Structural Engineering and Mechanics, Vol. 38, No. 3 (2011) 283-300
DOI: http://dx.doi.org/10.12989/sem.2011.38.3.283

Dynamics of a bridge beam under a stream of moving elements. Part 1 – Modelling and numerical integration

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(Received October 20, 2010, Accepted December 17, 2010)

Abstract. A new conception of fundamental tasks in dynamics of the bridge-track-train systems (BTT), with the aim to evaluate moving load's models adequacy, has been developed. The 2D physical models of BTT systems, corresponding to the fundamental tasks, have been worked out taking into account one-way constraints between the moving unsprung masses and the track. A method for deriving the implicit equations of motion, governing vibrations of BTT systems' models, as well as algorithms for numerical integration of these equations, leading to the solutions of high accuracy and relatively short times of simulations, have been also developed. The derived equations and formulated algorithms constitute the basis for numerical simulation of vibrations of the considered systems.

Keywords: bridge-track-train system; fundamental tasks; bridge beam; moving elements streams; one-way contact; implicit equations of motion; numerical integration.

1. Introduction

The bridge-track-train dynamic systems (BTT) are characterized by the following fundamental features (Podworna 2005a, b):

√ fast-varying configuration of the system resulting from a service velocity of the train,
√ parametric – forced excitation of bridge vibrations,
√ a 3D bridge superstructure with stepped mass and stiffness distributions,
√ nonlinear elastic and damping properties of the ballast layer as well as of rail fasteners,
√ 3D superstructures of rail-vehicles creating a train,
√ two-stage elastic-damping spatial suspensions of rail-vehicles,
√ deviation from symmetry of the bridge superstructure and rail-vehicles with respect to the vertical plane coinciding the track axis,
√ snaking and lateral impacts of wheel sets of rail-vehicles,
√ one-way constraints between rotating wheels and rail heads,
√ fluctuations of track stiffness,
√ an infinite length of a track.

Such a great degree of complexity of the discussed dynamic systems results in ongoing theoretical research into planar and spatial modelling of these systems adequate to reality (Klasztorny 2005).

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The author has developed theoretical research oriented to evaluation of the influence of selected factors on the dynamic response of steel beam bridges subjected to a Shinkansen high-speed train (Podworna 2004). Based on numerical simulations performed for the physically nonlinear 2D models of the BTT systems with two-way constraints between the wheels and the rails, it was proved that:

√ vibrations of the BTT system meet the condition of small displacements (the system is geometrically linear but physically nonlinear),
√ the Timoshenko effects (rotary inertia, shear deformation) in the bridge superstructure are slight,
√ dynamic effects resulting from nonlinear properties of the ballast are small,
√ the influence of the compliance of rail fasteners is slight,
√ the influence of the wheel sets' inertia forces can be significant,
√ the influence of the compliance of the track outside the bridge deck is slight.

This theoretical research has been carried out on a series-of-types of single-span single-track steel bridges.

An extensive review on papers published before 2001, concerning dynamics of the BTT systems, (Klasztorny 2005) can be concluded in such a way that many writers assumed simplified models of a high-speed train, in the form of concentrated forces (Klasztorny 1990, Yau 2001), concentrated unsprung masses (Nelson 1971, Benedetti 1974, Klasztorny 1990, Yang 1997, Cheng 2001, Fryba 2001), single- or double-mass oscillators (Chu 1979, Klasztorny 1990), without evaluation of an adequacy degree of these models to reality. These references were developed under the assumption of double-sided constraints between moving elements and the track. This assumption may lead to the results that differ significantly from reality.

The most extensive analysis of a simply-supported Euler beam loaded by representative types of moving load models (Klasztorny 1990) applies the algorithms for determining transient and steady-state vibrations for an infinite stream of discrete moving elements, based on the Lagrange – Ritz method and the explicit equations of motion. In the case of moving mass particles or moving viscoelastic oscillators, such approach leads to substantially longer times of numerical simulations than the approach of the implicit algorithms developed in this paper.

Recent references (since 2004) have shown that simplified analytic and analytic – numerical methods in dynamics of structures under moving loads are still developed. The writers adopt simplified models of vehicles or trains in the form of streams of forces, mass particles or oscillators without verification of adequacy of these models to reality. It gives the reasons for reconsidering this topic. The first recent paper (Cojocaru 2004) considered vibrations of a bridge beam loaded by a moving train modelled by a second elastic beam moving with a constant speed. Both, the elastic stiffness and the mass of such model of the train are taken into account in extension of the usual model of distributed forces.

The next recent paper (Garinei 2006) considers a bridge beam loaded by moving loads in the form of a single constant/harmonic force or a series of equidistant constant/harmonic forces. The analysis in focused on the effects of the phase and frequency as well as the critical velocity of the load. Subsequent studies (Yau 2007, Fryba 2009) consider a problem of vibration of a suspension bridge due to moving loads of equidistant constant forces and shaken by vertical support motions caused by earthquake. The suspension bridge is modelled as a single-span suspended beam. The writers applied the decomposition and Galerkin's methods.

Next writers (Yau 2008) presented a study on vibration of a suspension bridge installed with a water pipeline and subjected to moving trains. The suspension bridge is modelled as a single span beam and the train is simulated as a sequence of equidistant moving constant forces. The Galerkin method was applied. The study (Bilello 2008) developed the correction procedure for dynamic analysis of linear, proportionally damped, continuous systems under travelling concentrated loads. Two cases of non-parametric (moving forces) and parametric (moving mass) loads are considered. Improvement in the evaluation of the dynamic response is obtained by separating the contribution of the low frequency modes from that of the high-frequency modes.

A stream of moving oscillators crossing a simply supported beam with arbitrary time law has been considered in the subsequent study (Muscolino 2009). The system is governed by sets of differential equations with time-dependent coefficients. The writers examined the bridge support effect on dynamic response of the system. An incremental-iterative procedure (Yau 2009) has been used to investigate the influence of ground settlement on dynamic interactions of train–bridge system. The train is simulated as a sequence of identical sprung mass units with equal intervals and the bridge system as a series of simple beams with identical properties. A new method of dynamic analysis on the bridge-vehicle interaction problem considering uncertainties has been developed in a recent paper (Wu 2010). The bridge is modeled as a simply supported Euler-Bernoulli beam with Gaussian random elastic modulus and mass density of material with moving forces on top. These forces are time varying with a coefficient of variation at each time instance and they are considered as Gaussian random processes.

A multiply supported continuous beams subjected to moving loads modelled either as moving forces or moving masses is considered in the next recent paper (De Salvo 2010). A dedicated variant of the component mode synthesis method is proposed in which the classical primary-secondary substructure approach is tailored to cope with slender (i.e., Euler-Bernoulli) continuous beams with arbitrary geometry. The whole structure is ideally decomposed in primary and secondary spans with convenient restraints, whose exact eigen-functions are used as assumed local modes. Vibration behaviour of a suspension bridge due to moving loads with vertical support motions caused by earthquake is studied in the extremely recent contribution (Liu 2011). The bridge system is governed by two coupled nonlinear cable-beam equations whereas traffic is modelled as a row of equidistant moving forces.

