Structural Engineering and Mechanics, Vol. 4, No. 1 (1996) 1-8
DOI: http://dx.doi.org/10.12989/sem.1996.4.1.001

A modified Zienkiewicz-Zhu error estimator

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Abstract. A new error measure for a static finite element analysis is proposed. This error measure is a modification to the Zienkiewicz and Zhu energy norm. The new error estimator is a global error measure for the analysis and is independent of finite element model size and internal stresses, hence it is readily transportable to other error calculations. It is shown in this paper the the new error estimator also produces conservative error measurements, making it a suitable procedure to adopt in commerical packages.

Key words: finite elements; static analysis; error estimation

1. Introduction

The error estimator first proposed by Zienkiewicz and Zhu (1987), (1992) is based upon calculating an energy norm as Eq. (1).

$$\|e\| = \left[\int_{\Omega} (\sigma^* - \sigma_h)^T D^{-1} (\sigma^* - \sigma_h) d\Omega \right]^{1/2} \tag{1}$$

The finite element analysis stresses is given by σ_h and a more accurate set of stresses is given by σ^*, and D is the material properties matrix.

In virtually all error estimation techiques the quantity to be assessed must be compared to a quantity of higher accuracy. For an exact error estimate σ^* would have to be the exact stress distribution through out the body of the structure. However, only for relatively simple cases is this quantity known.

Hinton and Campbell (1974) devised a techique of improving the accuracy of the stress distribution through out a finite element model. This is based on the results of Barlow (1976) who identifies the Gauss points of an element as the more accurate points for stress retrieval.

Wieberg an Li (1994) have more recently extended the procedure of stress retrieval to estimate a new stress distribution over an element patch as opposed to a patch around each node in the finite element model. This work has been shown to give accurate results and is computationally easier to implement for a global error estimation of a static finite element analysis.

The term σ^*-σ_h in Eq. (1) gives a measure of the difference in the finite element analysis stress. The product of this term and the inverse of the material properties matrix results in a measure of the strain difference. Taking the square root of the integral of the multiple of the strain difference and the stress difference over the body of the model results in a measure for the energy difference of the model.

This error measure is effective and convergent (Ainsworth, Zhu, Craig and Zienkiewicz 1989),

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but only produces a quantity that is specific for the particular model as it is dependent on the dimensions plus the stresses and strains through out the structure. This error measure is only suitable in comparing on mesh quality with another, such as in automatic mesh refinement processes where it is mainly used. Also only one element type can be used in this error measure.

A more robust error measure is to be transportable to various situations, such as predicting the error in the geometric stiffness matrix in a buckling analysis. The new error measure should also be able to cope with multiple element types such that it is valid when comparing finite element models with different element formulations.

2. New error measure

The finite element analysis approximates the potential energy of the structure. The potential energy of the finite element analysis can be written as Eq. (2).

$$\pi = \frac{1}{2}\{u\}^T[K]\{u\} - \{P\}^T\{u\} \tag{2}$$

Where $[K]$ is the elastic stiffness matrix $\{u\}$ are the nodal displacements and $\{P\}$ are the applied loads. As the loads are applied linearly in a quasi-static manner the work done is equal to the strain energy in the normal manner.

$$\frac{1}{2}\{u\}^T[K]\{u\} = \frac{1}{2}\{P\}^T\{u\} \tag{3}$$

Following substitution of Eq. (3) into Eq. (2).

$$\pi = -\frac{1}{2}\{u\}^T[K]\{u\} \tag{4}$$

Extrapolating Eq. (4) into the continuous medium.

$$\pi = -\frac{1}{2}\int_\Omega \sigma^T \varepsilon d\Omega \tag{5}$$

Where σ and ε represents the exact stresses and strains in the structure.

On completion of a finite element analysis the approximations to the stresses and strains are produced and are denoted as σ_h and ε_h respectively. The integration of their product over the body of the model will produce an approximation to the true energy of the structure. This approximation can be written as Eq. (6).

$$\pi_h = -\frac{1}{2}\int_\Omega \sigma_h{}^T \varepsilon_h d\Omega \tag{6}$$

The error in energy can be used as a relative measure of the overall error to the finite element analysis. This error can be written as Eq. (7).

$$\varepsilon = \frac{\pi_h - \pi}{\pi} \tag{7}$$

This measure is dimensionless as any dimensional variables present in the numerator will be cancelled by the same dimensional variables in the denominator.

