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Equivalent stiffness method for nonlinear analysis of stay cables

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Abstract. In the famous equivalent elasticity modulus method proposed by Ernst for the geometrical nonlinear analysis of stay cables, the cable shape was assumed as a parabolic curve, and only a part of the gravity load normal to the chord was taken into account with the other part of gravity load parallel to the chord being ignored. Using the actual catenary curve and considering the entire gravity load of stay cables, the present study has derived the equivalent stiffness method to analyze the sag effect of stay cables in cable-stayed bridges. The derived equivalent stiffness can be degenerated into Ernst's equivalent elasticity modulus method with some approximations. Therefore, the Ernst's method is a special and approximate formulation of the present method. The derived equivalent stiffness provides a theoretical explanation for the famous Ernst's formula.

Keywords: equivalent stiffness method; equivalent elasticity modulus method; catenary curve; stay cable; geometrical nonlinearity; sag effect

1. Introduction

Sag effect of stay cables is an important source of nonlinearity, which must be included in structural analysis of cable-stayed bridges with long cables (Calcada *et al.* 2005, Wu *et al*. 2007, Wang *et al*. 2010). In the past decades, many methods were proposed and documented in the literature. Ernst (1965) presented the famous equivalent elasticity modulus method for the geometric nonlinear analysis of stay cables, in which the cable shape was assumed as a parabolic curve, and only part of the gravity load normal to the chord was taken into account with the other part of gravity load parallel to the chord being ignored. Therefore, Ernst's formula has been generally used in highly stressed cables with small sag effects (Wu and Cai 2009, 2010). Based on the same assumptions, the secant elasticity modulus method was suggested (Gimsing 1997). Jayaraman and Knudson (1981) presented a kind of curved elements for the analysis of cable structures in suspension bridges with small strains. In the analyses of net structures, Tang *et al*. (1995) used

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isoparametric elements of five-nodes to consider the geometric nonlinearity of cables. In essence, these methods did not fully capture the sag effect, and the accuracy depended on the iteration procedures. Yang and Chen (2003a, b) presented a two-node catenary element to analyze cable structures, for which the Modified Lagrangian interpolating function was constructed. This method adopted analytical solutions into the interpolating function for the cable sag. Yang and Tsay (2008) also used catenary elements in the cable analysis.

In the present study, to overcome the shortcomings of Ernst's formula and extend the application limit of the equivalent elasticity modulus method, an equivalent stiffness method is developed to fully reflect the sag effect of cable structures, in which a catenary curve is used to describe the actual cable shape instead of using a parabolic curve, and the entire gravity load effect of cables is taken into account instead of just the partial gravity load normal to the chord of cables. The derived equivalent stiffness provides a theoretical explanation for the famous Ernst's formula.

2. Catenary equation for flexible cables

For stay cables of cable-stayed bridges, only tension force can be loaded, and the flexural stiffness is usually ignored. As shown in Fig. 1, the catenary equation of the cable shape can be expressed using $ch(x)$ function as

$$y = a\left[ch\left(\frac{x}{a}+c\right)-ch(c)\right] \tag{1}$$

where $a = H/q$ is the ratio of horizontal force H and distributed force q along the cable (such as the gravity load); $c = sh^{-1}\left(\frac{h}{2a\cdot sh\left(\frac{L}{2a}\right)}\right)-\frac{L}{2a}$ is an integral constant; L is the horizontal projection length of cable; and h is the vertical projection length of cable.

At the end points of A and B, the tangent values can be written as

$$y'|_{x=0} = \tan\theta_A = sh(c) \tag{2}$$

$$y'|_{X=L} = \tan\theta_B = sh\left(\frac{L}{a}+c\right) \tag{3}$$

By integration, we can obtain the arc length along the axial line of cable from Eq. (1) as

$$S = \int_0^L \sqrt{1+(y')^2}dx = \sqrt{h^2+4a^2sh^2\left(\frac{L}{2a}\right)} \tag{4}$$

3. Equivalent stiffness of stay cables

To derive the equivalent stiffness method for geometric nonlinear analysis of cables, a hinged end at point A and a roller end at point B are assumed, as shown in Fig. 1. When an incremental