Part 1 of this study presents a new concept of fundamental tasks in dynamics of the BTT systems. The 2D physical models of the BTT systems corresponding to the fundamental tasks have been developed taking into consideration one-way constraints between rail-vehicles' wheels and rail heads. A method for deriving the implicit equations of motion governing vibrations of the BTT systems' models, based on combination of the previous methods with some extensions, has been developed. Computer algorithms for numerical integration of the implicit equations of motion of the BTT systems, leading to the solutions of high accuracy and relatively short simulation times, have been elaborated. The numerical analysis of the problem will be presented in Part 2 of this study.

2. The fundamental tasks in dynamics of the BTT systems

2.1 A concept of the fundamental tasks

The following fundamental tasks are proposed (Fig. 1):

1) transient and quasi-steady-state vibrations of an Euler-Bernoulli beam under a cyclic stream of concentrated forces (model P),

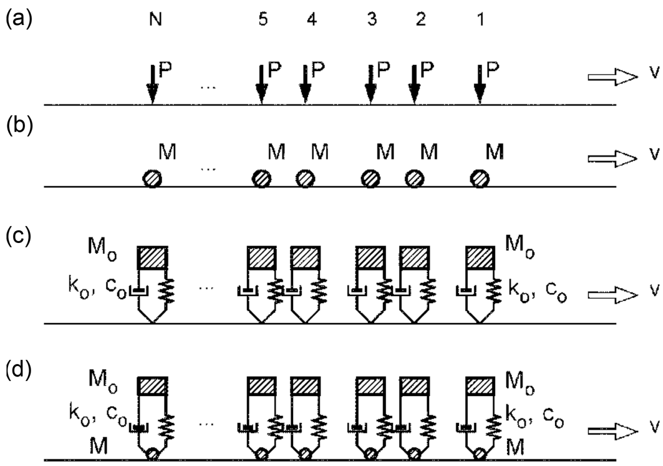

Fig. 1 Moving load models in the fundamental tasks; (a) model P, (b) model M, (c) model M_o, (d) model MM_o

2) transient and quasi-steady-state vibrations of an Euler-Bernoulli beam under a cyclic stream of concentrated unsprung masses (model M),
3) transient and quasi-steady-state vibrations of an Euler-Bernoulli beam under a cyclic stream of single-mass viscoelastic oscillators (model M_o),
4) transient and quasi-steady-state vibrations of an Euler-Bernoulli beam under a cyclic stream of a double-mass viscoelastic oscillators (model MM_o).

An Euler beam is a simplified model of a single-span simply-supported railway bridge. Cyclic streams consist of repeatable cycles of concentrated moving elements representing a simplified model of a high-speed train. Subsequent pairs of moving elements represent a single rail-vehicle supported on two-axle trucks, e.g., an ICE-3 or Shinkansen rail-vehicle. Each truck is reflected by one moving element. Different intervals between moving elements in the stream reflect the main horizontal dimensions of rail-vehicles.

The fundamental tasks are formulated in the viscoelastic range. One-way constraints are to be considered only in the case of the M and MM_o models. Separation of oscillators from the track does not occur for the M_o model what has been confirmed in the numerical analysis (Podworna 2010). For of P and M_o models the BTT system is linear, whereas for M and MM_o models linearity of the system is partitioned.

2.2 Assumptions and matrix equations of motion of the system

The following assumptions are adopted:
1) the beam-moving load system is linear both physically and geometrically in respective subsequent time intervals,
2) an Euler beam is prismatic, inertial, deformable in flexure and made of linearly viscoelastic material,
3) a constant damping decrement for all modes of the beam is assumed (Langer 1980),
4) there are considered vertical vibrations of the beam and moving elements,
5) a structural rise of the beam axis is selected so as the beam axis under the dead load is rectilinear,

6) there are considered isothermal processes,
7) the moving load constitutes a stream of N concentrated elements in the intervals reflecting a truck base and a length of a repeatable rail-vehicle,
8) the load moves along the track at a constant service velocity,
9) for M and MM_o models one-way constraints between the track and the moving load are assumed, i.e. they only transmit compression,
10) the track is smooth (track irregularities are neglected),
11) at the initial moment the moving load is located outside the beam (the first moving element is located over the left support of the beam),
12) at the initial moment the beam and moving oscillators are in the static equilibrium,
13) the track outside the beam is rectilinear and rigid.

Significantly great number of concentrated moving elements allows to bring the system to resonance or out-of-resonance quasi-steady-state vibrations.

In order to derive the implicit matrix equations of motion governing vibrations of the beam-moving elements stream system (B-S) the Lagrange-Ritz method (Klasztorny 1990) and the Klasztorny method (Klasztorny 2005) will be combined in the following way:

1) the vertical deflections of the bridge beam are approximated globally with a series of functions satisfying the Ritz conditions (a kinematically admissible complete set),
2) the implicit equations of motion are formulated separately for the beam using the 2^{nd} kind Lagrange's equations and for the M, M_o or MM_o moving load using d'Alembert principle,
3) the gradient of the beam load's work, referred to M, M_o or MM_o stream, is calculated for the implicit interactions,
4) the M unsprung masses in the M and MM_o models are represented with sprung masses via introducing contact springs of adequately great stiffness, k_M (the Hertz contact problem for the truck),
5) the vertical interactions for M and MM_o models are formulated taking into account one-way constraints,
6) equations of motion are formulated using matrix calculus.

The following dimensional and non-dimensional parameters describe the B-S systems:

l - a span length [m],
m - beam mass per unit length [kg/m],
E - a Young's modulus of the beam material [Pa],
I_b - an inertia moment of the beam cross-section in relation to the horizontal central axis [m^4],
EI_b - bending stiffness of the beam [$N \cdot m^2$],
γ - a damping ratio for the beam,
v - a horizontal velocity of the moving load (a service velocity) [m/s],
P - a concentrated force in model P [N],
M - a concentrated unsprung mass in models M and MM_o [kg],
M_o - a concentrated sprung mass in models M and MM_o [kg],
k_o - suspension stiffness for the M_o mass [N/m],
c_o - a suspension damping coefficient for the M_o mass [N·s/m],
k_M - contact stiffness,
b_1, b_2 - intervals between concentrated moving elements representing a truck base (b_1) and a length of a repeatable vehicle ($b_1 + b_2$).