The exact stresses and strains are unknown, but a stress and strain distribution that is more accurate than the finite element results is achieved using a patch recovery technique. The Zienkiewicz-Zhu (1987), (1992) method uses patches of elements around the nodes of the central element, but Wieberg and Li (1994) more effectively use a patch of elements around the central element to produce a more accurate stress approximation to the exact solution through out each element.

The new stress solution can be achieved by taking a weighted least squares fit to the stresses at the Gauss points of the central and surrounding elements in the patch (Barlow 1976). These stress locations are more accurate than the stresses obtained at any other points in the element. In C^0 elements the stresses are not continuous at the boundaries of the finite elements as the stresses are derived from the first differential of the finite element shape function. A weighted least squares fit of the stresses will improve the discontinuity effect and give a more accurate stress distribution throughout each element.

The energy of the finite element model requires the strains to be calculated. This can be done by multiplying the strains by the inverse of the material properties matrix. Hence the improved energy approximation for the model can be written as Eq. (8).

$$\pi^* = -\frac{1}{2}\int_{\Omega} \sigma^{*T} D^{-1} \sigma^* \, d\Omega \tag{8}$$

where σ^* is the improved stresses in the finite element model using the patch recovery technique on the Gauss point stresses in the elements.

As the improved energy approximation is more accurate than the standard finite element energy approximation it can be used to measure the relative error in the analysis. The error measure can be written as Eq. (9).

$$e^* = \frac{\pi_h - \pi^*}{\pi^*} \tag{9}$$

This error measure is again dimensionless and can be transferred to other situations where error measures are required.

3. Example

The work by Zienkiewicz and Zhu on stress recovery examines the stresses obtained in an infinte plate with a circular hole subject to a unidirectional load. Only a finite portion of this structure is modelled and is shown in Fig. 1.

This example was chosen due to the exact solution being in existence (Sokolnikoff 1956). The solution for the displacements and the stresses under a unit uniaxial stress is given by Eqs. (10)-(14).

$$u_r = \frac{1+\nu}{4Er}\left[(2-4\nu)r^2 + 2\alpha^2 + 2\left[\alpha^2(4-4\nu)) + r^2 - \frac{\alpha^2}{r^2}\right]\cos 2\theta\right] \tag{10}$$

$$u_\theta = -\frac{1+\nu}{2Er}\left[r^2 + \alpha^2(2-4\nu) + \frac{\alpha^2}{r^2}\right]\sin 2\theta \tag{11}$$

$$\sigma_x = 1 - \frac{\alpha^2}{r^2}\left(\frac{3}{2}\cos 2\theta + \cos 4\theta\right) + \frac{3\alpha^4}{2r^4}\cos 4\theta \tag{12}$$

$$\sigma_y = -\frac{\alpha^2}{r^2}\left(\frac{1}{2}\cos 2\theta - \cos 4\theta\right) - \frac{3\alpha^4}{2r^4}\cos 4\theta \tag{13}$$

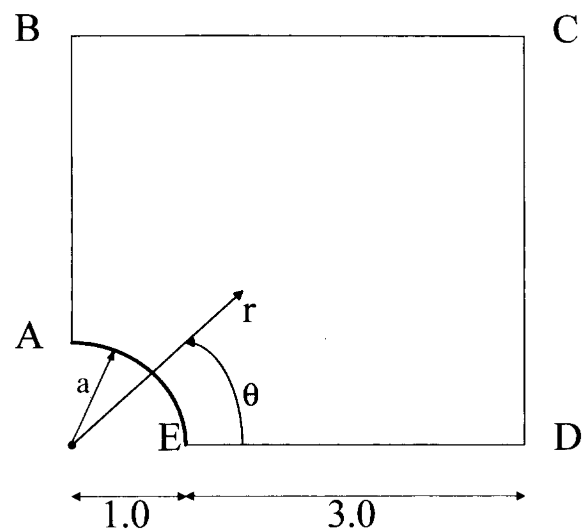

Fig. 1 Dimensions for the model of a hole in an infinite plate

$$\tau_{xy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\sin 2\theta - \cos 4\theta\right) + \frac{3a^4}{2r^4}\sin 4\theta \tag{14}$$

The appropriate boundary conditions are applied to lines AB and ED to represent symmetry. Zienkiewicz uses edge tractions applied to lines BC and DC to simulate the analytical solution to the infinite plate. However, the finite element analysis package STRAND 6.1 used in this work is capable of applying nodal constraints where the final displacement of a set of nodes can be entered into the analysis and the remainder of nodes have their displacements calculated based on these constrained displacements.