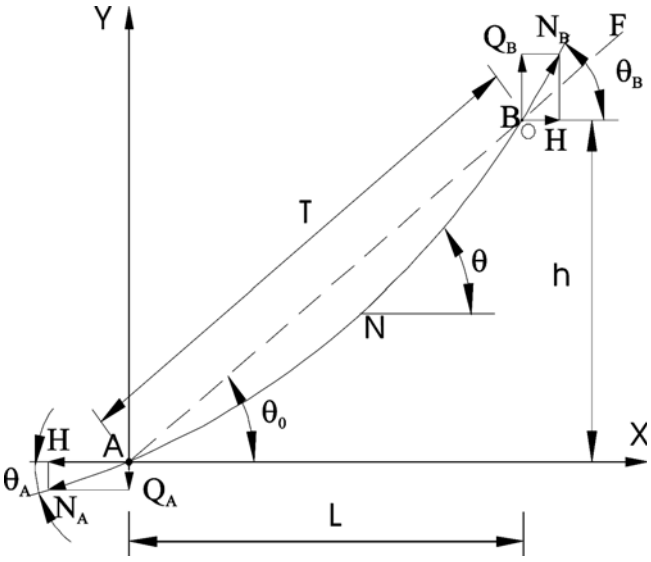

Fig. 1 Catenary curve for stay cable

tension force is applied to the cable, there is an elastic elongation along the initial shape and the sag decreases between points A and B, which results in the cable sliding toward the point B slightly. In this study, the elastic stiffness and gravity stiffness are introduced to describe these deformations.

When the force increment ΔF of F parallel to the chord is applied, the incremental axial force ΔN of cable at any arbitrary point can be written as

$$\Delta N = \frac{\Delta F \cos\theta_0}{\cos\theta} \tag{5}$$

Using the moment-area method, the flexibility, namely the deformation due to a unit force, of the cable is written as

$$\delta = \int_0^S \frac{(\Delta N)^2}{EA} ds = \frac{\cos^2\theta}{EA} S\left[1 + \frac{1}{3}(\tan^2\theta_A + \tan^2\theta_B + \tan\theta_A \tan\theta_B)\right] \tag{6}$$

where S is the arc length of the cable shown in Eq. (4), θ_0, θ_A, and θ_B are the angles shown in Fig. 1. EA is the tension stiffness, i.e., elasticity modulus, of stay cables. Based on the relationship between the flexibility and stiffness, one can obtain the elasticity stiffness from Eq. (6) as

$$K_e^{Equ} = \frac{1}{\delta} = \frac{EA}{\cos^2\theta_0 S\left[1 + \frac{1}{3}(\tan^2\theta_A + \tan^2\theta_B + \tan\theta_A \tan\theta_B)\right]} \tag{7}$$

Under the condition of initial shape of cables, one can obtain the difference S between the arc length and chord length from Eq. (4) as

$$\Delta S = S - T = \sqrt{h^2 + 4a^2 sh^2\left(\frac{L}{2a}\right)} - T \tag{8}$$

where T is the chord length.

Because the horizontal component H is constant, we can determine it at any arbitrary point such as $H = F\cos\theta_0$. When the tension force of cable increases, the arc length decreases. The gravity stiffness is used to establish the relationship between F and ΔS as

$$K_g^{Equ} = -\frac{\partial F}{\partial(\Delta S)} = -\frac{\partial F}{\partial H}\frac{\partial H}{\partial(\Delta S)} = \frac{H\sqrt{h^2 + 4a^2 sh^2\left(\frac{L}{2a}\right)}}{\cos\theta_0\left\{2aLsh\left(\frac{L}{2a}\right)ch\left(\frac{L}{2a}\right) - \left[2ash\left(\frac{L}{2a}\right)\right]^2\right\}} \tag{9}$$

4. Proposed equivalent stiffness

From Eqs. (7) and (9), we can obtain the total cable deformations under a unit force increment ($\Delta F = 1$) as

$$\delta_T = \frac{1}{K_e^{Equ}} + \frac{1}{K_g^{Equ}} \tag{10}$$

and the combined total equivalent stiffness can be written as

$$K_{eg}^{Equ} = \frac{1}{\delta_T} = \frac{K_e^{Equ}}{1 + \frac{K_e^{Equ}}{K_g^{Equ}}} \tag{11}$$

For the convenience of finite element analysis, we can obtain the stiffness matrix of cable element from Eq. (11) as

$$[K] = K_{eg}^{Equ} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{12}$$

5. Approximation and relationship to Ernest' formula

When the cable structures are simulated using finite elements, the Ernst's equivalent elasticity modulus is commonly used to form the element stiffness. As shown in Fig. 1, the chord length is assumed as T between points A and B. The Ernst's formula can be re-written in the following stiffness format as (Ernst 1965)

$$K_{eg}^{Ernst} = \frac{\frac{EA}{T}}{1 + \frac{(qL)^2 AE}{12F^3}} = \frac{\frac{EA}{T}}{1 + \frac{\frac{EA}{T}}{\frac{12F^3}{(qL)^2 T}}} = \frac{K_e^{Ernst}}{1 + \frac{K_e^{Ernst}}{K_g^{Ernst}}} \tag{13}$$

where the elastic stiffness $K_e^{Ernst} = \frac{EA}{T}$, and the gravity stiffness $K_g^{Ernst} = \frac{12F^3}{(qL)^2 T}$.