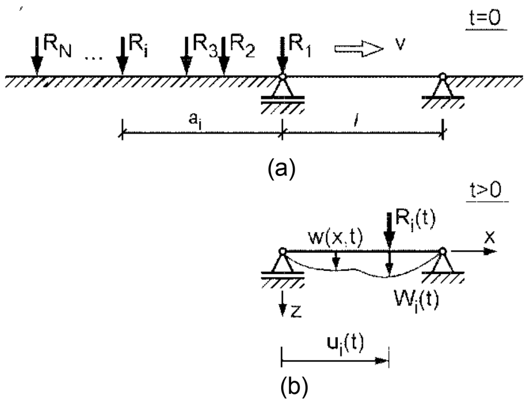

Fig. 2 The bridge beam loaded with the interactions' stream; (a) the system configuration at the initial instant, (b) the dynamic deflection, $w(x, t)$, and location of the interaction $R_i(t)$

Vertical deflections of the beam are approximated with a sine series fulfilling Ritz's conditions, i.e. (Fig. 2)

$$w(x,t) = \mathbf{q}^T(t)\mathbf{s}(x) = \mathbf{s}^T(x)\mathbf{q}(t) \tag{1}$$

where

$$\mathbf{q}(t) = [q_1(t), q_2(t), \ldots, q_n(t)]^T$$

$$\mathbf{s}(x) = [\sin\pi\xi, \sin 2\pi\xi, \ldots, \sin n\pi\xi]^T, \quad \xi = \frac{x}{l} \tag{2}$$

with

x - an abscissa in the xz planar coordinate system,
t - a time variable,
$\mathbf{q}(t)$ - a Lagrange's general coordinates vector for the beam,
$\mathbf{s}(x)$ - an approximate functions vector (a sine series).

The 2nd kind Lagrange's equations related to the beam have the form (Langer 1980)

$$\frac{d}{dt}[\mathrm{grad}E_k(\dot{\mathbf{q}})] + \mathrm{grad}\Phi(\dot{\mathbf{q}}) + \mathrm{grad}E_p(\mathbf{q}) = \mathrm{grad}L(\mathbf{q}) \tag{3}$$

where $(\dot{\cdot}) = \frac{d}{dt}$ and

$E_k(\dot{\mathbf{q}})$ - kinetic energy of the beam,
$\Phi(\dot{\mathbf{q}})$ - damping power of the beam,
$E_p(\mathbf{q})$ - elastic deformation energy of the beam,
$L(\mathbf{q})$ - work of the beam's external load on displacements $w(x, t)$.

In general, the external load of the beam takes the form of a stream of concentrated moving forces $R_1(t), R_2(t), \ldots, R_N(t)$, being dynamic pressures of moving elements onto the track, Fig. 2. When moving elements M, MM_o separate from the track, the system is regarded as linear in respective time intervals.

The kinetic energy of the beam amounts to (Klasztorny 2001)

$$E_k = \frac{1}{2}\int_0^l m\left(\frac{\partial w}{\partial t}\right)^2 dx = \frac{1}{2}\dot{\mathbf{q}}^T m\int_0^l \mathbf{s}\mathbf{s}^T dx\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{B}\dot{\mathbf{q}}^T \tag{4}$$

where

$$\mathbf{B} = m\int_0^l \mathbf{s}\mathbf{s}^T dx = ml\int_0^l \mathbf{s}\mathbf{s}^T d\xi = \frac{ml}{2}\mathbf{I} = \{\mathbf{B}\}$$

$$\mathbf{I} = \mathrm{diag}(1, 1, \ldots, 1), \ \ \dim\mathbf{I} = n\times n \tag{5}$$

The elastic energy of the beam amounts to (Klasztorny 2001)

$$E_P = \frac{1}{2}\int_0^l EI_b\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\mathbf{q}^T EI_b\int_0^l \frac{\partial^2\mathbf{s}}{\partial x^2}\frac{\partial^2\mathbf{s}^T}{\partial x^2}dx\mathbf{q} = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q} \tag{6}$$

where

$$\mathbf{K} = EI_b\frac{1}{2}\int_0^l \frac{\partial^2\mathbf{s}}{\partial x^2}\frac{\partial^2\mathbf{s}^T}{\partial x^2}dx = \frac{EI_b}{l^3}\{\mathbf{d}\}^2\int_0^1 \mathbf{s}\mathbf{s}^T d\xi\{\mathbf{d}\}^2 = \frac{EI_b}{2l^3}\{\mathbf{d}\}^4 = \{\mathbf{K}\}$$

$$\{\mathbf{d}\} = \mathrm{diag}\{\pi, 2\pi, \ldots, n\pi\} \tag{7}$$

The inertia matrix, **B**, and the stiffness matrix, **K**, are diagonal, thus in this case the approximate basis, **s**, is the eigenfunction basis. Then, vibration damping in the beam can be described according to a constant decrement damping model (Langer 1980) in which all beam modal systems have the same damping coefficient γ, while $\gamma_{cr} = 1$. The beam vibration damping power, Φ, as well as the damping matrix, **D**, are then defined by the formulae

$$\Phi = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{D}\dot{\mathbf{q}}$$

$$\mathbf{D} = \{\mathbf{D}\} = 2\gamma\sqrt{\{\mathbf{K}\}\{\mathbf{B}\}} = \gamma\frac{\sqrt{EI_b m}}{l}\{\mathbf{d}\}^2 \tag{8}$$

The abscissa and the beam deflection following location of interaction $R_i(t)$, according to Fig. 2 and Eqs. (1), (2), amount to

$$u_i(t) = vt - a_i, \quad \frac{u_i(t)}{l} = \tau - \alpha_i$$

$$W_i(t) = w[u_i(t), t] = \mathbf{q}^T(t)\mathbf{s}[u_i(t)] = \mathbf{q}^T(t)\mathbf{S}_i(t) \tag{9}$$

where

$$\tau = \frac{vt}{l}, \quad \alpha_i = \frac{a_i}{l}$$

$$\mathbf{S}_i(t) = \mathbf{s}[u_i(t)] = \left[\sin\frac{\pi u_i(t)}{l}, \sin\frac{2\pi u_i(t)}{l}, \ldots, \sin\frac{n\pi u_i(t)}{l}\right]^T =$$

$$= \begin{cases} [\sin\pi(\tau-\alpha_i) \ \ \sin 2\pi(\tau-\alpha_i) \ \ldots \ \sin n\pi(\tau-\alpha_i)]^T, & \text{for } \ \tau-\alpha_i \in [0, 1] \\ \mathbf{0}, & \text{for } \ \tau-\alpha_i \notin [0, 1] \end{cases} \tag{10}$$

Quantity $a_i > 0$ determines a distance of the i-th concentrated moving element from the beam left support at the initial instant with $a_i = 0$. The following vector, $\mathbf{S}_i(t)$, is determined due to possible moving elements' positions either on the beam or outside it. The τ variable is dimensionless and determines the relative position of the first interaction in relation to the beam left support. The use of the τ variable makes possible to put on time histories of the given quantity for different service velocities as well as to put on the dynamic curve on a quasi-static one.

