to form a pseudo body force, ${}^{AB}\overline{f}$.

$$ {}^{AB}_{R}\overline{f} = {}^{AB}_{R}J\ {}^{A}_{R}\overline{f} \tag{40} $$

${}^{A}\overline{f}$ is a force density. It is sometimes convenient to express this as acting through some agent property of the material. Examples of this are gravity acting through the mass density and electric forces acting through the charge density. Then we would write

$$ {}^{A}\overline{f} = {}^{A}\beta\ {}^{A}\overline{b} \tag{41} $$

where ${}^{A}\beta$ is the agent property of the material and ${}^{A}\overline{b}$ is the field that acts through ${}^{A}\beta$.

Exactly the same kind of argument as was used in section for mass density may be applied to the agent density here. The conclusion may be drawn that the volume Jacobian is the ratio of the agent property in the two configurations. For such forces it may be preferred to keep the transformed equations in terms of a pseudo agent density and an active field. The term pseudo agent density is used since there is no general guaranteed constancy to the agent quantity as there is with mass. The body force term would appear as

$$ \begin{aligned} {}^{AB}_{B}J\ {}^{A}_{B}\overline{f} \cdot {}^{A}_{B}\overline{v} &= {}^{AB}_{B}J\ {}^{A}_{B}\mu\ {}^{A}_{B}\overline{b} \cdot {}^{A}_{B}\overline{v} \\ {}^{AB}_{B}J\ {}^{A}_{B}\overline{f} \cdot {}^{A}_{B}\overline{v} &= {}^{AB}_{B}\beta\ {}^{A}_{B}\overline{b} \cdot {}^{A}_{B}\overline{v} \end{aligned} \tag{42} $$

where the pseudo agent density, ${}^{AB}_{B}\mu$, is

$$ {}^{AB}_{B}\beta = {}^{AB}_{B}J\ {}^{A}_{B}\beta \tag{43} $$

Alternatively, it may be preferred to keep the agent density unchanged and use a pseudo active field. This second approach is less popular.

3.5. Transformed Rate of Work Principle

All terms may now be combined to give the strong form of the Rate of Work Principle transformed to C_B.

$$ 0 = -\frac{d\,{}^{AB}_{B}\kappa}{dt} + \frac{\partial}{\partial\,{}^{B}_{B}\overline{x}} \cdot \left({}^{AB}_{B}\overline{\overline{T}} \cdot {}^{A}_{B}\overline{v} \right) - {}^{AB}_{B}\overline{\overline{S}} : \frac{d\,{}^{AB}_{B}\overline{\overline{E}}}{dt} + {}^{AB}_{B}\overline{f} \cdot {}^{A}_{B}\overline{v} \tag{44} $$

For calculation this is often kept in integral form with the divergence term expressed as a surface integral.

$$ 0 = \int_{B\Omega} \left\{ -\frac{d\,{}^{AB}_{B}\kappa}{dt} + {}^{AB}_{B}\overline{f} \cdot {}^{A}_{B}\overline{v} - {}^{AB}_{B}\overline{\overline{S}} : \frac{d\,{}^{AB}_{B}\overline{\overline{E}}}{dt} \right\} {}^{B}_{B}dV + \int_{B\partial\Omega} {}^{B}_{B}d\overline{a} \cdot \left({}^{AB}_{B}\overline{\overline{T}} \cdot {}^{A}_{B}\overline{v} \right) \tag{45} $$

The differential surface vector, ${}^{B}_{B}d\overline{a}$, can be expanded so that only the area is in differential form, ${}^{B}_{B}da\ {}^{B}_{B}\hat{n}$. Then the unit outward normal is taken in a dot product with the first Piola-Kirchhoff pseudo-stress tensor to give the (pseudo-) tranction on the surface, scaled and rotated to suit C_B, ${}^{AB}_{B}\overline{T}$. This allows the Eq. (45) to be expressed as

$$0=\int_{B_\Omega}\left\{-\frac{d\,{}^{AB}_{B}\kappa}{dt}+{}^{AB}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v}-{}^{AB}_{B}\bar{\bar{S}}:\ \frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt}\right\}{}^{B}_{B}dV+\int_{B_{\partial\Omega}}{}^{B}_{B}da\,{}^{AB}_{B}\bar{T}\cdot{}^{A}_{B}\bar{v} \tag{46}$$

This is one of the more popular expressions of the rate of work principle, properly transformed and referred.