This is a more suitable procedure to represent the continuation of the structure beyond its modelled volume. The perimeter nodes being constrained will exactly represent the effect of the continuous structure. Surface tractions applied at the boundary of the model will not cause the boundary nodes to behave exactly as if there were structure beyond the model.

A number of meshes were used in the analysis of the structure consiting of four node and eight node isoparametric elements. One such mesh is shown in Fig. 2.

The Zienkiewicz-Zhu error measure was calculated and plotted in Figs. 3 and 4 for the four node and eight node isoparametric elements respectively. Figs. 5 and 6 following is the plot of the new error measure for the respective models. As the exact distribution is known, the error measure is plotted for both the exact stress solution and the stresses retrieved using a patch recovery technique.

In both error measures the values obtained using the patch recovery technique on the stresses and the exact stress solution converge to each other at higher numbers of degrees of freedom. This suggests that both error measures are accurate.

The Zienkiewicz-Zhu error measure will contain dimensions specific to the problem at hand. This error measure can not be transferred to other problems. The new error measure is a relative error of the potential energy of the model and can be used as a reference error measure as it is independent of dimensions and magnitudes of stress.

Another very significant aspect of the new error measure is that it produces a conservative estimate for the true error of the analysis. This error measure is ideal for practical error estimations

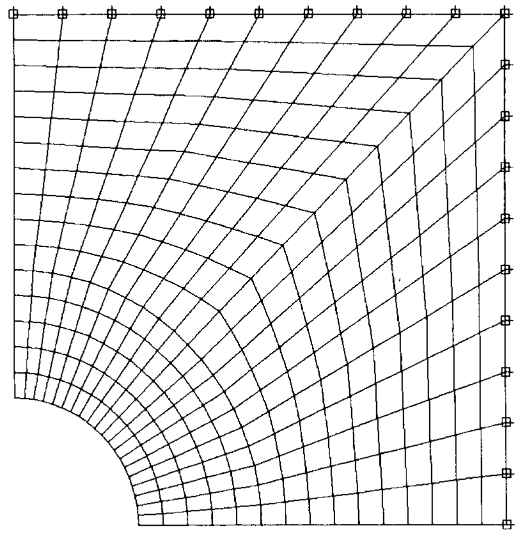

Fig. 2 Finite element mesh and constraints for the hole in the infinite plate model

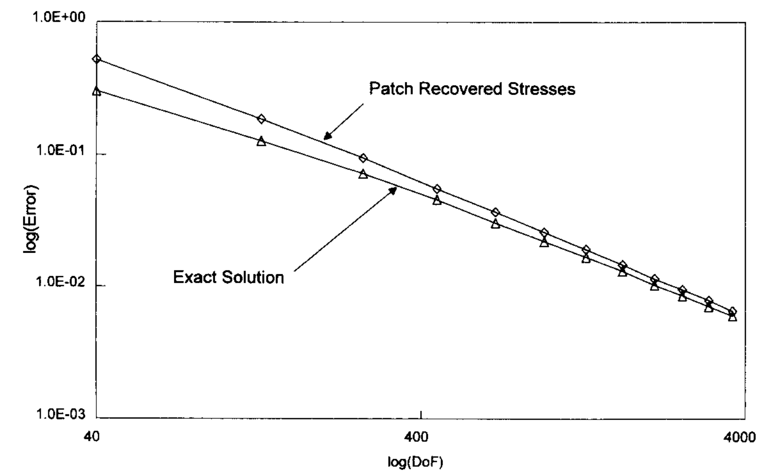

Fig. 3 Zienkiewicz-Zhu error measure for four node elements

as the analysis error will be over estimated and on the safe side for structural design.