The $R_i(t)$ force operates only when it acts onto the beam. The work of the $R_i(t)$ force in the implicit form on displacement $w_i(t)$ amounts to (Klasztorny 2005)

$$L_i(\mathbf{q}) = R_i(t)w_i(t) = w_i(t)R_i(t) = \mathbf{q}(t)\mathbf{S}_i(t)R_i(t) \tag{11}$$

The total work of the stream of moving forces $R_1(t)$, $R_2(t)$, …, $R_N(t)$ equals to

$$L(\mathbf{q}) = \sum_{i=1}^{N} L_i(\mathbf{q}) = \mathbf{q}^T\mathbf{F} \tag{12}$$

where

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{S}_i(t)R_i(t) = \mathbf{SR} \tag{13}$$

while

$$\mathbf{S} = [\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_N], \quad \mathbf{R} = [R_1, R_2, \ldots, R_n]^T \tag{14}$$

A gradient of the $L(\mathbf{q})$ work must be calculated for vector $\mathbf{R}$ in the implicit form (Klasztorny 2005), i.e.

$$\mathrm{grad}L(\mathbf{q}) = \mathbf{F} = \mathbf{SR} \tag{15}$$

After inserting Eqs. (4), (6), $(8)_1$, (12) into Eq. (3), one obtains the general matrix equation of motion governing vibrations of the bridge beam loaded with a stream of interaction forces $R_1(t)$, $R_2(t)$, …, $R_N(t)$, in the following implicit form

$$\mathbf{B}\ddot{\mathbf{q}}(t) + \mathbf{D}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{SR} \tag{16}$$

Eq. (16) constitutes a system of normal, linear, heterogeneous differential equations of the second order and constant coefficients. Vibration coupling between the beam and moving elements M, $\mathrm{M_o}$, $\mathrm{MM_o}$ is hidden in the $\mathbf{R}$ generalized load vector. The initial conditions for the beam have the form

$$\mathbf{q}(0) = \mathbf{0}, \quad \dot{\mathbf{q}}(0) = \mathbf{0} \tag{17}$$

2.3 Formulating the interaction vectors and additional equations of motion

In task 1 (model P), the beam is loaded with a stream of N moving forces of constant value P. The interaction vector is constant in time and amounts to

$$\mathbf{R} = P\mathbf{1} \tag{18}$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$, dim $\mathbf{1} = N$. An additional equation of motion disappears. Eq. (16) is explicit.

In task 2 (model M) the beam is loaded with a stream of concentrated moving masses of $M_i = M$ value. Up to now, in the considered task, a double-sided constraint between concentrated moving

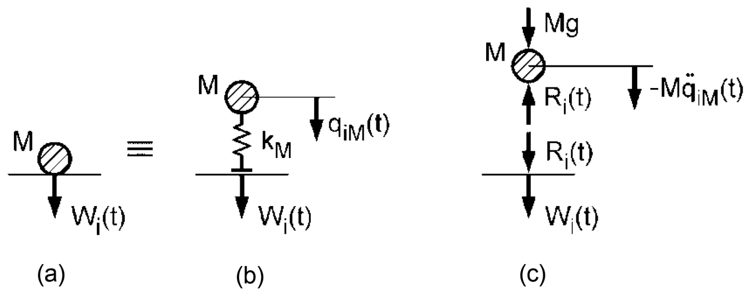

Fig. 3 Modelling out of a concentrated unsprung mass to a concentrated sprung mass (a) unsprung mass, (b) equivalent sprung mass, (c) forces action on mass M

mass M and the track was assumed. Interaction $R_i(t)$ was calculated as a sum of the gravity force, Mg, where $g = 9{,}81$ m/s^2 - the gravity acceleration, and the vertical inertia force, $-M\ddot{W}(t)$. Such an approach results in both a matrix equation of motion with time-varying coefficients (parametric-forced excitation) and in complex and time-consuming numerical integration of these equations (Klasztorny 1990).

In the present paper, an unsprung mass, M, is replaced with a sprung mass with the use of a one-way spring of large stiffness k_M modelling the Hertz contact stiffness in the wheel set – rails system (Fig. 3). The contact stiffness can be calculated from an approximate formula

$$k_M = \frac{P}{\delta} \tag{19}$$

where P is the pressure force of two wheel sets on the track, and δ is a contact deformation corresponding to P. Assuming that $P = 320000\,N$, $\delta = 0.4$ mm $= 0.0004$ m, one obtains $k_M = 8 \cdot 10^8$ N/m (Klasztorny 2005).

Let us introduce the additional generalized coordinates vector in the form

$$\mathbf{q}_M(t) = [q_{1M}(t), q_{2M}(t), \ldots, q_{NM}(t)]^T \tag{20}$$

where $q_{iM}(t)$ represents the vertical displacement of i-th mass M (Fig. 3). At the initial instant mass M does not produce any vertical vibrations. The contact spring is shortened by Mg/k_M value as a result of the gravity force, Mg. Therefore, the initial conditions for the moving masses set take the form

$$\mathbf{q}_M(0) = \left(\frac{Mg}{k_M}\right)\mathbf{1}, \quad \dot{\mathbf{q}}_M(0) = \mathbf{0} \tag{21}$$

The interaction amounts to (Fig. 3)

$$R_i(t) = \begin{cases} k_M[q_{iM}(t) - W_i(t)], & \text{dla } q_{iM}(t) - W_i > 0 \\ 0, & \text{dla } q_{iM}(t) - W_i \le 0 \end{cases} \tag{22}$$

while $W_i(t)$ is determined by Eqs. (9), (10). Eq. (22) takes into account one-way acting of the contact spring.

An additional matrix equation of motion of the elastic oscillators with parameters M, k_M can be determined via considering the dynamic equilibrium of masses M according to the d'Alembert principle (Fig. 3(c)). The result has the following implicit form

$$\{\mathbf{M}\}\ddot{\mathbf{q}}_M = \mathbf{G} - \mathbf{R} \tag{23}$$

where

$$\begin{gathered} \{\mathbf{M}\} = M\mathbf{I},\ \mathbf{I} = \mathrm{diag}(1, 1, \ldots, 1), \quad \dim\mathbf{I} = N \times N \\ \mathbf{G} = Mg\mathbf{1}, \quad \mathbf{R} = [R_1(t), R_2(t), \ldots, R_N(t)]^T \end{gathered} \tag{24}$$

In task 3 (model M_o) the beam is loaded with a uniform stream of single-mass viscoelastic oscillators, Fig. 1(c). In this case, a double-sided constraint is assumed because of the weightless bottom end of the suspension; this a priori assumption was confirmed by numerical simulations (Podworna 2010). The additional generalized coordinates vector is introduced in the following form

$$\mathbf{q}_o(t) = [q_{1o}(t), q_{2o}(t), \ldots, q_{No}(t)]^T \tag{25}$$

where $q_{io}(t)$ represents the vertical displacement of i-th mass M_o (Fig. 4(a)). At the initial instant mass M_o does not produce any vertical vibrations. The static shortening of spring k_o amounts to $M_o g/k_o$. Therefore, the initial conditions for the moving oscillators' set are in the form of

$$\mathbf{q}_o(0) = (M_o g/k_o)\mathbf{1}, \quad \dot{\mathbf{q}}_o(0) = \mathbf{0} \tag{26}$$