4. Incremental form

For the purpose of incremental investigations one could take the velocity based form, Eq. (46), and multiply it by a small time, *dt*. In examining the right hand side it is convenient to distribute the factor of *dt* and inspect each term separately.

4.1. Increment of kinetic energy density

This increment may be expressed as

$$\frac{d\,{}^{AB}_{B}\kappa}{dt}\,dt={}^{AB}_{B}d\kappa \tag{47}$$

Put this back into the form

$${}^{AB}_{B}d\kappa={}^{B}_{B}\rho\,{}^{A}_{B}\bar{v}\cdot\left(\frac{d\,{}^{A}_{B}\bar{v}}{dt}\right)dt \tag{48}$$

The last two factors combine to give the increment of velocity, ${}^{A}_{B}d\bar{v}$. This may be put into a form involving incremental displacement.

$${}^{AB}_{B}d\kappa={}^{B}_{B}\rho\,{}^{A}_{B}\bar{v}\cdot{}^{A}_{B}d\bar{v}={}^{B}_{B}\rho\left(\frac{d\,{}^{A}_{B}\bar{v}}{dt}\right){}^{A}_{B}d\bar{u} \tag{49}$$

Recognize the rate of velocity as the acceleration and substitute the rate of displacement for velocity. So Eq. (47) may be rewritten as

$${}^{AB}_{B}d\kappa={}^{B}_{B}\rho\,{}^{A}_{B}\bar{a}\cdot{}^{A}_{B}d\bar{u} \tag{50}$$

4.2. Increment of strain energy density

Express the third term of Eq. (46) as

$${}^{AB}_{B}\bar{\bar{S}}:\ \frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{dt}\,dt={}^{AB}_{B}d\alpha \tag{51}$$

Where Eq. (51) defines an incremental change in an energy density. This density is called the strain energy density, ${}^{AB}_{B}\alpha$. Combine the rate with the time increment to find

$${}^{AB}_{B}d\alpha={}^{AB}_{B}\bar{\bar{S}}:\ {}^{AB}_{B}d\bar{\bar{E}} \tag{52}$$

The last factor in Eq. (52) has benefitted from this exposé by using a notation that keeps the uses of each configuration explicit. We may divide the increment of the pseudo-strain in a linear and a nonlinear part. The linear part is

$$ {}^{2G}_{R}\bar{\bar{E}}^{L}=\frac{1}{2}\left\{ {}^{21,G}_{R}\bar{u}+{}^{21,GT}_{R}\bar{\bar{u}}^{T}+{}^{1G,G}_{R}\bar{u}\cdot{}^{21,G}_{R}\bar{\bar{u}}^{T}+{}^{21,G}_{R}\bar{\bar{u}}+{}^{1G,G}_{R}\bar{\bar{u}}^{T}\right\} \tag{53} $$

and the nonlinear part is

$$ {}^{2G}_{R}\bar{\bar{E}}^{NL}=\frac{1}{2}\left\{ {}^{21,G}_{R}\bar{\bar{u}}\cdot{}^{21,G}_{R}\bar{\bar{u}}^{T}\right\} \tag{54} $$

where C_1 is the configuration that the body moved from
C_2 is the configuration that the body moved to
C_G is the configuration where gradients are taken
C_R is the reference configuration

For the usual Updated Lagrangian Formulation the linear part becomes particularly simple. The configuration of reference, the configuration where an increment of motion begins and the configuration where gradients are taken are the same. So the linear increment of pseudo-strain becomes

$$ {}^{21}_{1}\bar{\bar{E}}^{L}=\frac{1}{2}\left\{ {}^{21,1}_{1}\bar{\bar{u}}+{}^{21,1}_{1}\bar{\bar{u}}^{T}\right\} \tag{55} $$

In previous attempts made with less explicit notations confusion arose between the configuration from which an increment of motion began and the configuration used for reference. The result was that spurious terms were included in ${}^{21}_{1}\bar{\bar{E}}^{L}$ that involved the gradient of displacement from an unstrained configuration to the current configuration.