A four node isoparametric element has a complete polynomial of order one and an eight node isoparametric element has a complete polynomial of order two, hence the convergence rate of the finite element solutions using the eight node element is twice as rapid as the four node element. This is reflected in the new error estimate as the gradient of the error against the number of degrees of freedom on a log-log graph, for the eight node elements is twice that of the four node elements. The Zienkiewicz-Zhu error measure both have a unit convergence gradient for both element types.

With the new error measure it will possible to incorporate elements with different shape functions. Figs. 7 and 8 show the potential energy of the finite element models with various mesh sizes for the four node and eight node elements respectively. The potential energy was calcuated

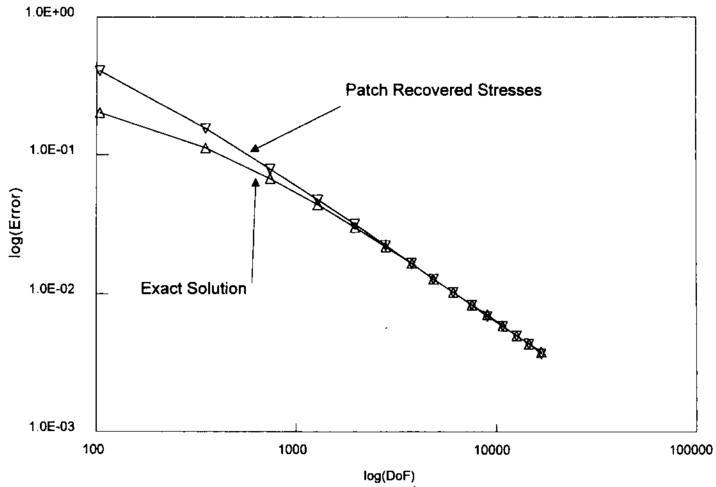

Fig. 4 Zienkiewicz-Zhu error measure for eight node elements

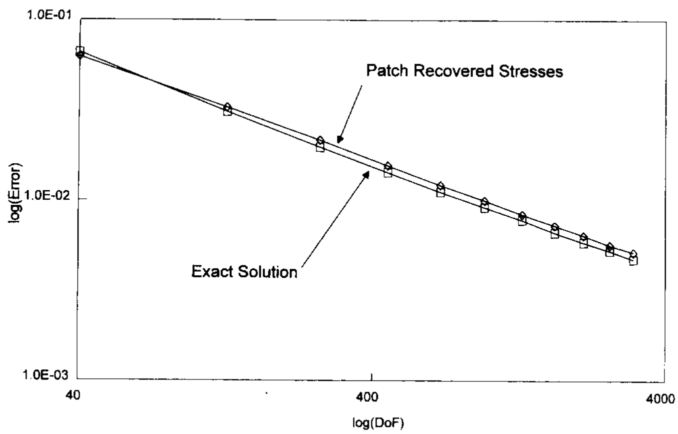

Fig. 5 New error measure for four node elements

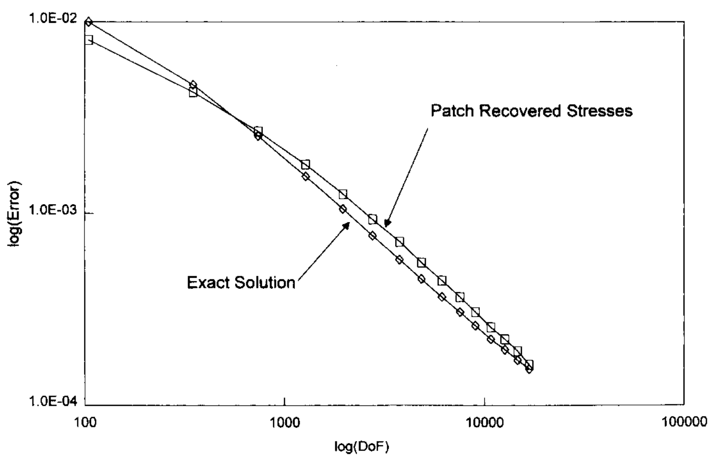

Fig. 6 New error measure for four eight elements

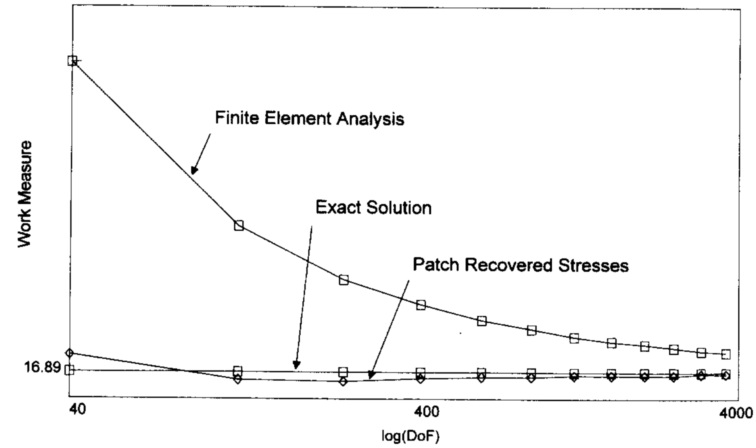

Fig. 7 Finite element analysis potential energies for four node elements

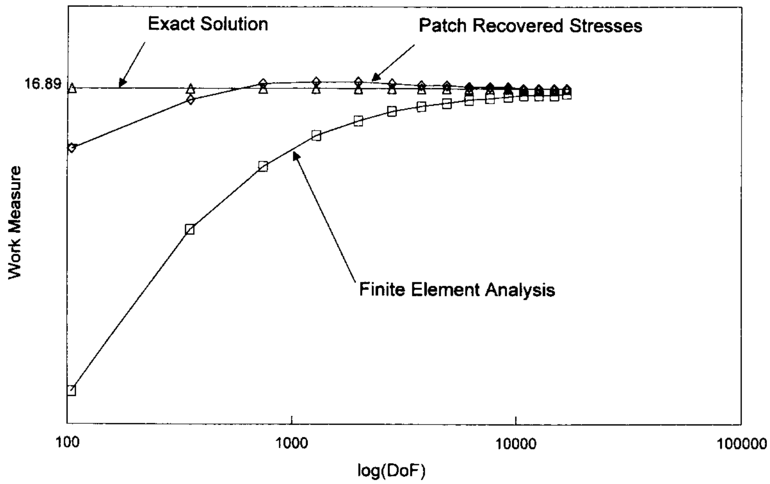

Fig. 8 Finite element analysis potential energies for eight node elements

using:

(1) The finite element analysis stresses.
(2) The patch recovered finite element analysis stresses.
(3) The exact stresses.

From both graphs it is clear that the potential energy calculated using the patch recovered stresses is closer to the exact solution than the potential energy calculated using the finite element analysis results. Hence using the potential energy derived from the patch recovered stresses as the assumed exact solution will produce an accurate error measure.

The potential energy derived using the patch recovered stresses and the finite element analysis energy have their errors on opposite sides of the exact solution. This is the reason why a conservative error estimate was produced using this quantity as the error measure in most cases.