An interaction between the moving oscillator and the track equals

$$R_i(t) = k_o[q_{io}(t) - W_i(t)] + c_o[\dot{q}_{io}(t) - \dot{W}_i(t)] \tag{27}$$

while the following displacement, $W_i(t)$ as well as the following velocity, $\dot{W}_i(t)$ are determined by Eqs. (9), (10), and

$$\dot{\mathbf{S}}_i(t) = \frac{v}{l}\{\mathbf{d}\}\mathbf{C}_i(t)$$

$$\mathbf{C}_i = \begin{cases} [\cos\pi(\tau-\alpha_i), \cos 2\pi(\tau-\alpha_i), \ldots, \cos n\pi(\tau-\alpha_i)]^T, & \text{dla } \tau-\alpha_i \in [0, 1] \\ \mathbf{0}, & \text{dla } \tau-\alpha_i \notin [0, 1] \end{cases}$$

$$\dot{W}_i(t) = \frac{d}{dt}[\mathbf{q}^T(t)\mathbf{S}_i(t)] = \dot{\mathbf{q}}^T(t)\mathbf{S}_i(t) + \mathbf{q}^T(t)\dot{\mathbf{S}}_i(t) \tag{28}$$

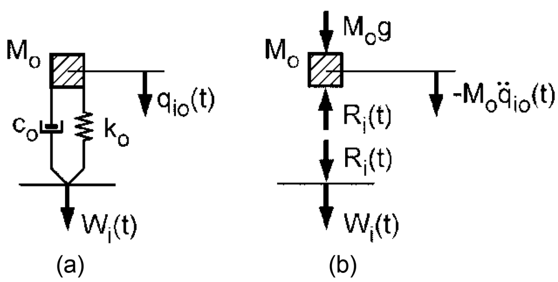

Fig. 4 (a) A single moving element in model M_o and (b) a set of forces influencing mass M_o

An additional matrix equation of motion of viscoelastic oscillators with parameters M_o, k_o, c_o can be determined with the use of the d'Alembert principle for masses M_o (Fig. 4(b)). The result has the following implicit form

$$\{\mathbf{M}_o\}\ddot{\mathbf{q}}_o = \mathbf{G}_o - \mathbf{R} \tag{29}$$

where

$$\{\mathbf{M}_o\} = M_o\mathbf{I}\,,\ \mathbf{I} = \mathrm{diag}(1, 1, \ldots, 1), \quad \dim\mathbf{I} = N\times N$$
$$\mathbf{G}_o = M_o g\mathbf{1}\,, \quad \mathbf{R} = [R_1(t), R_2(t), \ldots, R_N(t)]^T \tag{30}$$

while dynamic interaction $R_i(t)$ is determined by Eq. (27).

In task 4 (model MM_o) the beam is loaded with a uniform stream of double-mass viscoelastic oscillators, Fig. 1(d). Unsprung mass M can be replaced by sprung mass M with the use of one-way contact spring of stiffness k_M like in task 2. Two additional generalized coordinates vectors are introduced (Fig. 5)

$$\mathbf{q}_M(t) = [q_{1M}(t), q_{2M}(t), \ldots, q_{NM}(t)]^T$$
$$\mathbf{q}_o(t) = [q_{1o}(t), q_{2o}(t), \ldots, q_{No}(t)]^T \tag{31}$$

where $q_{iM}(t)$, q_{io} represent the displacements of masses M and M_o in the i-th moving element, respectively.

At the initial instant the oscillator is in the static equilibrium state, thus masses M and M_o do not produce any vertical vibrations. The shortening amounts to $M_o g/k_o$ for the upper spring, and to $(M_o + M)g/k_M$ for the lower one. The initial conditions for the double-mass oscillators set have the form of

$$\mathbf{q}_o(0) = [M_o g/k_o + (M_o + M)g/k_M]\mathbf{1}\,, \quad \dot{\mathbf{q}}_o(0) = 0$$
$$\mathbf{q}_M(0) = [(M_o + M)g/k_M]\mathbf{1}\,, \quad \dot{\mathbf{q}}_M(0) = 0 \tag{32}$$

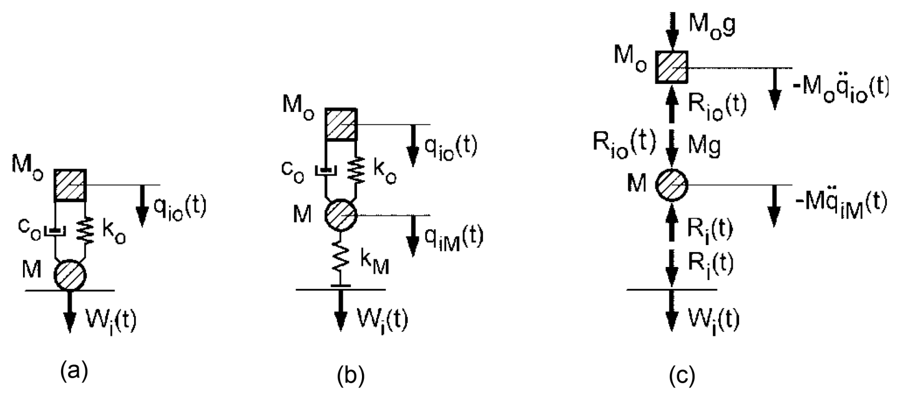

Fig. 5 A single moving element in model MM_o (a) the element before the modelling out, (b) modelling out of mass M to sprung mass, (c) a set of vertical forces influencing masses M and M_o

The interactions amount to (Fig. 5)

$$R_i(t) = \begin{cases} k_M[q_{iM}(t) - W_i(t)], & \text{dia } q_{iM}(t) - W_i > 0 \\ 0, & \text{dia } q_{iM}(t) - W_i \le 0 \end{cases}$$

$$R_{io}(t) = k_o[q_{io}(t) - q_{iM}(t)] + c_o[\dot{q}_{io}(t) - \dot{q}_{iM}(t)] \tag{33}$$

while the following displacement, $W_i(t)$ is determined by Eqs. (9), (10).