When some approximations are introduced to Eqs. (7) and (9), we can degenerate Eq. (11) into Eq. (13). In other words, the equivalent elasticity modulus formula, i.e., the Ernst's formula, is a special and approximated formulation of the equivalent elasticity stiffness derived in the present study. These approximations are summarized as follows

(1) $\tan\theta_A \doteq \tan\theta_B \doteq \tan\theta_0$

(2) $S \doteq T$

(3) $2ash\left(\frac{L}{2a}\right) = 2a\left[\frac{L}{2a} + \frac{1}{3!}\left(\frac{L}{2a}\right)^3 + \frac{1}{5!}\left(\frac{L}{2a}\right)^5 + \ldots\right] \doteq L + \frac{L^3}{24a^2}(\text{or } L)$

(4) $Lch\left(\frac{L}{2a}\right) = L\left[1 + \frac{1}{2!}\left(\frac{L}{2a}\right)^2 + \frac{1}{4!}\left(\frac{L}{2a}\right)^4 + \ldots\right] \doteq L + \frac{L^3}{8a^2}$

(5) $\sqrt{h^2 + \left[2ash\left(\frac{L}{2a}\right)\right]^2} = 2ash\left(\frac{L}{2a}\right)\sqrt{1 + \left[\frac{h}{2ash\left(\frac{L}{2a}\right)}\right]^2} \doteq 2ash\left(\frac{L}{2a}\right)\sqrt{1 + \left(\frac{h}{L}\right)^2} = 2ash\left(\frac{L}{2a}\right)\sec\theta_0$

Substituting approximations (1) and (2) into Eq. (7), the elasticity stiffness K_e^{Equ} can be degenerated into K_e^{Ernst}; substituting approximations (3), (4) and (5) into Eq. (9), the gravity stiffness K_g^{Equ} can be degenerated into K_g^{Ernst}. As a result, Eq. (11) becomes the same as the Ernst's formula, Eq. (13).

6. Examples

For the structural analyses of cable-stayed bridges, the Ernst's equivalent elasticity modulus method is often used. In the first example, the Ernst's method and the present method are compared. For this cable, the gravity load is $q = 0.987$ kN/m, horizontal projection length is $L = 127.506$ m, vertical projection length is $h = 75.977$ m, and tension stiffness is $EA = 2.409 \times 10^6$ kN. The

Table 1 Comparison of elasticity and gravity stiffness for different horizontal forces

H	K_e^{Ernst}	K_e^{Equ}	K_g^{Ernst}	K_g^{Equ}	K_{eg}^{Ernst}	K_{eg}^{Equ}
194.68	16230.31	15199.55	59.41	58.53	59.20	58.31
283.54	16230.31	15734.71	183.54	182.25	181.49	180.16
457.79	16230.31	16038.10	772.50	770.41	737.41	735.10
1319.42	16230.31	16207.08	18495.50	18489.46	8644.51	8636.60
1749.29	16230.31	16217.10	43101.96	43093.95	11790.52	11782.95
2179.02	16230.31	16221.04	83309.99	83300.02	13583.91	13577.71
2608.68	16230.31	16224.40	142947.80	142935.90	14575.41	14570.52
3038.31	16230.31	16226.02	225845.20	225831.30	15142.12	15138.33
3467.90	16230.31	16226.86	335825.60	335809.70	15482.07	15478.89
773842.40	16230.31	16230.00	3.73×10^{12}	3.73×10^{12}	16230.31	16230.00

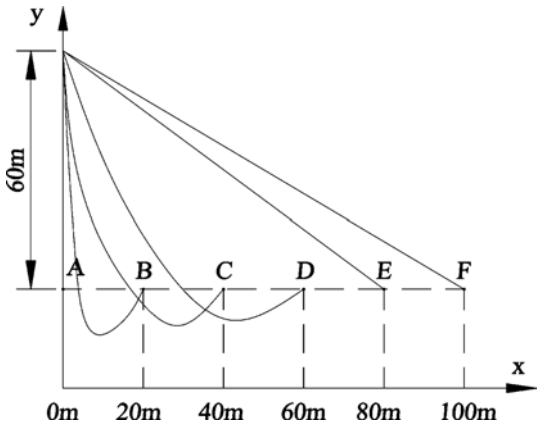

Fig. 2 Example 2

Table 2 Comparison of vertical and horizontal forces

Position	Vertical force Q (kN)				Horizontal force H (kN)			
	Jayaraman (1981)	Tang (2003)	Yang (2003)	Present results	Jayaraman (1981)	Tang (2003)	Yang (2003)	Present Results
Point A	20.02	20.02	20.02	20.02	0.00	0.00	0.00	0.00
Point B	19.93	19.93	19.93	19.94	3.06	3.06	3.06	3.06
Point C	19.24	19.24	19.24	19.25	9.17	9.17	9.17	9.18
Point D	15.73	15.73	15.73	15.74	22.15	22.15	22.15	22.16
Point E	-328.80	-328.80	-328.87	-330.61	504.00	504.00	504.10	506.48
Point F	-2511000	-2511000	-2553385	-2751172	4170000	4170000	4255724	4585190

horizontal forces, along with the corresponding elasticity stiffness, gravity stiffness, and equivalent elastic stiffness are listed and compared in Table 1.