Here the increment of pseudo strain becomes

$$ d\,{}^{AB}_{B}\bar{\bar{E}}=\frac{1}{2}\left\{ {}^{AB,B}_{B}\bar{\bar{u}}+{}^{AB,B}_{B}\bar{\bar{u}}^{T}+{}^{AB,B}_{B}\bar{\bar{u}}\cdot{}^{AB,B}_{B}\bar{\bar{u}}^{T}\right\} \tag{56} $$

4.3. Increment of volume work density

Here we get a volume work density, ${}^{AB}_{B}\tau$, and its increment is

$$ {}^{AB}_{B}d\tau={}^{AB}_{B}\bar{f}\cdot{}^{A}_{B}d\bar{u} \tag{57} $$

4.4. Increment of surface work density

Here we define a surface work density, ${}^{AB}_{B}\tau$, and its increment as

$$ {}^{AB}_{B}d\tau={}^{AB}_{B}\bar{T}\cdot{}^{A}_{B}d\bar{u} \tag{58} $$

4.5. Combined increments

These four energy density increments may be combined to express the incremental form of the Rate of Work Principle as

$$ 0=\int_{B\Omega}\left\{-{}^{AB}_{B}d\kappa+{}^{AB}_{B}d\omega-{}^{AB}_{B}d\alpha\right\}{}^{B}_{B}dV+\int_{B\partial\Omega}da\,{}^{AB}_{B}d\tau \tag{59} $$

It is common practice to modify this by lumping all terms. The two work densities are often

as a single term defining the incremental work, ${}^{AB}_{B}dW$, as

$$ {}^{AB}_{B}dW+\int_{B\Omega} {}^{AB}_{B}d\omega \; {}^{B}_{B}dV+\int_{B\partial\Omega} {}^{B}_{B}da \, {}^{AB}_{B}d\tau \tag{60} $$

Eq. (59) may be expressed as

$$ 0=-\int_{B\Omega} {}^{AB}_{B}d\kappa \, {}^{B}_{B}dV-\int_{B\Omega} {}^{AB}_{B}d\alpha \, {}^{B}_{B}dV+{}^{AB}_{B}dW \tag{61} $$

The first two terms define the increment to the kinetic energy, ${}^{AB}_{B}K$, and the increment to the strain energy, ${}^{AB}_{B}U$, respectively. So use

$$ {}^{AB}_{B}dK=\int_{B\Omega} {}^{AB}_{B}d\kappa \; {}^{B}_{B}dV \tag{62} $$

to define the increment to the kinetic energy and

$$ {}^{AB}_{B}dU=\int_{B\Omega} {}^{AB}_{B}d\alpha \, {}^{B}_{B}dV \tag{63} $$

to define the increment to the strain energy. Then the rate of work principle can be stated as

$$ 0=-{}^{AB}_{B}dK-{}^{AB}_{B}dU+{}^{AB}_{B}dW \tag{64} $$

Another popular contraction is to lump together the increments to strain energy and work under the name potential energy. Define this potential energy, ${}^{AB}_{B}\Pi$, as

$$ {}^{AB}_{B}\Pi={}^{AB}_{B}U-{}^{AB}_{B}W \tag{65} $$

So, the principle becomes

$$ 0=-{}^{AB}_{B}d\Pi-{}^{AB}_{B}dK \tag{66} $$

This is a famous form and the starting off point for many analyses.