Additional equations of motion of double-mass viscoelastic oscillators with parameters M_o, M, k_o, c_o, k_M can be derived from of the d'Alembert principle for masses M_o and M, i.e. (Fig. 5(c))

$$\begin{aligned} M_o g - M_o \ddot{q}_{io} - R_{io} = 0 &\Rightarrow M_o \ddot{q}_{io} + R_{io} = M_o g \\ Mg - M\ddot{q}_{iM} + R_{io} - R_i = 0 &\Rightarrow M\ddot{q}_{iM} - R_{io} = Mg - R_i \end{aligned} \tag{34}$$

After inserting Eq. $(33)_2$ into Eq. (34) one obtains

$$\begin{aligned} M_o \ddot{q}_{io} + c_o(\dot{q}_{io} - \dot{q}_{iM}) + k_o(q_{io} - q_{iM}) &= M_o g, \quad i = 1, 2, \ldots, N \\ M\ddot{q}_{iM} - c_o(\dot{q}_{io} - \dot{q}_{iM}) - k_o(q_{io} - q_{iM}) &= Mg - R_i, \quad i = 1, 2, \ldots, N \end{aligned} \tag{35}$$

Based on Eq. (35) the additional matrix equation of motion for the double-mass oscillators stream has the following implicit form

$$\mathbf{B}_s \ddot{\mathbf{q}}_s(t) + \mathbf{D}_s \dot{\mathbf{q}}_s(t) + \mathbf{K}_s \mathbf{q}_s(t) = \mathbf{F}_s \tag{36}$$

where

$$\mathbf{B}_s = \begin{bmatrix} \{\mathbf{M}_o\} & \mathbf{0} \\ \mathbf{0} & \{\mathbf{M}\} \end{bmatrix}, \quad \mathbf{D}_s = \begin{bmatrix} \{\mathbf{c}_o\} & -\{\mathbf{c}_o\} \\ -\{\mathbf{c}_o\} & \{\mathbf{c}_o\} \end{bmatrix}, \quad \mathbf{K}_s = \begin{bmatrix} \{\mathbf{k}_o\} & -\{\mathbf{k}_o\} \\ -\{\mathbf{k}_o\} & \{\mathbf{k}_o\} \end{bmatrix}$$

$$\mathbf{q}_o = \begin{bmatrix} \mathbf{q}_o \\ \mathbf{q}_M \end{bmatrix}, \quad \mathbf{F}_s = \begin{bmatrix} \mathbf{G}_0 \\ \mathbf{G} - \mathbf{R} \end{bmatrix}$$

$$\begin{aligned} &\{\mathbf{M}_o\} = M_o\mathbf{I}, \quad \{\mathbf{M}\} = M\mathbf{I}, \quad \{\mathbf{c}_o\} = c_o\mathbf{I}, \quad \{\mathbf{k}_o\} = k_o\mathbf{I} \\ &\{\mathbf{G}_o\} = M_o g\mathbf{1}, \quad \mathbf{G} = Mg\mathbf{1}, \quad \mathbf{R} = [R_1(t), R_2(t), \ldots, R_N(t)]^T \end{aligned} \tag{37}$$

while

$$\begin{aligned} \mathbf{I} &= \operatorname{diag}(1, 1, \ldots, 1), \quad \dim\mathbf{I} = N \times N \\ \mathbf{1} &= [1, 1, \ldots, 1]^T, \quad \dim\mathbf{I} = N \end{aligned} \tag{38}$$

2.4 Analysis of the equations governing vibrations of the beam – moving elements stream system

Beam vibrations induced by a moving forces stream (model P) are described by Eq. (16) being a matrix equation of motion with constant matrix coefficients. The interaction vector is defined by Eq. (18). The explicit equations of motion are integrated numerically with the explicit type algorithm. The initial conditions are determined by Eq. (17).

Beam vibrations induced by an unsprung masses stream (model M) are described by Eqs. (16), (23) being matrix equations of motion with constant matrix coefficients. The interaction vector, **R**, is described by Eqs. $(14)_2$, (22). Incorporating the contact springs representing the rail-vehicles' wheel sets stiffness has simplified the form of equations of motion and simultaneously has enabled taking one-way constraints into consideration. The implicit equations of motion are integrated numerically according to the implicit type algorithm. The initial conditions are determined by Eqs. (17), (21).

Beam vibrations induced by a single-mass viscoelastic oscillators stream (model M_o) are described by Eqs. (16), (29) of constant matrix coefficients. The interaction vector, **R**, is determined by Eqs. $(14)_2$, (27). Separation of the oscillators from the track does not occur. The implicit equations of motion are integrated numerically in accordance with the implicit type algorithm. The initial conditions are determined by Eqs. (17), (26).

Beam vibrations induced by a double-mass viscoelastic oscillators stream (model MM_o) are governed by Eqs. (16), (36) also of constant matrix coefficients. The interaction vector, **R**, is determined by Eqs. $(37)_2$, $(33)_1$. The contact springs of stiffness k_M allow to take one-way constraints into account and to simplify the mathematical description of the B-S system. The implicit equations of motion are integrated numerically with the implicit type algorithm. The initial conditions are determined by Eqs. (17), (32).

3. Algorithms for numerical integration of equations of motion

To date, a great deal of one-step and multi-step methods for numerical integration of equations of motion of discrete systems have been developed. One-step methods are natural for these equations due to initial conditions. One-step methods are divided into two groups, i.e., the methods without numerical damping (e.g., a set of Newmark's methods with parameter $\gamma_N = 1/2$) and the methods with numerical damping (e.g., a set of Newmark's methods with parameter $\gamma_N \neq 1/2$) (Newmark 1959).

Methods without numerical damping are analysed due to the stability limit, the amplitude error and the period error (Langer 1986). A finite stability limit causes a rapid growth of numerical integration errors nearby this limit. The amplitude error influences the simulation accuracy, however, it does not accumulate itself during the integration process. The period error crucially influences the simulation accuracy since it accumulates itself during the integration process.

The Newmark average acceleration method with parameters $\beta_N = 1/4$, $\gamma_N = 1/2$ is selected to be applied in tasks 1÷4. In the case of explicit equations, this method is unconditionally stable. The amplitude error vanishes, while the period error is close to the error for the central differences method. The influence of the period error can be freely reduced via assuming a relatively small integration step determined from the initial numerical tests.

In task 1 (model P) the numerical integration algorithm for Eq. (18) is explicit. The dynamic response of the system is determined in equidistant time points in the $[0, T_p]$ interval, i.e.

$$t_{j+1} = (j+1)h, \quad j = 0, 1, \ldots, N_p - 1 \tag{39}$$

where T_p is the dynamic process duration time, $h = \Delta t$ is a time step, $N_p = T_p/h$ is number of integration steps. The following discrete values are introduced

$$\mathbf{q}_j = \mathbf{q}(t_j)\,, \quad \dot{\mathbf{q}}_j = \dot{\mathbf{q}}(t_j)\,, \quad \ddot{\mathbf{q}}_j = \ddot{\mathbf{q}}(t_j)\,, \quad \mathbf{S}_{j+1} = \mathbf{S}(t_{j+1}) \tag{40}$$

and the Newmark recurrent formulae for $\beta_N = 1/4$, $\gamma_N = 1/2$ are applied (Newmark 1959), i.e.

$$\begin{cases} \mathbf{q}_{j+1} = \mathbf{q}_j + h\dot{\mathbf{q}}_j + \frac{1}{4}h^2(\ddot{\mathbf{q}}_j + \ddot{\mathbf{q}}_{j+1}) \\ \dot{\mathbf{q}}_{j+1} = \dot{\mathbf{q}}_j + \frac{1}{2}h(\ddot{\mathbf{q}}_j + \ddot{\mathbf{q}}_{j+1}) \end{cases} \tag{41}$$