In both the Zienkiewicz-Zhu and the new error measure the error derived using the patch recovered stresses is close to the exact solution. The Zienkiewicz-Zhu error estimate limits to the exact solution at a faster rate than the new error measure. However, the new error measure produced a conservative estimate for the finite element analysis when compared to the exact solution for most meshes. The Zienkiewicz-Zhu error estimator was not conservative.

The non conservative error measures for the analyses were obtained for quite coarse meshes. This is reflected by the energy associated with the patch recovered stresses shifting from above to below the exact value in Fig. 7 and visa versa in Fig. 8 for the non conservative estimate ranges.

Although there were non conservative error estimates using the new error measure for relatively coarse meshes, the error measures obtained were still quite accurate for these analyses.

4. Conclusions

The static finite element analysis error measure proposed in this paper is as effective as that of Zienkiewicz and Zhu. However this new error measure has the advantage of being dimensionless and thus its value more meaningful for transportation to other error calculations.

A major benefit of using the proposed error measure in this paper is that the results produced give a conservative estimate for the true error of the analysis for most meshes. This is opposed to the non conservative values using the Zienkiewicz-Zhu error estimates. The conservative error estimate makes the proposed error measure highly suitable for commercial use. Also with the new error measure it will be possible to determine error levels for analyses that incorporate elements with different orders of shape functions.

Acknowledgements

The first author is supported financially by the Australian Postgraduate Research Award (Industry) in conjunction with G+D Computing of Sydney.

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