5. Static analysis

For static analyses some special conditions should be noted. First the kinetic energy should be examined. The incremental form of the kinetic energy density is

$$ {}^{AB}_{B}d\kappa={}^{B}_{B}\rho \; {}^{A}_{B}\bar{a} \cdot {}^{A}_{B}d\bar{u} \tag{67} $$

In a static analysis the structure involved is not in motion. So the displacement is fixed and has no time derivatives of any order. So the acceleration vector in Eq. (67) is identically zero. So the kinetic energy density and the kinetic energy are identically zero. The rate of work principle becomes

$$ 0={}^{AB}_{B}d\Pi \tag{68} $$

It is also important to ask at this point what the meaning of the increment of potential energy can be if all is static. Clearly this cannot represent an incremental change as the body moves

along some path. Instead the possible changes are choices of configuration. This then allows us to express a principle for static displacements in meanigful terms. That is Eq. (68) should be

$$d^{AB}_{B}\Pi=0 \tag{69}$$

This is the condition that will allow a choice for C_A among neighbouring configurations. Eq. (69) can also written as

$$d^{AB}_{B}\Pi=d^{A}_{B}\bar{x}\cdot\frac{\delta^{AB}_{B}\Pi}{\delta^{A}_{B}\bar{x}}=0 \tag{70}$$

This shows the choice of C_A affecting the value of ${}^{AB}_{B}\Pi$. The position of C_A can also be expressed as a displacement from some other configuration. Common choices for this other configuration are the initial configuration and a nearby configuration. The nearby configuration is one that has already been calculated and that would be occupied were the loading slightly different. The position of a body point in C_A is ${}^{A}\bar{x}$ and we can write

$${}^{A}\bar{x}={}^{C}\bar{x}+{}^{AC}\bar{u} \tag{71}$$

where ${}^{C}\bar{x}$ is in the other configuration mentioned above. A differential change in ${}^{A}\bar{x}$ would give a new position

$${}^{A}\bar{x}+d^{A}\bar{x}={}^{C}\bar{x}+{}^{AC}\bar{u}+d^{A}\bar{u} \tag{72}$$

where $d^{A}\bar{u}$ is the differential change in displacement. In a vectorial sense $d^{A}\bar{u}$ is from the previously supposed position of X to a putative new position. Two things are clear: (1) $d^{A}\bar{u}$ is not a function of C_C, and (2) $d^{A}\bar{u}$ is identical to $d^{A}\bar{x}$. This leads us to the realization that Eq. (70) can be written as

$$d^{AB}_{B}\Pi=d^{A}_{B}\bar{u}\cdot\frac{\delta^{AB}_{B}\Pi}{\delta^{A}_{B}\bar{u}}=0 \tag{73}$$

There are three possibilities for the fulfillment of Eq. (73)

(1) $d^{A}_{B}\bar{u}$ is zero

(2) $d^{A}_{B}\bar{u}$ is normal to $\dfrac{\delta^{AB}_{B}\Pi}{\delta^{A}_{B}\bar{u}}$

(3) $\dfrac{\delta^{AB}_{B}\Pi}{\delta^{A}_{B}\bar{u}}$ is zero

The first possibility can be discarded because we clearly are considering variations. The configurational derivative of ${}^{AB}_{B}\Pi$ is a vector is a particular direction; whereas $d^{A}_{B}\bar{u}$ could be in any direction. So we cannot rely on a normality condition and must reject the second possibility. This leaves the third possibility. So, the equilibrium condition can be stated as

$$\frac{\delta^{AB}_{B}\Pi}{\delta^{A}_{B}\bar{u}}=\bar{0} \tag{74}$$

6. Conclusions

A careful examination of the static and dynamic equilibrium principles has been presented. This is done using a notation that keeps the use of each placement explicit. Such a notation is more cumbersome than earlier methods. It has shown its value in giving detailed and correct forms for an arbitrary Lagrangian approach to calculating the motion of a body. Particular attention is paid to the Updated Lagrangian Formulation with the result that a correction to the usual term for the increment of strain energy has been discovered.

References

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