The acceleration vector is determined from the condition of collocation at the end of the time step what leads to an algebraic system of linear equations (Langer 1986), i.e.

$$\mathbf{A}\ddot{\mathbf{q}}_{j+1} = \mathbf{V}_{j+1} \Rightarrow \ddot{\mathbf{q}}_{j+1}\mathbf{A}^{-1}\mathbf{V}_{j+1} \tag{42}$$

where

$$\mathbf{A} = \mathbf{B} + \frac{1}{2}h\mathbf{D} + \frac{1}{4}h^2\mathbf{K}$$

$$\mathbf{V}_{j+1} = \mathbf{S}_{j+1}\mathbf{R}_{j+1} - \mathbf{D}\left(\dot{\mathbf{q}}_j + \frac{1}{2}h\ddot{\mathbf{q}}_j\right) - \mathbf{K}\left(\mathbf{q}_j + h\dot{\mathbf{q}}_j + \frac{1}{4}h^2\ddot{\mathbf{q}}_j\right) \tag{43}$$

Matrix **A** is reversed only once. Vector **R** is constant in time. In the extended version the initial conditions defined by Eq. (17) take the form

$$\mathbf{q}_0 = \mathbf{0}\,, \quad \dot{\mathbf{q}}_0 = \mathbf{0}, \quad \ddot{\mathbf{q}}_0 = \mathbf{0} \tag{44}$$

In task 2 (model M), the algorithm for numerical integration of Eqs. (16), (23) is implicit due to the interaction vector, **R**. Eqs. (40)-(43) for the unsprung moving masses are in the form

$$\mathbf{q}_{M,j} = \mathbf{q}_M(t_j), \quad \dot{\mathbf{q}}_{M,j} = \dot{\mathbf{q}}_M(t_j), \quad \ddot{\mathbf{q}}_{M,j} = \ddot{\mathbf{q}}_M(t_j), \quad \mathbf{R}_j = \mathbf{R}(t_j)$$

$$\begin{cases} \mathbf{q}_{M,j+1} = \mathbf{q}_{M,j} + h\dot{\mathbf{q}}_{M,j} + \frac{1}{4}h^2(\ddot{\mathbf{q}}_{M,j} + \ddot{\mathbf{q}}_{M,j+1}) \\ \dot{\mathbf{q}}_{M,j+1} = \dot{\mathbf{q}}_{M,j} + \frac{1}{2}h(\ddot{\mathbf{q}}_{M,j} + \ddot{\mathbf{q}}_{M,j+1}) \end{cases} \tag{45}$$

$$\ddot{\mathbf{q}}_{M,j+1} = \{\mathbf{M}\}^{-1}(\mathbf{G} - \mathbf{R}_{j+1})$$

There is applied a linear prediction of vector **R** (Podworna 2005b), i.e.

$$\mathbf{R}^p_{j+1} = 2\mathbf{R}_j - \mathbf{R}_{j-1} \tag{46}$$

while $\mathbf{R}_{-1} = \mathbf{R}_0$ due to the static equilibrium of the system in the $t \le 0$ interval. In the extended version the initial conditions defined by Eqs. (17), (21) take the form

$$\mathbf{q}_0 = \mathbf{0}\,, \quad \dot{\mathbf{q}}_0 = \mathbf{0}, \quad \ddot{\mathbf{q}}_0 = \mathbf{0}$$

$$\mathbf{q}_{M,0} = (Mg/k_M)\mathbf{1}\,, \quad \dot{\mathbf{q}}_{M,0} = \mathbf{0}\,, \ddot{\mathbf{q}}_{M,0} = \mathbf{0}$$

$$\mathbf{R}_0 = \mathbf{G} = Mg\mathbf{1} \tag{47}$$

The implicit algorithm for numerical integration of Eqs. (16), (23) consists of the following calculation stages:

1) prediction of $\mathbf{R}^p_{j+1}$ according to Eq. (46),
2) calculation of $\ddot{\mathbf{q}}_{j+1}, \ddot{\mathbf{q}}_{M,j+1}$ from Eqs. (42), $(45)_7$,
3) calculation of $\mathbf{q}_{j+1}, \dot{\mathbf{q}}_{j+1}, \mathbf{q}_{M,j+1}, \dot{\mathbf{q}}_{M,j+1}$ from Eqs. (41), $(45)_{5,6}$,
4) correction of $\mathbf{R}^c_{j+1}$ according to the formula resulting from Eq. (22), i.e.

$$R^c_{i,j+1} = \begin{cases} k_M(q_{iM,j+1} - W_{i,j+1}), & \text{dla } q_{iM,j+1} - W_{i,j+1} > 0 \\ 0, & \text{dla } q_{iM,j+1} - W_{i,j+1} \le 0 \end{cases} \tag{48}$$

where

$$W_{i,j+1} = \mathbf{q}^T_{j+1}\mathbf{S}_{i,j+1}, \quad \mathbf{S}_{i,j+1} = \mathbf{S}_i(t_{j+1}) \tag{49}$$

5) control of the iteration ending condition

$$\left|R^c_{i,j+1} - R^p_{i,j+1}\right| \le \varepsilon, \quad i = 1, 2, \ldots, N \tag{50}$$

where $\varepsilon > 0$ is an accuracy parameter;
if Eq. (50) is met then go to the next time step;
if Eq. (50) does not met then substitute

$$\mathbf{R}^p_{j+1} := \mathbf{R}^c_{j+1} \tag{51}$$

and go to stage 2.

In task 3 (model M_o) the algorithm for numerical integration of Eqs. (16), (23) is also implicit because of the interaction vector, **R**. Eqs. (40)-(43) for the single-mass viscoelastic oscillators stream take the form

$$\mathbf{q}_{o,j} = \mathbf{q}_o(t_j), \quad \dot{\mathbf{q}}_{o,j} = \dot{\mathbf{q}}_o(t_j), \quad \ddot{\mathbf{q}}_{o,j} = \ddot{\mathbf{q}}_o(t_j), \quad \mathbf{R}_j = \mathbf{R}(t_j)$$

$$\begin{cases} \mathbf{q}_{o,j+1} = \mathbf{q}_{o,j} + h\dot{\mathbf{q}}_{o,j} + \frac{1}{4}h^2(\ddot{\mathbf{q}}_{o,j} + \ddot{\mathbf{q}}_{o,j+1}) \\ \dot{\mathbf{q}}_{o,j+1} = \dot{\mathbf{q}}_{o,j} + \frac{1}{2}h^2(\ddot{\mathbf{q}}_{o,j} + \ddot{\mathbf{q}}_{o,j+1}) \end{cases}$$

$$\ddot{\mathbf{q}}_{o,j+1} = \{\mathbf{M}_o\}^{-1}(\mathbf{G}_o - \mathbf{R}_{j+1}) \tag{52}$$