From the results shown in Table 1, we can conclude that the Ernst's equivalent elasticity modulus K_{eg}^{Ernst} slightly overestimates the stiffness of stay cable for this particular example. For cables with high stress, the results of two methods are nearly identical.

In the second example shown in Fig. 2, the initial arc length with zero stress is $S_0 = 100$ m, the tension stiffness is $EA = 3 \times 10^7$ kN, the gravity load along x-axis line is $q = 1$ kN/m, and the coefficient of thermal expansion is $\alpha = 6.5 \times 10^{-6}$. When the cable is heated to 100°C, the bottom point of the cable is moved to points A, B, C, D, E and F, which are 0 m, 20 m, 40 m, 60 m, 80 m and 100 m apart from the starting point A, respectively. The horizontal and vertical forces are predicted and listed in Table 2.

We can see from Table 2 that the results of the present method agree well with those reported in the literature except for point F where the present method predicts about 7.8% higher vertical and horizontal forces than the other methods.

The comparison with available methods reveals that there is some difference between the present method and the other methods in some cases. The present method is based on rational derivations and can be deemed correct. The proposed equivalent stiffness method can be used to analyze the sag effect of stay cables, especially for the long stays in long-span cable-stayed bridges. It can also

be used for static analysis of transmission lines.

7. Conclusions

This paper presents an equivalent stiffness method to analyze the sag effect for stay cables in cable-stayed bridges, which eliminates the Ernst's assumptions for equivalent elasticity modulus method. The cable shape is modeled with the actual catenary curve, and the whole gravity load is considered, instead of only the partial gravity load normal to the chord. With some approximations, this method can be degenerated to the Ernst's equivalent elasticity modulus method. From this point of view, Ernst's formula is an approximated and special formulation of the present method. The derived equivalent stiffness provides a theoretical explanation for the famous Ernst formula.

References

Calcada, R., Cunha, A. and Delgado, R. (2005), "Analysis of traffic-induced vibrations in a cable-stayed bridge. Part : numerical modeling and stochastic simulation", *J. Bridge Eng.*, **10**(4), 386-397.

Ernst, H.J. (1965), "Der E-Modal von Seilen unter Berücksichtigung des Durchhanges", *Der Bauingenieur*, **40**(2), 52-55.

Gimsing, N.L. (1997), *Cable Supported Bridge-concept and Design*, John Wiley & Sons Ltd., England.

Jayaraman, H.B. and Knudson, W.C. (1981), "A curved element for the analysis of cable structures", *Comput. Struct.*, **14**(3), 325-333.

Tang, J.M., Shen, Z.Y. and Qian, R.J. (1995), "A nonlinear finite element method with five-node curved element for analysis of cable structures", *Proceeding of IASS International Symposium*, **2**, 929-935.

Tang, M.L. (2003), "3D geometric nonlinear analysis of long-span suspension bridge and its software development", Ph.D. Thesis, China Southwest Jiaotong University, 146-158. (in Chinese)

Yang, M.G. and Chen, Z.Q. (2003), "Nonlinear analysis of cable structures using A two-node curved cable element of high precision", *Chinese J. Eng. Mech.*, **20**(1), 42-47. (in Chinese)

Yang, Y.B. and Tsay, J.Y. (2008), " Wind-induced aerostatic instability of cable-supported bridges by a two-stage geometric nonlinear analysis", *Interact. Mult. Mech.*, **1**(3).

Wang, P.H., Liu, M.Y., Huang, Y.T. and Lin, L.C. (2010), "Influence of lateral motion of cable stays on cable-stayed bridges", *Struct. Eng. Mech.*, **34**(6).

Wu, Q.X., Takahashi, K. and Chen, B. (2007), "Influence of cable loosening on nonlinear parametric vibrations of inclined cables", *Struct. Eng. Mech.*, **25**(2), 219-238.

Wu, W.J. and Cai, C.S. (2009), "Comparison of deck-anchored damper and clipped tuned mass damper on cable vibration reduction", *Struct. Eng. Mech.*, **32**(6), 741-754.

Wu, W.J. and Cai, C.S. (2010), "Cable vibration control with a semiactive MR damper - numerical simulation and experimental verification", *Struct. Eng. Mech.*, **34**(5), 611-62.