Eq. (46) defining linear prediction of vector **R** remains valid and $\mathbf{R}_{-1} = \mathbf{R}_0$ due to the static equilibrium of the system in the $t \le 0$ interval. In the extended version the initial conditions defined by Eqs. (17), (26) take the form

$$\mathbf{q}_0 = \mathbf{0}, \quad \dot{\mathbf{q}}_0 = \mathbf{0}, \quad \ddot{\mathbf{q}}_0 = \mathbf{0}$$

$$\mathbf{q}_{o,0} = (M_o g/k_o)\mathbf{1}, \quad \dot{\mathbf{q}}_{o,0} = \mathbf{0}, \quad \ddot{\mathbf{q}}_{o,0} = \mathbf{0}$$

$$\mathbf{R}_0 = \mathbf{G}_o = M_o g\mathbf{1} \tag{53}$$

The implicit algorithm for numerical integration of Eqs. (16), (29) consists of the following stages:

1) prediction of $\mathbf{R}^p_{j+1}$ from Eq. (46),
2) calculation of $\ddot{\mathbf{q}}_{j+1}, \ddot{\mathbf{q}}_{o,j+1}$ from Eqs. (42), $(52)_7$,
3) calculation of $\mathbf{q}_{j+1}, \dot{\mathbf{q}}_{j+1}, \mathbf{q}_{o,j+1}, \dot{\mathbf{q}}_{o,j+1}$ from Eqs. (41), $(52)_{5,6}$,
4) correction of $\mathbf{R}^c_{j+1}$ according to the formula resulting from Eq. (27), i.e.

$$R^c_{i,j+1} = k_o(q_{io,j+1} - W_{i,j+1}) + c_o(\dot{q}_{io,j+1} - \dot{W}_{i,j+1}) \tag{54}$$

where

$$W_{i,j+1} = \mathbf{q}^T_{j+1}\mathbf{S}_{i,j+1}, \quad \dot{W}_{i,j+1} = \dot{\mathbf{q}}^T_{j+1}\mathbf{S}_{i,j+1} + \mathbf{q}^T_{j+1}\dot{\mathbf{S}}_{i,j+1}$$
$$\mathbf{S}_{i,j+1} = \mathbf{S}_i(t_{j+1}), \quad \dot{\mathbf{S}}_{i,j+1} = \dot{\mathbf{S}}_i(t_{j+1}) \tag{55}$$

5) control of the iteration ending condition according to Eq. (50).

In task 4 (model $\mathrm{MM_o}$), the algorithm for numerical integration of Eqs. (16), (36) is also implicit. Eqs. (40)-(43) for the double-mass viscoelastic oscillators stream take the form

$$\mathbf{q}_{s,j} = \mathbf{q}_s(t_j), \quad \dot{\mathbf{q}}_{s,j} = \dot{\mathbf{q}}_s(t_j), \quad \ddot{\mathbf{q}}_{s,j} = \ddot{\mathbf{q}}_s(t_j), \quad \mathbf{R}_j = \mathbf{R}(t_j)$$
$$\begin{cases} \mathbf{q}_{s,j+1} = \mathbf{q}_{s,j} + h\dot{\mathbf{q}}_{s,j} + \frac{1}{4}h^2(\ddot{\mathbf{q}}_{s,j} + \ddot{\mathbf{q}}_{s,j+1}) \\ \dot{\mathbf{q}}_{s,j+1} = \dot{\mathbf{q}}_{s,j} + \frac{1}{2}h(\ddot{\mathbf{q}}_{s,j} + \ddot{\mathbf{q}}_{s,j+1}) \end{cases}$$
$$\mathbf{A}_s\ddot{\mathbf{q}}_{s,j+1} = \mathbf{V}_{s,j+1} \Rightarrow \ddot{\mathbf{q}}_{s,j+1} = \mathbf{A}_s^{-1}\mathbf{V}_{s,j+1}$$
$$\mathbf{A}_s = \mathbf{B}_s + \frac{1}{2}h\mathbf{D}_s + \frac{1}{4}h^2\mathbf{K}_s$$
$$\mathbf{V}_{s,j+1} = \mathbf{F}_s(\mathbf{R}_{j+1}) - \mathbf{D}_s\left(\dot{\mathbf{q}}_{s,j} + \frac{1}{2}h\ddot{\mathbf{q}}_{s,j}\right) - \mathbf{K}_s\left(\mathbf{q}_{s,j} + h\dot{\mathbf{q}}_{s,j} + \frac{1}{4}h^2\ddot{\mathbf{q}}_{s,j}\right) \tag{56}$$

Eq. (46) defining linear prediction of vector $\mathbf{R}$ remains valid and $\mathbf{R}_{-1} = \mathbf{R}_0$ due to the static equilibrium of the system in the $t \le 0$ interval. The extended initial conditions defined by Eqs. (17), (32) equal

$$\mathbf{q}_0 = \mathbf{0}, \quad \dot{\mathbf{q}}_0 = \mathbf{0}, \quad \ddot{\mathbf{q}}_0 = \mathbf{0}$$
$$\mathbf{q}_{s,0} = \begin{bmatrix} [M_o g/k_o + (M_o + M)g/k_M]\mathbf{1} \\ [(M_o + M)g/k_M]\mathbf{1} \end{bmatrix}, \quad \dot{\mathbf{q}}_{s,0} = \mathbf{0}, \quad \ddot{\mathbf{q}}_{s,0} = \mathbf{0}$$
$$\mathbf{R}_0 = \mathbf{G}_o + \mathbf{G} = (M_o + M)g\mathbf{1} \tag{57}$$

The implicit algorithm for numerical integration of Eqs. (16), (36) consists of the following stages:
1) prediction of $\mathbf{R}^p_{j+1}$ according to Eq. (46),
2) calculation of $\ddot{\mathbf{q}}_{j+1}, \ddot{\mathbf{q}}_{s,j+1}$ according to Eqs. (42), $(56)_7$,
3) calculation of $\mathbf{q}_{j+1}, \dot{\mathbf{q}}_{j+1}, \mathbf{q}_{s,j+1}, \dot{\mathbf{q}}_{s,j+1}$ from Eqs. (41), $(56)_{5,6}$,
4) correction of $\mathbf{R}^c_{j+1}$ according to Eqs. (48), (49) resulting from Eq. $(33)_1$,
5) control of the iteration ending condition according to Eq. (50).

4. Conclusions

A new conception of the fundamental tasks in dynamics of the bridge-track-train systems has been developed. Four fundamental tasks corresponding to moving load models in the form of a moving forces cyclic stream (model P), an unsprung masses cyclic stream (model M), a single-mass viscoelastic oscillators cyclic stream (model M_o) and a double-mass viscoelastic oscillators cyclic stream (model MM_o) have been considered.

A new approach to the moving load problem has been proposed in the form of modelling out of unsprung masses to sprung masses using one-way contact springs. A method for formulating implicit matrix equations of motion, governing vibrations of bridge-track-train systems, has been developed that results in very effective numerical simulation algorithms.

The governing equations and the numerical integration algorithms constitute a base for numerical simulation of the considered systems. These simulations and the analysis of the results is presented in the second part of this study (Podworna 2010).

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